# Green's Functions and Comparison Theorems for Differential Equations on Measure Chains

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**Abstract**

We are concerned with the self-adjoint equation $Lx(t) = [r(t)x^{\Delta}(t)]^{\Delta} + q(t)x(\sigma(t)) = 0$. We will study certain Green's functions associated with this equation. Comparison theorems for initial value problems and boundary value problems will be given.

Key words: *measure chains,time scale, Green's function*
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## 1 Introduction

We are concerned with the self-adjoint equation

$$Lx(t) = [r(t)x^{\Delta}(t)]^{\Delta} + q(t)x(\sigma(t)) = 0.$$

To understand this so-called differential equation on a measure chain(time scale) we need some preliminary definitions.
**Definition** Let T be a closed subset of the real numbers $R$. We assume throughout that T has the topology that it inherits from the standard topology on the real numbers $R$. For t < supT, define the forward jump operator by

$$\sigma(t) := inf\{\tau > t : \tau \in T\} \in T$$

and for t > inf T define the backwardjump operator by

$$\rho(t) := sup\{\tau < t : \tau \in T\} \in T$$

for all $t \in$ T. If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$ we say t is left scattered. If $\sigma(t) = t$ we say t is right dense, while if $\rho(t) = t$ we say t is left dense.

Throughout this paper we make the blanket assumption that a $\leq$ b are points in T.

**Definition** Define the interval in T

$$[a,b] := \{t \in T \ \ such \ \ that \ \ a \leq t \leq b\}.$$

Other types of intervals are defined similarly.

We are concerned with the calculus on measure chains which is a unified approach to continuous and discrete calculus. An excellent introduction to this subject is given by S. Hilger [7]. See also the monograph by Kaymakcalan, Laksmikantham, and Sivasudaram [8]. Agarwal and Bohner [1] refer to it as calculus on time scales. Other papers in this area include Agarwal and Bohner [3], Agarwal, Bohner, and Wong [4], and Hilger and Erbe [6]. In a forthcoming paper the authors will apply the techniques of this paper to prove the existence of positive solutions to general two-point boundary value problems. To do this we will use fixed point theorems for operators defined on appropriate cones in a Banach space.

**Definition** Assume $x : T \to R$ and fix $t \in T$ (if t=supT assume t is not left-scattered), then we define $x^{\Delta}$(t) to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,$$

for all s $\in$ U. We call $x^{\Delta}(t)$ the delta derivative of x(t).
It can be shown that if $x : T \to R$ is continuous at $t \in T$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

If t is right dense, then

$$x^{\Delta}(t) = \lim_{s \to t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s}.$$

**Definition** We say $x : T \to R$ is right dense continuous on T provided it is continuous at all right dense points and at points that are left dense and right scattered we just assume the left hand limit exists (and is finite). We denote this by x $\in C_{rd}(T)$.

**Definition** Define the set D to be the set of all functions $x : T \to R$, such that $x^{\Delta}(t)$ is continuous on T and $[r(t)x^{\Delta}(t)]^{\Delta}$ is right dense continuous on T.

We say that a function x is a solution of Lx=0 on T provided x $\in$ D and Lx(t)=0 for all t $\in$ T.

**Example 1** *If $T = Z$, the set of integers, then*

$$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t).$$

*Furthermore the equation Lx=0 reduces to the self-adjoint difference equation*

$$\Delta[r(t)\Delta x(t)] + q(t)x(\sigma(t) = 0.$$

*See the books [2] and [9] and the references there for results concerning this self-adjoint difference equation.*

**Example 2** *If $T = R$, then the equation Lx=0 reduces to the self-adjoint differential equation*

$$Lx(t) = [r(t)x'(t)]' + q(t)x(t) = 0,$$

*which has been studied extensively over the years.*

**Example 3** *Consider the differential equation*

$$x^{\Delta\Delta}(t) + \pi^2 x(\sigma(t)) = 0,$$

*on the measure chain*

$$T = [0, 1] \cup \cup_{n=2}^{\infty}\{n\}.$$

*Let x(t) be the solution of the above equation satisfying the initial conditions $x(0) = 0, x^{\Delta}(0) = 1$. One can show that*

$$x(t) = \begin{cases} \frac{1}{\pi} sin\pi t, & 0 \leq t \leq 1 \\ -1, & t=2 \\ (2 - \pi^2)x(t-1)-x(t-2), & t=3,4,\ldots \end{cases}$$

*Note that our solution "pieces" together a solution of a differential equation and a solution of a difference equation.*

**Example 4** *Consider the differential equation*

$$x^{\Delta\Delta}(t) + \pi^2 x(\sigma(t)) = 0,$$

*on the measure chain*

$$T = \cup_{n=0}^{\infty}[2n, 2n + 1].$$

*Let x(t) be the solution of the above equation satisfying the initial conditions $x(0) = 0, x^{\Delta}(0) = 1$. One can show that this solution on [0,1]∪[2,3] is given by*

$$x(t) = \begin{cases} \frac{1}{\pi} sin\pi t, & 0 \leq t \leq 1 \\ -cos\pi t+\frac{\pi^2-1}{\pi} sin\pi t, & 2 \leq t \leq 3. \end{cases}$$

In the next example our differential equation on a measure chain leads to a discrete problem with variable step size.

**Example 5** *Solve the IVP*

$$x^{\Delta\Delta}(t) + x(\sigma(t)) = 0,$$

$$x(0) = a, x^{\Delta}(0) = b,$$

*on the measure chain*

$$T = \cup_{k=0}^{\infty} T_k$$

*where*

$$T_k = \cup_{n=1}^{\infty}\{k + \frac{n-1}{n}\}.$$

*For $t \in T_0$, this solution is determined by the IVP*

$$x(\frac{n+1}{n+2}) + \frac{1-2n(n+1)^3}{n(n+1)^2(n+2)}x(\frac{n}{n+1}) + \frac{n}{n+2}x(\frac{n-1}{n}) = 0.$$

$$x(0) = a, \quad x^\Delta(0) = b.$$

*For $t \in T_1$, using Maple we get that this solution is determined by the IVP*

$$x(1+\frac{n+1}{n+2}) + \frac{1-2n(n+1)^3}{n(n+1)^2(n+2)}x(1+\frac{n}{n+1}) + \frac{n}{n+2}x(1+\frac{n-1}{n}) = 0.$$

$$x(1) = lim_{n\to\infty}x(\frac{n-1}{n}) \approx -.897a + 1.584b$$

$$x^\Delta(1) = lim_{n\to\infty}x^\Delta(\frac{n-1}{n}) \approx -1.637a + .801b.$$

*In particular, note that on each set $T_k$ the solution x(t) of the given IVP solves the same difference equation, where for $k \geq 1$ the initial conditions on $T_k$ are determined by the values of the solution on $T_{k-1}$.*

**Definition** If $F^\Delta(t) = f(t)$, then we define an integral by

$$\int_a^t f(\tau)\ \Delta\tau = F(t) - F(a).$$

In this paper we will use elementary properties of this integral that either are in the references [1,3-8] or are easy to verify.

**Definition** We say x(t,s) is the Cauchy function for Lx=0 provided for each fixed s $\in$ T, x(t,s) is the solution of the IVP Lx(t,s)=0,

$$x(\sigma(s), s) = 0,$$

$$x^\Delta(\sigma(s), s) = \frac{1}{r(\sigma(s))}.$$

It is easy to verify the following example.

**Example 6** *The Cauchy function for $Lx = [r(t)x^\Delta(t)]^\Delta = 0$ is given by*

$$x(t,s) = \int_{\sigma(s)}^t \frac{1}{r(\tau)}\Delta\tau.$$

We will use the following result in the next section, whose proof is a straight forward consequence of the definition of the delta derivative.

**Lemma 7** *Let a, $b \in T$ and assume $f^\Delta(t,s)$ is continuous on [a, $\sigma(b)$] $\times$ [a,b], then*

$$\{\int_a^t f(t,\tau)\Delta\tau\}^\Delta = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t),$$

$$\{\int_t^b f(t,\tau)\Delta\tau\}^\Delta = \int_t^b f^\Delta(t,\tau)\Delta\tau - f(\sigma(t),t).$$

## 2 Main Results

We will use the first formula in Lemma 7 to prove the following variation of constants formula.

**Theorem 8** *(Variation of constants formula) Assume h(t) is continuous on [a,b] and x(t,s) is the Cauchy function for Lx(t)=0, then it follows that*

$$x(t) := \int_a^t x(t,s)h(s)\Delta s$$

*is the solution of the IVP*

$$Lx(t) = h(t),$$
$$x(a) = 0, \quad x^{\Delta}(a) = 0.$$

*Proof:* Let x(t,s) be the Cauchy function for Lx=0 and set

$$x(t) = \int_a^t x(t,s)h(s)\Delta s.$$

Note that x(a)=0. Using the first formula in Lemma 7, we get that

$$x^{\Delta}(t) = \int_a^t x^{\Delta}(t,s)h(s)\Delta s + x(\sigma(t),t)h(t)$$
$$= \int_a^t x^{\Delta}(t,s)h(s)\Delta s,$$

since $x(\sigma(t),t)$=0. Hence, $x^{\Delta}(a)$=0. Also

$$r(t)x^{\Delta}(t) = \int_a^t r(t)x^{\Delta}(t,s)h(s)\Delta s.$$

Again using the first formula in Lemma 7, we get that

$$[r(t)x^{\Delta}(t,s)]^{\Delta} = \int_a^t [r(t)x^{\Delta}(t,s)]^{\Delta}h(s)\Delta s + r(\sigma(t))x^{\Delta}(\sigma(t),t)h(t)$$
$$= \int_a^t [r(t)x^{\Delta}(t,s)]^{\Delta}h(s)\Delta s + h(t).$$

It follows that

$$Lx(t) = \int_a^t Lx(t,s)h(s)\Delta s + h(t)$$
$$= h(t).$$

**Definition** Let a, b $\in$ T. We want to consider Lx(t)=0 on the interval $[a,\sigma^2(b)]$. We say a nontrivial solution of Lx=0 has a generalized zero at a iff x(a)=0. We say a nontrivial solution x(t) has a generalized zero at $t_0 \in (a,\sigma^2(b)]$, provided either $x(t_0)$=0 or $x(\rho(t_0))x(t_0) < 0$. Finally we say Lx=0 is disconjugate on $[a,\sigma^2(b)]$ provided there is no nontrivial solution of Lx=0 with two(or more) generalized zeros in $[a,\sigma^2(b)]$.

**Theorem 9** *(Comparison theorem for IVP's) Assume* $Lx = 0$ *is disconjugate on* $[a, \sigma^2(b)]$. *If* $u,\ v \in D$ *are functions satisfying*

$$Lu(t) \geq Lv(t), \qquad t \in [a, b],$$

$$u(a) = v(a), \quad u^\Delta(a) = v^\Delta(a),$$

*it follows that*

$$u(t) \geq v(t), \qquad t \in [a, \sigma^2(b)].$$

*Proof:* Let u(t), v(t) be as in the statement of this theorem and set

$$w(t) := u(t) - v(t),$$

for t $\in [a, \sigma^2(b)]$. Then

$$h(t) := Lw(t) = Lu(t) - Lv(t) \geq 0,$$

for t $\in [a, b]$. It follows that w(t) solves the IVP

$$Lw(t) = h(t),$$

$$w(a) = w^\Delta(a) = 0.$$

Hence by the variation of constants formula

$$w(t) = \int_a^t x(t, s)h(s)\Delta s.$$

Since Lx=0 is disconjugate on [a,$\sigma^2(b)$], x(t,s) $\geq 0$ for t$\geq \sigma(s)$. Note that

$$w(t) = \int_a^{\rho(t)} x(t, s)h(s)\Delta s + \int_{\rho(t)}^t x(t, s)h(s)\Delta s.$$

If t is left scattered, then

$$w(t) = \int_a^{\rho(t)} x(t, s)h(s)\Delta s + x(t, \rho(t))h(\rho(t))$$

$$= \int_a^{\rho(t)} x(t, s)h(s)\Delta s.$$

Since x(t,s) $\geq 0$ for t $\geq \sigma(s)$ we get that

$$w(t) \geq 0$$

which implies the desired result. If t is left dense it is easy to see that w(t) $\geq 0$.

Now consider the general two-point BVP

$$Lx = 0,$$

$$\alpha x(a) - \beta x^{\Delta}(a) = 0,$$
$$\gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0,$$

where we assume throughout that $\alpha, \beta, \gamma, and\ \delta$ are contants such that

$$\alpha^2 + \beta^2 > 0,$$

$$\gamma^2 + \delta^2 > 0.$$

**Theorem 10** *Assume that the BVP*

$$Lx = 0,$$
$$\alpha x(a) - \beta x^{\Delta}(a) = 0,$$
$$\gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0,$$

*has only the trivial solution. For each fixed s* $\in$ *[a,b], let u(t,s) be the unique solution of the BVP Lu(t,s)=0,* $\alpha u(a,s) - \beta u^{\Delta}(a,s) = 0$, $\gamma u(\sigma(b),s) + \delta u^{\Delta}(\sigma(b),s) = -\gamma x(\sigma(b),s) - \delta x^{\Delta}(\sigma(b),s))$, *where x(t,s) is the Cauchy function for Lx=0. Then*

$$G(t,s) = \begin{cases} u(t,s), & t \leq s \\ u(t,s)+x(t,s), & \sigma(s) \leq t \end{cases}$$

*is the Green's function for the BVP*

$$Lx = 0,$$
$$\alpha x(a) - \beta x^{\Delta}(a) = 0,$$
$$\gamma x(\sigma(b)) + \delta x^{\Delta}(\sigma(b)) = 0.$$

*Proof:* It is easy to see, for each fixed s $\in$ [a,b], that u(t,s) is uniquely determined by the BVP given in the statement of the theorem. Let u(t,s), x(t,s), and G(t,s) be as in the statement of this theorem and assume h(t) is a given continuous function on [a,b]. Then define

$$x(t) := \int_a^{\sigma(b)} G(t,s)h(s)\Delta s.$$

for $t \in [a, \sigma^2(b)]$. First note that

$$x(t) = \int_a^{\sigma(b)} G(t,s)h(s)\Delta s.$$

$$= \int_a^t G(t,s)h(s)\Delta s + \int_t^{\sigma(b)} G(t,s)h(s)\Delta s.$$

Using the definition of G(t,s) we get that

$$x(t) = \int_a^t [u(t,s) + x(t,s)]h(s)\Delta s + \int_t^{\sigma(b)} u(t,s)h(s)\Delta s.$$

Hence,

$$x^{\Delta}(t)=\int_a^t[u^{\Delta}(t,s)+x^{\Delta}(t,s)]h(s)\Delta s+[u(\sigma(t),t)+x(\sigma(t),t)]h(t)+\int_t^{\sigma(b)}u^{\Delta}(t,s)h(s)\Delta s-u(\sigma(t),t)h(t)$$

Simplifying we get that

$$x^{\Delta}(t)=\int_a^t[u^{\Delta}(t,s)+x^{\Delta}(t,s)]h(s)\Delta s+\int_t^{\sigma(b)}u^{\Delta}(t,s)h(s)\Delta s.$$

It follows that

$$x^{\Delta}(a)=\int_a^{\sigma(b)}G^{\Delta}(a,s)h(s)\Delta s$$

and

$$x^{\Delta}(\sigma(b))=\int_a^{\sigma(b)}G^{\Delta}(\sigma(b),s)h(s)\Delta s.$$

Therefore

$$\begin{aligned}\alpha x(a)-\beta x^{\Delta}(a)&=\int_a^{\sigma(b)}[\alpha G(a,s)-\beta G^{\Delta}(a,s)]h(s)\Delta s,\\ &=\int_a^{\sigma(b)}[\alpha u(a,s)-\beta u^{\Delta}(a,s)]h(s)\Delta s,\\ &=0.\end{aligned}$$

$$\gamma x(\sigma(b))+\delta x^{\Delta}(\sigma(b))=\int_a^{\sigma(b)}[\gamma G(\sigma(b),s)+\delta G^{\Delta}(\sigma(b),s)]h(s)\Delta s,$$

$$=\int_a^{\sigma(b)}\{\gamma[x(\sigma(b),s)+u(\sigma(b),s)]+\delta\;[x^{\Delta}(\sigma(b),s)+u^{\Delta}(\sigma(b),s)\}h(s)\Delta s,$$

$$=0.$$

Hence we have shown that x(t) satisfies the boundary conditions.
From above we get that

$$r(t)x^{\Delta}(t)=\int_a^t[r(t)u^{\Delta}(t,s)+r(t)x^{\Delta}(t,s)]h(s)\Delta s+\int_t^{\sigma(b)}r(t)u(t,s)h(s)\Delta s.$$

Therefore,

$$\begin{aligned}[r(t)x^{\Delta}(t)]^{\Delta}&=\int_a^t\{[r(t)u^{\Delta}(t,s)]^{\Delta}+[r(t)x^{\Delta}(t,s)]^{\Delta}\}h(s)\Delta s\\ &+r(\sigma(t))u^{\Delta}(\sigma(t),t)h(t)+r(\sigma(t))x^{\Delta}(\sigma(t),t)h(t)\\ &+\int_t^{\sigma(b)}[r(t)u^{\Delta}(t,s)]^{\Delta}h(s)\Delta s-r(\sigma(t))u^{\Delta}(\sigma(t),t)h(t)\\ &=\int_a^{\sigma(b)}[r(t)u^{\Delta}(t,s)]^{\Delta}h(s)\Delta s+\int_a^t[r(t)x^{\Delta}(t,s)]^{\Delta}h(s)\Delta s+h(t).\end{aligned}$$

From earlier in the proof we get that

$$x(t) = \int_a^{\sigma(b)} u(t,s)h(s)\Delta s + \int_a^t x(t,s)h(s)\Delta s.$$

It follows that

$$q(t)x(\sigma(t)) = \int_a^{\sigma(b)} q(t)u(\sigma(t),s)h(s)\Delta s + \int_a^{\sigma(t)} q(t)x(\sigma(t),s)h(s)\Delta s.$$

If t is right dense, then we get that

$$q(t)x(\sigma(t)) = \int_a^{\sigma(b)} q(t)u(\sigma(t),s)h(s)\Delta s + \int_a^t q(t)x(\sigma(t),s)h(s)\Delta s.$$

If t is right scattered, then

$$\int_a^{\sigma(t)} q(t)x(\sigma(t),s)h(s)\Delta s = \int_a^t q(t)x(\sigma(t),s)h(s)\Delta + \int_t^{\sigma(t)} q(t)x(\sigma(t),s)h(s)\Delta$$

Using the fact that $x(\sigma(t),t) = 0$, we also get that the second term is zero. Hence in either case we get that

$$q(t)x(\sigma(t)) = \int_a^{\sigma(b)} q(t)u(\sigma(t),s)h(s)\Delta s + \int_a^t q(t)x(\sigma(t),s)h(s)\Delta s.$$

Finally, we obtain

$$\begin{aligned} Lx(t) &= [r(t)x^{\Delta}(t)]^{\Delta} + q(t)x(\sigma(t)) \\ &= \int_a^{\sigma(b)} Lu(t,s)h(s)\Delta s + \int_a^t Lx(t,s)h(s)\Delta s + h(t) = h(t), \end{aligned}$$

for $t \in [a,b]$.

**Corollary 11** *(Green's function for the conjugate problem) Assume $Lx = 0$ is disconjugate on $[a, \sigma^2(b)]$. Let x(t,s) be the Cauchy function for $Lx = 0$ and for each fixed $s \in T$ let u(t,s) be the unique solution of the BVP $Lx = 0, u(a,s) = 0, u(\sigma^2(b), s) = -x(\sigma^2(b), s)$. Then*

$$G(t,s) := \begin{cases} u(t,s), & t \le s \\ u(t,s)+x(t,s), & \sigma(s) \le t \end{cases}$$

*is the Green's function for the BVP*

$$Lx(t) = 0,$$
$$x(a) = 0, \quad x(\sigma^2(b)) = 0.$$

*Proof:* Note that, since Lx=0 is discojugate on [a,$\sigma^2$(b)], it follows that the BVP Lx=0, x(a)=0, x($\sigma^2(b)$)=0 has only the trivial solution. If $\sigma(b)$ is right scattered then this Corollary follows from Theorem 9 with

$$\alpha \neq 0,\ \beta = 0,\ \delta = [\sigma^2(b) - \sigma(b)]\gamma,\ \gamma \neq 0.$$

On the other hand if $\sigma(b)$ is right dense then this result follows from Theorem 9 with

$$\alpha \neq 0,\ \beta = 0,\ \gamma \neq 0,\ \delta = 0.$$

**Corollary 12** *The Green's function for the BVP*

$$Lx = [r(t)x^\Delta(t)]^\Delta = 0,$$

$$x(a) = 0,$$

$$x(\sigma^2(b)) = 0,$$

*is given by*

$$G(t,s) = \begin{cases} -\dfrac{\int_a^t \frac{1}{r(\tau)}\Delta\tau \int_{\sigma(s)}^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}, & t \leq s \\ \\ -\dfrac{\int_a^{\sigma(s)} \frac{1}{r(\tau)}\Delta\tau \int_t^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}, & \sigma(s) \leq t \end{cases}$$

*Proof:* It is easy to see that

$$Lx = [r(t)x^\Delta(t)]^\Delta = 0$$

is disconjugate on [a, $\sigma^2(b)$]. By Corollary 11, the u(t,s) in the statement of Corollary 11 for each fixed s $\in$ [a,b] solves the BVP $Lx = 0$,

$$u(a,s) = 0,$$

$$u(\sigma^2(b), s) = 0.$$

Since

$$x_1(t) = 1 \quad and \quad x_2(t) = \int_a^t \frac{1}{r(\tau)}\Delta\tau$$

are solutions of Lx=0,

$$u(t,s) = \alpha(s) * 1 + \beta(s)\int_a^t \frac{1}{r(\tau)}\Delta\tau.$$

Using the boundary conditions $u(a,s) = 0$ and $u(\sigma^2(b)) = 0$ it can be shown that

$$u(t,s) = -\frac{\int_a^t \frac{1}{r(\tau)}\Delta\tau \int_{\sigma(s)}^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}.$$

Hence G(t,s) has the desired form for t≤ s. By Corollary 11 for t $\geq \sigma(s)$,

$$G(t,s) = x(t,s) + u(t,s),$$

where $x(t,s)$ is the Cauchy function for $Lx = [r(t)x^{\Delta}(t)]^{\Delta}$. From Example 6

$$x(t,s) = \int_{\sigma(s)}^{t} \frac{1}{r(\tau)} \Delta\tau.$$

Therefore

$$G(t,s) = \int_{\sigma(s)}^{t} \frac{1}{r(\tau)} \Delta\tau - \frac{\int_a^t \frac{1}{r(\tau)}\Delta\tau \int_{\sigma(s)}^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}.$$

Getting a common denominator,

$$G(t,s) = \frac{\int_{\sigma(s)}^t \frac{1}{r(\tau)}\Delta\tau \int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau - \int_a^t \frac{1}{r(\tau)}\Delta\tau \int_{\sigma(s)}^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}.$$

$$= \frac{\int_{\sigma(s)}^t \frac{1}{r(\tau)}\Delta\tau [\int_a^t \frac{1}{r(\tau)}\Delta\tau + \int_t^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau] - \int_a^t \frac{1}{r(\tau)}\Delta\tau \int_{\sigma(s)}^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}.$$

$$= \frac{\int_a^t \frac{1}{r(\tau)}\Delta\tau [-\int_t^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau] + \int_{\sigma(s)}^t \frac{1}{r(\tau)}\Delta\tau \int_t^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}.$$

$$= -\frac{\int_t^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau [\int_a^t \frac{1}{r(\tau)}\Delta\tau - \int_{\sigma(s)}^t \frac{1}{r(\tau)}\Delta\tau]}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}.$$

Hence for $t \geq \sigma(s)$,

$$G(t,s) = -\frac{\int_a^{\sigma(s)} \frac{1}{r(\tau)}\Delta\tau \int_t^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau}{\int_a^{\sigma^2(b)} \frac{1}{r(\tau)}\Delta\tau},$$

which is the desired result.

For the special case of Corollary 12 where T=$Z$, the Green's function is given in Example 1.26 in [2]. See Section 7.5 in [2] for the vector case.

**Theorem 13** *(Comparison theorem for conjugate BVP's) Assume* $Lx = 0$ *is disconjugate on* $[a, \sigma^2(b)]$. *If u(t), v(t) are functions in D satisfying*

$$Lu(t) \leq Lv(t), \qquad t \in [a,b],$$

$$u(a) \geq v(a), \quad u(\sigma^2(b)) \geq v(\sigma^2(b)),$$

*it follows that*

$$u(t) \geq v(t), \qquad t \in [a, \sigma^2(b)].$$

*Proof:* Let u(t) and v(t) be as in the statement of the theorem and set

$$w(t) = u(t) - v(t),$$

for t $\in [a, \sigma^2(b)]$. Set

$$h(t) = Lw(t), \qquad t \in [a, b],$$

then

$$h(t) = Lu(t) - Lv(t) \leq 0, \qquad t \in [a, b].$$

It follows that w(t) solves the BVP

$$Lw(t) = h(t), \qquad t \in [a, \sigma^2(b)],$$

$$w(a) = A, \quad w(\sigma^2(b)) = B,$$

where

$$A := u(a) - v(a) \geq 0, \quad B := u(\sigma^2(b)) - v(\sigma^2(b)) \geq 0.$$

It follows that

$$w(t) = \phi(t) + \int_a^{\sigma(b)} G(t, s)h(s)\Delta s,$$

where $\phi(t)$ solves the BVP

$$L\phi(t) = 0$$

$$\phi(a) = A, \quad \phi(\sigma^2(b)) = B.$$

From the disconjugacy

$$\phi(t) \geq 0, \quad t \in [a, \sigma^2(b)],$$

and

$$G(t, s) \leq 0,$$

$$t \in [a, \sigma^2(b)], \quad s \in [a, b].$$

It follows that

$$w(t) \geq 0,$$

which gives the desired result.

The following result shows that a nontrivial solution of Lx=0 can not have a "double zero".

**Theorem 14** *If x(t) is a nontrivial solution of Lx=0 such that x(t) $\geq$ 0 on [a,$\sigma^2(b)$], then x(t)) $> 0$ on (a,$\sigma(b)$).*

*Proof:* Assume that x(t) is a nontrivial solution of Lx=0 such that x(t) $\geq 0$ on [a,$\sigma^2(b)$]. We will assume there is a $\tau$ in (a,$\sigma^2$(b)) such that x($\tau$)=0 and show that this leads to a contradiction. First we show that

$$x^\Delta(\tau) > 0.$$

If $\tau$ is right scattered, then

$$x^{\Delta}(\tau) = \frac{x(\sigma(\tau)) - x(\tau)}{\sigma(\tau) - \tau}$$

$$= \frac{x(\sigma(\tau))}{\sigma(\tau) - \tau} > 0,$$

since x(t) is a nontrivial solution. On the other hand if $\tau$ is right dense, then

$$x^{\Delta}(\tau) = \lim_{s \to \tau} \frac{x(\sigma(\tau)) - x(s)}{\sigma(\tau) - s}$$

$$= \lim_{s \to \tau} \frac{-x(s)}{\tau - s}$$

$$= \lim_{s \to \tau^{+}} \frac{-x(s)}{\tau - s}$$

$$\geq 0.$$

Since $x(\tau)$ =0 and x(t) is a nontrivial solution we have that

$$x^{\Delta}(\tau) > 0.$$

Hence in all cases we have that this last inequality holds. To get a contradiction we consider the two possibilities that $\tau$ is left scattered or $\tau$ is left dense. First assume $\tau$ is left scattered. Using x(t) is a solution we get after an integration that

$$\{r(t)x^{\Delta}(t)\}_{\rho(\tau)}^{\tau} = -\int_{\rho(\tau)}^{\tau} q(t)x(\sigma(t))\Delta t.$$

$$= -q(\rho(\tau))x(\sigma(\rho(\tau)))$$

$$= -q(\rho(\tau))x(\tau) = 0.$$

Hence

$$r(\tau)x^{\Delta}(\tau) = r(\rho(\tau))x^{\Delta}(\rho(\tau)).$$

This implies that

$$x^{\Delta}(\rho(\tau)) > 0.$$

But

$$x^{\Delta}(\rho(\tau)) = \frac{x(\tau) - x(\rho(\tau))}{\tau - \rho(\tau)}$$

$$= \frac{-x(\rho(\tau))}{\tau - \rho(\tau)} \leq 0,$$

which is a contradiction.
Finally consider the case where we assume $\tau$ is left dense. In this case using the fact that $x^{\Delta}(t)$ is left continuous at $\tau$ we get that there is a $\delta > 0$ such that

$$x^{\Delta}(t) > \frac{x^{\Delta}(\tau)}{2}$$

when t $\in [\tau-\delta, \tau]$. Let $\tau_0 \in (\tau-\delta,\, \tau)$ and integrate from $\tau_0$ to $\tau$ to get that

$$x(\tau) - x(\tau_0) = \frac{x^\Delta(\tau)}{2}\int_{\tau_0}^{\tau} \Delta s \geq 0.$$

Since $x(\tau)$=0 we get that

$$x(\tau_0) \leq 0,$$

which is a contradiction.

**Corollary 15** *If, in Theorem 8, we either assume in addition that $Lu(t) < Lv(t)$ on a subset of [a,b] of positive measure or if we assume in addition that one of the inequalities $u(a) \geq v(a)$, $u(\sigma^2(b)) \geq v(\sigma^2(b))$ is strict, then we get that*

$$u(t) > v(t)$$

*for $t \in (a, \sigma^2(b))$.*

*Proof:*

Next we consider the focal BVP

$$Lx = 0,$$

$$x(a) = 0, \quad x^\Delta(\sigma(b)) = 0.$$

The following result follows from Theorem 10 with

$$\beta = \gamma = 0.$$

**Corollary 16** *(Green's function for focal BVP) Assume the BVP $Lx = 0, x(a) = 0, x^\Delta(\sigma(b)) = 0$ has only the trivial solution. For each fixed $s \in$ [a,b] let u(t,s) be the solution of the BVP $Lu(t,s) = 0, u(a,s) = 0, u^\Delta(\sigma(b),s) = -x^\Delta(\sigma(b),s)$, where x(t,s) is the Cauchy function of Lx=0. Then*

$$G(t,s) := \begin{cases} u(t,s), & t \leq s \\ u(t,s)+x(t,s), & \sigma(s) \leq t \end{cases}$$

*is the Green's function for the focal BVP*

$$Lx(t) = 0,$$

$$x(a) = 0, \;\; x^\Delta(\sigma(b)) = 0.$$

**Corollary 17** *The Green's function for the focal BVP*

$$Lx = [r(t)x^\Delta(t)]^\Delta = 0,$$

$$x(a) = 0,$$

$$x^{\Delta}(\sigma(b)) = 0,$$

*is given by*

$$G(t,s) = \begin{cases} -\int_a^t \frac{1}{r(\tau)}\Delta\tau, & t \le s \\ -\int_a^{\sigma(s)} \frac{1}{r(\tau)}\Delta\tau, & \sigma(s) \le t \end{cases}$$

*Proof:* It is easy to see that the focal BVP given in the statement of this theorem has only the trivial solution. Hence we can apply Corollary 11 to find the focal Green's function G(t,s). For $t \le s$, G(t,s)=u(t,s), where for each fixed s, u(t,s) solves the BVP

$$[r(t)x^{\Delta}(t)]^{\Delta} = 0,$$

$$u(a,s) = 0,$$

$$u(\sigma(b)) = -x(\sigma(b),s).$$

Solving this BVP we get that

$$u(t,s) = -\int_a^t \frac{1}{r(\tau)}\Delta\tau,$$

which is the desired expression for G(t,s) for $t \le s$. For $t \ge \sigma(s)$,

$$G(t,s) = u(t,s) + x(t,s),$$

$$= -\int_a^t \frac{1}{r(\tau)}\Delta\tau + \int_{\sigma(s)}^t \frac{1}{r(\tau)}\Delta\tau.$$

Simplifying we get the desired conclusion

$$G(t,s) = -\int_a^{\sigma(s)} \frac{1}{r(\tau)}\Delta\tau,$$

for $t \ge \sigma(s)$.

**Definition** We say Lx=0 is disfocal on [a,$\sigma^2(b)$] provided there is no nontrivial solution x(t) such that x(t) has a generalized zero in [a,$\sigma^2(b)$] followed by a generalized zero of $x^{\Delta}(t)$ in [a,$\sigma^2(b)$].

The proof of the following result is similar to the proof of some earlier results in this paper and will be omitted.

**Corollary 18** *(Comparison theorem for focal BVP's) Assume u,v $\in$ D, Lx=0 is disfocal on* $[a,\sigma^2(b)]$,

$$Lu(t) \le Lv(t), \quad t \in [a,b],$$

*and*

$$u(a) \ge v(a), \quad u^{\Delta}(\sigma(b)) \ge v^{\Delta}(\sigma(b))$$

*then u(t) $\ge$ v(t) for t $\in$* $[a,\sigma^2(b)]$.

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