Square Integrability of Gaussian Bells on Time Scales

Lynn Erbe[1], Allan Peterson[1] and Moritz Simon[2]

[1]Department of Mathematics, University of Nebraska-Lincoln
Lincoln, NE 68588-0323
lerbe@math.unl.edu
apeterso@math.unl.edu

[2]Department of Mathematics, Munich University of Technology
Boltzmannstraße 3, D-85747 Garching
ms.skabba@t-online.de
he-man@soundfreaks.de

Abstract

We will consider a generalization $\mathbf{E}(x)$ of the continuum Gaussian bell $e^{-x^2/2}$ on a time scale $\mathbb{T}$. A crucial question is whether this Gaussian bell is square integrable on the corresponding time scale. However, this is not the case in general. We will establish sufficient and necessary conditions for the time scale $\mathbb{T}$ to satisfy $\mathbf{E} \in L^2(\mathbb{T})$.

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1 Motivation of the Subject

In one-dimensional quantum mechanics, trying to compute the probability density $|\psi(x)|^2$ of measuring a particle signal at position $x \in \mathbb{R}$, one is usually led to the well-known stationary Schrödinger equation

$$\mathcal{H}\psi(x) = \left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = -\psi''(x) + V(x)\psi(x) = E\psi(x) \quad \forall\, x \in \mathbb{R}, \tag{1}$$

where the operator $-d^2/dx^2$ represents the kinetic energy and the operator $V(x)$ corresponds to the potential energy. (Of course this is only true after some rescaling procedures in the

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argument. The "operator" $V(x)$ is to be understood as $[V(x)\psi](x) \equiv V(x)\psi(x)$ — a multiplication process.) E is thus the total energy of the particle — an eigenvalue of the Hamiltonian operator $\mathcal{H}$.

For a quadratic potential $V(x) = x^2$, occurring e.g. in harmonic oscillator interactions, the eigenvalue problem (1) can be solved by the following *ladder formalism*: Decompose the Hamiltonian $\mathcal{H}$ into

$$\mathcal{H} = \mathcal{A}^\dagger\mathcal{A} + \mathcal{I} \equiv \left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right) + \mathcal{I}, \tag{2}$$

$\mathcal{I}$ denoting the identity operator. The operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ are adjoint to each other, being densely defined in $L^2(\mathbb{R})$. Furthermore they satisfy the following commutator relation:

$$[\mathcal{A}, \mathcal{A}^\dagger] = \mathcal{A}\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{A} = 2\mathcal{I}. \tag{3}$$

Exploiting (3), we easily see that, if we have an eigenfunction φ_n of $\mathcal{A}^\dagger\mathcal{A}$ corresponding to some eigenvalue λ_n, we obtain another one with eigenvalue $\lambda_{n+1} = \lambda_n + 2$, defined by $\varphi_{n+1} = \mathcal{A}^\dagger\varphi_n$:

$$\mathcal{A}^\dagger\mathcal{A}\varphi_{n+1} = \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger\varphi_n = \mathcal{A}^\dagger(\mathcal{A}^\dagger\mathcal{A}\varphi_n + 2\varphi_n) = \mathcal{A}^\dagger(\lambda_n + 2)\varphi_n = (\lambda_n + 2)\varphi_{n+1}.$$

A "first" eigenfunction φ_0 of $\mathcal{A}^\dagger\mathcal{A}$ with respect to $\lambda_0 = 0$ is fixed by $\mathcal{A}\varphi_0 \equiv 0$, which is solved by the *Gaussian bell* $e^{-x^2/2}$. Now, keeping in mind the decomposition (2) of $\mathcal{H}$, we have at hand many functions φ_n satisfying the Schrödinger equation for a quadratic potential, namely

$$\varphi_n \equiv (\mathcal{A}^\dagger)^n\varphi \quad \forall n \in \mathbb{N}_0 \quad \text{where} \quad \varphi(x) = e^{-x^2/2}.$$

The corresponding eigenvalues are $E_n = \lambda_n + 1 = 2n + 1$. It can be shown that the eigenfunctions φ_n constitute an orthogonal basis of $L^2(\mathbb{R})$ — indicating that every probability density $|\psi|^2$ solving (1) in this case can be written in terms of the normalized eigenfunctions ψ_n:

$$\psi(x) = \sum_{n=0}^{\infty} c_n\psi_n(x).$$

The physical interpretation of this is that the coefficients c_n of the state ψ show that the probability of measuring the particle at energy level $E_n = 2n + 1$ is $|c_n|^2$.

The Gaussian bell $\varphi_0 \equiv \varphi$ is the so-called ground state of the physical particle. It is essential that this function be *square integrable*, since it must be possible to normalize its square to a probability density. Similar ladder formalisms have been constructed in order to consider discrete generalizations of this Schrödinger scenario, e.g. on so-called *unitary linear lattices*, being a certain mixture of q-linear grids and equidistant lattices — see [5] for details. In any case, it is of interest to find such discrete formalisms, since many potentials $V(x)$ in quantum mechanics do not allow an analytic treatment; for instance $V(x) = x^4$ cannot even be handled by perturbation methods. The only way to tackle these obstacles is to *discretize*. Of special importance are grids which are dense around several points and increasingly scattered at some distance from those points. (Such a "point" can be thought of as a proton attracting an electron — the probability density $|\psi(x)|^2$ of the ground state has then a peak close to the position of the proton. This is of course not an exact modelling, but quite reasonable for the development of a quantum intuition.) The q-linear grids, respectively, unitary lattices, satisfy this property.

Motivated by these tasks, we are going to consider a "Gaussian bell" on a general closed subset of the real numbers, known as a *time scale* $\mathbb{T} \subset \mathbb{R}$. (Notice that the lattices mentioned above are just special cases of time scales.) However, we will not state all the basic facts of the time scales calculus here. Those who are not familiar with this calculus are referred to the introductory book [2] and its sequel [3], or the fundamental article [4]. Our main question shall be:

Is the Gaussian bell square integrable on the time scale $\mathbb{T}$?

2 Definition of the Gaussian Bell and Properties

In order to define a generalized Gaussian bell on a time scale $\mathbb{T}$, we first have to consider some basics about *exponential functions* on $\mathbb{T}$. In the book [2] it is shown that the dynamic initial value problem

$$f^{\Delta}(x) = p(x)f(x), \quad f(x_0) = 1 \tag{4}$$

has a unique solution $f(x) \equiv e_p(x, x_0)$ if the function $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous (continuous at rd-points and at ld-points left hand limits exist and are finite) and *regressive*, which means

$$1 + \mu(x)p(x) \neq 0 \quad \forall\, x \in \mathbb{T}. \tag{5}$$

The regressivity condition (5) is also necessary. The exponential function $e_p(x, x_0)$ is then explicitly given by

$$e_p(x, x_0) = e^{\int_{x_0}^{x} \tilde{p}(t)\Delta t},$$

where the cylinder transform $\tilde{p} \equiv \xi_\mu(p)$ of the function p is defined by

$$\xi_\mu(p)(x) = \begin{cases} \frac{\mathrm{Log}(1+\mu(x)p(x))}{\mu(x)} & \text{if} \quad \mu(x) > 0 \\ p(x) & \text{if} \quad \mu(x) = 0, \end{cases}$$

where $\mathrm{Log}(z)$ denotes the principal logarithm of $z \neq 0$. If p is, in addition, a *positively regressive* function, i.e.

$$1 + \mu(x)p(x) > 0 \quad \forall\, x \in \mathbb{T}, \tag{6}$$

then it is easy to check that the solution of (4) is positive for all time: $e_p(x, x_0) > 0$ on $\mathbb{T}$. Our Gaussian bell shall be a particular exponential function. Since we want it to be positive — preserving the respective property of its continuum counterpart —, we need to guarantee the function p under consideration is positively regressive. The following result is due to Bohner and Akın-Bohner, see [1]:

Theorem 1 *The set $\mathcal{R}^+$ of all rd-continuous and positively regressive functions $p : \mathbb{T} \to \mathbb{R}$, supplied with the "circle-plus" addition*

$$p \oplus q(x) = p(x) + q(x) + \mu(x)p(x)q(x)$$

and the "circle-dot" multiplication

$$\alpha \odot p(x) = \begin{cases} \frac{(1+\mu(x)p(x))^{\alpha}-1}{\mu(x)} & if \quad \mu(x) > 0 \\ \alpha p(x) & if \quad \mu(x) = 0, \end{cases}$$

is a real vector space $(\mathcal{R}^+, \oplus, \odot)$.

One can even define the "$\odot$-product" of a function $f : \mathbb{T} \to \mathbb{R}$ and a positively regressive function $p : \mathbb{T} \to \mathbb{R}$ (not necessarily rd-continuous) in a pointwise sense:

$$(f \odot p)(x) \equiv f(x) \odot p(x) = \begin{cases} \frac{(1+\mu(x)p(x))^{f(x)}-1}{\mu(x)} & \text{if} \quad \mu(x) > 0 \\ f(x)p(x) & \text{if} \quad \mu(x) = 0. \end{cases}$$

It is straightforward to see that $f \odot p$ still satisfies (6). In particular, one can conclude that the function $x \odot p(x)$ is in $\mathcal{R}^+$ if $p \in \mathcal{R}^+$. Now let us motivate the definition of our generalized Gaussian bell.

The continuum Gaussian bell $\varphi(x) = e^{-x^2/2}$ satisfies the initial value problem

$$\varphi'(x) = -x\varphi(x), \quad \varphi(0) = 1. \tag{7}$$

Furthermore it is an even function. In order to take this over to the time scales setting, the time scale $\mathbb{T}$ under consideration should be symmetric with respect to 0 and contain 0. Therefore, all time scales appearing throughout the remainder of the paper will be assumed to satisfy these properties. In any case, later on we are only going to consider the nonnegative part of the time scale. Let us denote the nonnegative part of $\mathbb{T}$ by

$$\mathbb{T}_+ \equiv \{x \in \mathbb{T} \mid x \geq 0\}. \tag{8}$$

The Gaussian bell $\mathbf{E}(x)$ — being an exponential function — should thus satisfy a relation of the form

$$\mathbf{E}^{\Delta}(x) = p(x)\mathbf{E}(x) \quad \forall\, x \in \mathbb{T}_+, \quad \mathbf{E}(0) = 1, \tag{9}$$

where $p(x)$ must be $-x$ in the case $\mathbb{T} = \mathbb{R}$. However, we cannot take $p(x) = -x$ in the general case, since this function does not fulfill (6). Rather we will take $p(x) = \ominus(x \odot 1)$ because $1 \in \mathcal{R}^+$ holds in any case and $p \in \mathcal{R}^+$ is then true by the above results. This leads us to the following

Definition. On a time scale $\mathbb{T}$, we define the *Gaussian bell* $\mathbf{E}(x)$ by

$$\mathbf{E}^{\Delta}(x) = \ominus(x \odot 1)\mathbf{E}(x) \quad \forall\, x \in \mathbb{T}_+, \quad \mathbf{E}(0) = 1 \tag{10}$$

and $\mathbf{E}(-x) = \mathbf{E}(x)$ on $\mathbb{T}$.

What are the properties of this function? By construction, $\mathbf{E}(x)$ is an even function and positive on the whole time scale. To find out more, we rewrite (10) as a recurrence relation: Using the definition of $\odot$ and $\ominus$, we obtain

$$\ominus(x \odot 1) = \frac{(1 + \mu(x))^{-x} - 1}{\mu(x)}$$

if $\mu(x) > 0$. Taking this into account, we recognize that (10) immediately yields the relation

$$\mathbf{E}(\sigma(x)) = (1+\mu(x))^{-x}\mathbf{E}(x) \quad \forall\, x \in \mathbb{T}_+, \quad \mathbf{E}(0) = 1. \tag{11}$$

Since $\mathbf{E}(x)$, in addition, satisfies the differential equation of the continuum Gaussian bell $\varphi(x)$ at right-dense points $x \in \mathbb{T}_+$, thus has a nonpositive derivative at those points, we can conclude that $\mathbf{E}(x)$ is nonincreasing on $\mathbb{T}_+$. On a *discrete* time scale $\mathbb{T}$, i.e. a time scale $\mathbb{T}$ containing no continuum intervals, we can write down the Gaussian bell explicitly — provided $\mathbb{T}$ is countable:

$$\mathbf{E}(x) = \prod_{t\in[0,x)} (1+\mu(t))^{-t} \quad \forall\, x \in \mathbb{T}_+. \tag{12}$$

(This expression is obtained by iteration of (11); the product may consist of infinitely many factors!) However, the most important property of the continuum Gaussian bell is the fact that it is *square integrable*. In what follows, we intend to work out under which conditions this is also the case on a general time scale $\mathbb{T}$. It suffices to consider $\mathbb{T} \equiv \mathbb{T}_+$, in view of the definition of $\mathbf{E}(x)$.

Remark. Since we have defined the Gaussian bell as an even function, one could use the ∇-integral on $\mathbb{T}_-$ rather than the Δ-integral. Another way might be to drop the symmetry property. However, these questions will not be addressed in the remainder of this paper.

3 Examples

Let us finally give some examples of Gaussian bells.

Example 2 *On the time scale $\mathbb{T} = \mathbb{R}$, the Gaussian bell is by construction the continuum one:*

$$\mathbf{E}(x) = e^{-x^2/2} \quad \forall\, x \in \mathbb{R}$$

The next example is concerned with equidistant lattices.

Example 3 *Consider $\mathbb{T} = h\mathbb{Z}$ where $h > 0$. Then $\mathbf{E}(x)$ can be computed using formula (12):*

$$\mathbf{E}(hn) = \prod_{k=0}^{n-1} (1+h)^{-kh} = (1+h)^{-h\sum_{k=0}^{n-1} k} = (1+h)^{hn(1-n)/2} \quad \forall\, n \in \mathbb{N}_0$$

Resubstituting $x = hn$, we get

$$\mathbf{E}(x) = \left[(1+h)^{1/h}\right]^{-x(x-h)/2} \quad \forall\, x \in \mathbb{T}_+.$$

Notice that $\mathbf{E}(x) \geq e^{-x^2/2}$ for large x. In addition, $\mathbf{E}(x)$ converges to the continuum Gaussian bell as $h \to 0$ in the following sense: Define for each time scale $\mathbb{T} = h\mathbb{Z}$ where $h > 0$ the "embedded" bell $E_h : \mathbb{R} \to \mathbb{R}$ by $E_h(x) = E_h(-x)$ and

$$E_h(x) \equiv \mathbf{E}(hn) \quad \forall\, x \in [hn, h(n+1)),\ n \in \mathbb{N}_0.$$

Then we obtain the pointwise limit

$$\lim_{h \to 0} E_h(x) = e^{-x^2/2} \quad \forall\, x \in \mathbb{R},$$

as can be verified easily by considering our formula for $\mathbf{E}(x)$*.*

Now we have a look at q-linear grids.

Example 4 *Let* $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ *where* $0 < q < 1$*. Again we compute* $\mathbf{E}(x)$ *via (12). The result is*

$$\mathbf{E}(q^n) = \prod_{k=n+1}^{\infty} \left[1 + (q^{-1} - 1)q^k)\right]^{-q^k} \quad \forall\, n \in \mathbb{Z}.$$

This time the "limit" of $\mathbf{E}(x)$ *as* $q \to 1$ *is not easily observed to be the continuum bell, but that can indeed be proved.*

Example 5 *Let* $\mathbb{T} = {\mathbb{N}_0}^{\alpha}$ *where* $\alpha > 0$*. We obtain the following Gaussian bell:*

$$\mathbf{E}(n^\alpha) = \prod_{k=0}^{n-1} [1 + (k+1)^\alpha - k^\alpha]^{-k^\alpha} \quad \forall\, n \in \mathbb{N}_0.$$

In the special case of squared integers, i.e. $\alpha = 2$*, the expression can be further simplified to*

$$\mathbf{E}(n^2) = 2^{n(1-n)(2n-1)/6} \prod_{k=0}^{n-1} (k+1)^{-k^2} \quad \forall\, n \in \mathbb{N}_0.$$

Another frequently considered time scale is the set of harmonic numbers.

Example 6 *Let* $\mathbb{T} = \{H_n \mid n \in \mathbb{N}_0\}$ *where* $H_n = \sum_{k=1}^{n} 1/k$*. Then the Gaussian bell is given by*

$$\mathbf{E}(H_n) = \prod_{k=1}^{n} \left(\frac{k}{k+1}\right)^{H_{k-1}} \quad \forall\, n \in \mathbb{N}_0.$$

In our last example we consider a certain "mixture" of discrete and continuum structures, namely a union of closed intervals. Time scales of this type are important in biology, e.g. when one tries to compute the growth of plant populations — see [2], pages 71-73, for instance.

Example 7 *Let* $\mathbb{T} = \bigcup_{n=0}^{\infty} [2n, 2n+1]$*. Then the Gaussian bell has the explicit representation*

$$\mathbf{E}(x) = \left(\frac{e}{2}\right)^{n^2} e^{(n-x^2)/2} \quad \forall\, x \in [2n, 2n+1]. \tag{13}$$

Proof: Let us prove this by mathematical induction. The statement is true for $n = 0$, since $\mathbf{E}(x)$ is just the continuum bell on $[0, 1]$. So let us assume (13) is true for $n \in \mathbb{N}_0$. Then $\mathbf{E}(x)$ fulfills the following relation "between" the intervals:

$$\mathbf{E}(2n+2) = (1+1)^{-(2n+1)} \mathbf{E}(2n+1) = 2^{-2n-1} \mathbf{E}(2n+1) \tag{14}$$

Inserting the value at $x = 2n + 1$ into (14), from the induction assumption, we obtain

$$\mathbf{E}(2n+2) = 2^{-2n-1} \cdot \left(\frac{e}{2}\right)^{n^2} e^{(-4n^2-3n-1)/2} = \left(\frac{e}{2}\right)^{(n+1)^2} e^{-(4n^2+7n+3)/2}. \tag{15}$$

Now on the continuum interval $[2n+2, 2n+3)$ the Gaussian bell satisfies the differential equation $\mathbf{E}'(x) = -x\mathbf{E}(x)$, which implies

$$\mathbf{E}(x) = \mathbf{E}(2n+2)e^{[(2n+2)^2-x^2]/2} = \cdots = \left(\frac{e}{2}\right)^{(n+1)^2} e^{[(n+1)-x^2]/2} \tag{16}$$

on the interval $[2n+2, 2n+3]$ — (13) for $n+1$. □

Notice that the function in this last example satisfies $\mathbf{E}(x) \geq e^{-x^2/2}$ — the discrete gaps seem to make the Gaussian bell decrease more slowly!

4 A Sufficient Condition for Square Integrability

In this section we intend to prove that the Gaussian bell is square integrable on $\mathbb{T} \equiv \mathbb{T}_+$ if the time scale fulfills a certain "growth condition". This condition is fulfilled by all the time scales we have mentioned up to now, which means it is a rather weak condition for applications. The first result towards the general theorem concerns *isolated* time scales.

Lemma 8 *Let $\varepsilon > 0$ be arbitrary but fixed, moreover $\mathbb{T} \equiv \mathbb{T}_+$ be isolated with $\mu(x) \geq \varepsilon$ on $\mathbb{T}$. Then $\mathbf{E}(x)$ is square integrable on $\mathbb{T}$ provided the time scale satisfies the growth condition*

$$\frac{\sigma(x)}{\varrho(x)} = \mathcal{O}(\log \nu(x)) \quad \Longleftrightarrow \quad \sigma(x) = \mathcal{O}(\varrho(x) \log \nu(x)). \tag{17}$$

Remark. The $\mathcal{O}$-condition (17) should be understood in the following way:

$$\exists\, x_0 \in \mathbb{T},\ C > 0 : \quad \sigma(x) \leq C\varrho(x) \log \nu(x) \quad \forall\, x \geq x_0$$

The $\mathcal{O}$ is the so-called "Landau big O". For time scales fulfilling $\nu(x) \sim x$, i.e. $\nu(x)/x \to 1$ as $x \to \infty$, (17) can be simplified to

$$\sigma(x) = \mathcal{O}(\varrho(x) \log x), \tag{18}$$

which is a bit easier to check.

Proof: We want to guarantee the following inequality, at least for large $x \in \mathbb{T}$:

$$\mathbf{E}(x)^2 \leq \gamma^{\sigma(x)} \quad \text{where} \quad \gamma \in (0,1) \tag{19}$$

Why does this help us? Suppose (19) is fulfilled on the whole time scale. Then we can estimate

$$\int_{\mathbb{T}} \mathbf{E}(x)^2 \Delta x \leq \sum_{x \in \mathbb{T}} \gamma^{\sigma(x)} \mu(x) \leq \sum_{x \in \mathbb{T}} x \gamma^x. \tag{20}$$

In view of $\mu(x) \geq \varepsilon$, denoting the time scale $\mathbb{T} \equiv \{x_n \mid n \in \mathbb{N}, x_n < x_{n+1}\}$, we can find an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of nonnegative integers such that

$$\varepsilon n_k \leq x_k < \varepsilon(n_k + 1) \quad \forall\, n \in \mathbb{N}.$$

Inserting this "embedding" into (20), we obtain

$$\int_{\mathbb{T}} \mathbf{E}(x)^2 \Delta x \leq \sum_{k=1}^{\infty} \varepsilon(n_k + 1)\gamma^{\varepsilon n_k},$$

or equivalently, substituting $\delta \equiv \gamma^{\varepsilon} \in (0, 1)$,

$$\int_{\mathbb{T}} \mathbf{E}(x)^2 \Delta x \leq \varepsilon \sum_{k=1}^{\infty} (n_k + 1)\delta^{n_k} \leq \varepsilon \sum_{n=0}^{\infty} (n + 1)\delta^{n}.$$

But the last sum is finite by the ratio test.

The arguments are almost the same if (19) is just satisfied for large x — one has to split up into a finite sum and a sum for which the condition holds. But when does the inequality (19) hold? Since our time scale is isolated, we have an explicit formula for the Gaussian bell:

$$\mathbf{E}(x) = \prod_{t\in[0,x)} (1 + \mu(t))^{-t} \quad \forall\, x \in \mathbb{T}$$

As all factors in this product are smaller than 1, we may estimate $\mathbf{E}(x)^2 \leq \mathbf{E}(x)$ by the last factor:

$$\mathbf{E}(x)^2 \leq (1 + \nu(x))^{-\varrho(x)} \quad \forall\, x \in \mathbb{T}$$

Now (18) is of course satisfied provided

$$(1 + \nu(x))^{-\varrho(x)} \leq \gamma^{\sigma(x)} \tag{21}$$

holds for large x. Taking logarithms, (21) can be rewritten as follows:

$$\sigma(x) \leq -\varrho(x)\frac{\log(1 + \nu(x))}{\log \gamma}.$$

Now recall that $\gamma \in (0, 1)$ is arbitrary. Therefore we just have to guarantee

$$\sigma(x) \leq C\varrho(x) \log(1 + \nu(x)) \tag{22}$$

for some $C > 0$. For large x, (22) is clearly equivalent to the growth condition (17). □

So far we have only considered an isolated time scale. But what about its "dense" counterpart? In section 3, when considering the examples, we already got some intuitive knowledge that Gaussian bells on time scales containing dense parts seem to be "more" square integrable than the ones on isolated time scales. Now we will make this intuitive idea more precise.

Lemma 9 *Let $\varepsilon > 0$ be arbitrary, furthermore assume $\mathbb{T} \equiv \mathbb{T}_+$ satisfies $\mu(x) < \varepsilon\ \forall\, x \in \mathbb{T}$. Then the Gaussian bell $\mathbf{E}(x)$ is square integrable on $\mathbb{T}$.*

Proof: Without loss of generality let us assume $\mathbb{T}$ is discrete, countable — possibly containing some dense points. (Also right- or left-dense points are allowed.) Again we have

$$\mathbf{E}(x) = \prod_{t\in[0,x)} (1+\mu(t))^{-t} \quad \forall\, x \in \mathbb{T}.$$

Now define $I_k \equiv [\varepsilon k, \varepsilon(k+1))\ \forall\, k \in \mathbb{N}_0$ and $J_n \equiv [x_n, x_{n+1})\ \forall\, n \in \mathbb{N}$, where $(x_n)_{n\in\mathbb{N}}$ are chosen to be the smallest possible points in $\mathbb{T}$ satisfying

$$x_1 \in I_1, \quad \varepsilon \leq x_{n+1} - x_n < 2\varepsilon \quad \forall\, n \in \mathbb{N}. \tag{23}$$

(Notice that if $x_n \in I_k$, then x_{n+1} may either lie in I_{k+1} or in I_{k+2}.) Furthermore we define the following function $\mathcal{E} : \mathbb{R}_0^+ \to \mathbb{R}^+$, not being a Gaussian bell in general:

$$\mathcal{E}(x) \equiv 1 \quad \text{on } [0, x_1] \tag{24}$$
$$\mathcal{E}(x) \equiv \text{const} \quad \text{on } J_n \quad \forall\, n \in \mathbb{N} \tag{25}$$
$$\mathcal{E}(x_{n+1}) \equiv \mathcal{E}(x_n)(1+\varepsilon)^{-x_n} \quad \forall\, n \in \mathbb{N}. \tag{26}$$

The fact $\mathcal{E}(x) \in L^2(\mathbb{R}_0^+)$ follows as in the proof of Lemma 8, performing similar estimations. However, we shall not go into the details here.

Now if we prove

$$\mathbf{E}(x) \leq \mathcal{E}(x) \quad \forall\, x \in \mathbb{T}, \tag{27}$$

we obtain immediately

$$\int_{\mathbb{T}} \mathbf{E}(x)^2 \Delta x \leq \int_0^\infty \mathcal{E}(x)^2 dx < \infty.$$

By the construction of $\mathcal{E}(x)$, showing (27) is equivalent to showing

$$\mathbf{E}(x_n) \leq \mathcal{E}(x_n) \quad \forall\, n \in \mathbb{N}. \tag{28}$$

Let us finally prove this inequality by induction. Since we have clearly $\mathbf{E}(x_1) \leq \mathcal{E}(x_1)$, assume for the induction step (28) is true for some $n \in \mathbb{N}$. On the one hand we thus get

$$\mathbf{E}(x_{n+1}) = \mathbf{E}(x_n) \prod_{t\in J_n} (1+\mu(t))^{-t} \leq \mathcal{E}(x_n) \left[\prod_{t\in J_n} (1+\mu(t))\right]^{-x_n},$$

because of $t \geq x_n\ \forall\, t \in J_n$. On the other hand we have

$$\mathcal{E}(x_{n+1}) = \mathcal{E}(x_n)(1+\varepsilon)^{-x_n}$$

by the recurrence relation (26). Therefore it suffices to show

$$\prod_{t\in J_n} (1+\mu(t)) \geq 1+\varepsilon,$$

or equivalently, taking logarithms,

$$\sum_{t\in J_n} \log(1+\mu(t)) \geq \log(1+\varepsilon). \tag{29}$$

(This can be done since the possibly infinite product exists as a positive number. Moreover, in the case of nonexistence of the product, there is nothing to prove!) By the inequality in (23) we have $\varepsilon \leq \sum_{t\in J_n} \mu(t)$, therefore we only need to guarantee the following:

$$\sum_{t\in J_n} \log(1+\mu(t)) \geq \log\left(1+\sum_{t\in J_n}\mu(t)\right). \tag{30}$$

In order to do this, consider the function $f(x) \equiv \log(1+x)$. It is subadditive on $\mathbb{R}_0^+$, i.e.

$$f(x+y) \leq f(x)+f(y) \quad \forall\, x,y \geq 0. \tag{31}$$

For the sake of completeness, let us give an easy proof of this property: Apply exp to (31), yielding

$$1+x+y \leq (1+x)(1+y) = 1+x+y+xy \quad \forall\, x,y \geq 0.$$

But this is certainly true, proving (31). Notice now that the sums in (30) exist and that f is a continuous function. That is why the subadditivity (31) implies (30) at once, which completes the proof. □

Remark. The reason for the "w.l.o.g." in the proof of Lemma 9 is the following: Continuum intervals in the time scale even cause a faster decay of the Gaussian bell, at least for x large enough. We have seen this intuitively in the examples in section 3. However, working this out in detail is quite tedious and therefore not done here. (One also has to consider Cantor sets which are discrete but not countable!) In any case, the statement remains true.

Notice also that the function $\mathcal{E}(x)$ in the proof of the lemma is very similar to a Gaussian bell on the time scale $\mathbb{T} = \varepsilon\mathbb{Z}$, indeed can be estimated from above (in the sense of integrals) by the Gaussian bell of an equidistant time scale $\mathbb{T} = \delta\mathbb{Z}$ with $\delta \approx \varepsilon$. This indicates again what we had mentioned above; putting in dense parts of the time scale makes the Gaussian bell even "more" square integrable.

Now we are ready to state the general theorem about square integrability under certain sufficient growth conditions.

Theorem 10 *Let $\mathbb{T} = \mathbb{T}^+$ be a time scale containing 0. Then the Gaussian bell $\mathbf{E}(x)$ exists and is square integrable on $\mathbb{T}$ if the time scale obeys the growth condition*

$$\frac{\sigma(x)}{\varrho(x)} = \mathcal{O}(\log \nu(x)) \quad \Longleftrightarrow \quad \sigma(x) = \mathcal{O}(\varrho(x)\log \nu(x)). \tag{32}$$

Proof: The idea is to decompose the time scale $\mathbb{T}$ for arbitrary $\varepsilon > 0$ into an isolated part

$$\mathbb{T}_\varepsilon^+ = \{x \in \mathbb{T} \mid \mu(x) \geq \varepsilon\}$$

and a "dense" part

$$\mathbb{T}_\varepsilon^- = \{x \in \mathbb{T} \mid \mu(x) < \varepsilon\}.$$

Then we use Lemma 8 for $\mathbb{T}_\varepsilon^+$, Lemma 9 for $\mathbb{T}_\varepsilon^-$, and combine those two results appropriately. To see how this works, we recall that we have seen in the last remark that $\mathbb{T}_\varepsilon^-$ rather accelerates the decay of $\mathbf{E}(x)$ in comparison to $\mathbb{T}_\varepsilon^+$. Hence the square integrability in Lemma 8 is even strengthened by the supplementary part $\mathbb{T}_\varepsilon^-$. However, we will not work out the tedious details. □

Let us now consider in more detail the growth condition (32), respectively (17). It is quite clear that the time scales $\mathbb{R}$, $h\mathbb{Z}$, ${\mathbb{N}_0}^\alpha$ and the harmonic numbers fulfill the condition. So does, of course, the union of closed intervals in Example 7. Let us rather consider the q-linear grid $\mathbb{T} = \overline{q^{\mathbb{Z}}}$: Here we have $\sigma(x) = q^{-1}x$ and $\varrho(x) = qx$. The relation (32) now reads

$$\frac{q^{-1}x}{qx} = \mathcal{O}(\log((1-q)x)) \quad \Longleftrightarrow \quad 1 = \mathcal{O}(\log x).$$

Hence the q-linear grid obeys the growth condition and has $\mathbf{E} \in L^2(\mathbb{T})$. Now we are going to consider some time scales of huge growth in order to investigate borderline cases.

Example 11 *Let $\mathbb{T}$ be the time scale generated by the sequence $(x_n = n^{\omega n})_{n\in\mathbb{N}}$ where $\omega > 0$. Since this time scale satisfies $\nu(x_n) \sim x_n$, we just need to take (18) into consideration. On the one hand we obtain*

$$\frac{\sigma(x_n)}{\varrho(x_n)} = \frac{(n+1)^{\omega(n+1)}}{(n-1)^{\omega(n-1)}} = (n+1)^{2\omega}\left(1 + \frac{2}{n-1}\right)^{\omega(n-1)} = \mathcal{O}(n^{2\omega}),$$

on the other hand

$$\log x_n = \log(n^{\omega n}) = \omega n \log n.$$

Now, since $\log n$ increases more slowly than any positive power of n, the condition (18) will be fulfilled iff $\omega \leq 1/2$.

We learn from this example that $(\sqrt{n^n})_{n\in\mathbb{N}}$ constitutes a borderline case with respect to the growth condition. Does it also lie on the borderline concerning square integrability?

5 Counterexamples and Necessary Conditions

We will first show that the answer to the above question is NO.

Proposition 12 *The time scale $\mathbb{T} = \{0\} \cup \{x_n = n^n \mid n \in \mathbb{N}\}$ does not obey the growth condition (18), but possesses a square integrable Gaussian bell $\mathbf{E}(x)$.*

Proof: We must consider the following Gaussian bell:

$$\mathbf{E}(x_{n+1}) = \prod_{k=1}^{n} (1 + x_{k+1} - x_k)^{-x_k}.$$

As all the factors (except the first one) in this product are smaller than 1 and as $\mu(x_n) \sim x_{n+1}$ on this special time scale, we can choose $N \in \mathbb{N}$ such that

$$\mathbf{E}(x_{n+1}) \leq {x_{n+1}}^{-x_n} = (n+1)^{-(n+1)n^n} \leq (n+1)^{-n^{n+1}} \quad \forall\, n \geq N.$$

Therefore we obtain

$$\mathbf{E}(x_{n+1})^2\mu(x_{n+1}) \leq \frac{(n+2)^{n+2}}{(n+1)^{-n^{n+1}}} \leq \frac{1}{(n+1)^{-n^{n-1}}} \quad \forall\, n \geq N.$$

(The last inequality certainly holds if N is large enough.) Now we can easily estimate the integral:

$$\begin{aligned}\int_{\mathbb{T}} \mathbf{E}(x)^2\Delta x &= \int_0^{x_N} E(x)^2\Delta x + \sum_{n=N}^{\infty} \mathbf{E}(x_{n+1})^2\mu(x_{n+1}) \\ &\leq \text{const} + \sum_{n=N}^{\infty}(n+1)^{-n^{n-1}} < \infty\end{aligned}$$

This establishes the proof — notice the rapid convergence of the series! □

We have now shown that the growth condition (17) is sufficient but not necessary. Proposition 12 might make us think that the Gaussian bell is square integrable on every time scale. But this is NOT the case — consider the following:

Proposition 13 *Define the time scale* $\mathbb{T} = \{x_n \mid n \in \mathbb{N}, x_n < x_{n+1}\}$ *recursively such that* $x_1 = 0$ *and*

$$x_{n+1} \geq x_n + \prod_{k=1}^{n-1}(1 + x_{k+1} - x_k)^{2x_k} \quad \forall\, n \in \mathbb{N}. \tag{33}$$

Then the time scale is well-defined and $\mathbf{E} \notin L^2(\mathbb{T})$.

Proof: To see that such a time scale is well-defined, we just have to ensure that $x_n < x_{n+1}\ \forall\, n \in \mathbb{N}$. But this is the case since the product in (33) equals at least 1 in each step. The trick in the above construction lies in the fact that $\mathbf{E}(x_n)$ only depends on $x_1, \ldots, x_n$, but not on x_{n+1}. Thus we are free to define x_{n+1} in terms of $\mathbf{E}(x_n)$ — in (33) we construct it to satisfy

$$\mu(x_n) = x_{n+1} - x_n \geq \prod_{k=1}^{n-1}(1 + x_{k+1} - x_k)^{2x_k} = \mathbf{E}(x_n)^{-2} \quad \forall\, n \in \mathbb{N}. \tag{34}$$

(Notice that the finite products cannot diverge!) The relation (34) is equivalent to

$$\mathbf{E}(x_n)^2\mu(x_n) \geq 1 \quad \forall\, n \in \mathbb{N},$$

which forces the integral of $\mathbf{E}(x)^2$ to diverge. □

Let us now consider two examples of the form $\mathbb{T} = \{x_n \mid n \in \mathbb{N}, x_n < x_{n+1}\}$, one for which the Gaussian bell is in "L^2" and one for which the Gaussian bell is not in "L^2". What is the asymptotic growth of these time scales?

Example 14 *Define* $\mathbb{T}_1$ *recursively such that* $x_1 = 0$ *and*

$$x_{n+1} = x_n + \prod_{k=1}^{n-1}(1 + x_{k+1} - x_k)^{2x_k} \quad \forall\, n \in \mathbb{N}. \tag{35}$$

Then $\mathbf{E}_1 \notin L^2(\mathbb{T}_1)$ *by Proposition 13. The growth of this time scale is huge:* $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 6$, $x_5 = 2506$, $x_6 \approx 2500^{13}$ *and so on. Because of this growth we can write down (35) asymptotically:*

$$x_{n+1} \sim C \prod_{k=N}^{n-1} {x_{k+1}}^{2x_k}.$$

Here the constant C *involves those factors up to some* $N-1$ *in the product, where* $(1pays + x_{k+1} - x_k)/x_{k+1} \approx 1$ *is not yet fulfilled. Taking into account the same asymptotic representation for* x_n *(product up to* $n-2$*) and resubstituting, we obtain*

$$x_{n+1} \sim x_n \cdot {x_n}^{2x_{n-1}} = {x_n}^{2x_{n-1}+1}. \tag{36}$$

This is the desired growth relation.

Example 15 *Define* $\mathbb{T}_2$ *recursively by* $x_1 = 0$ *and*

$$x_{n+1} = x_n + \frac{1}{n^2} \prod_{k=1}^{n-1} (1 + x_{k+1} - x_k)^{2x_k} \quad \forall\, n \in \mathbb{N}. \tag{37}$$

Then $\mathbf{E}_2(x_n)^2 \mu(x_n) = 1/n^2\ \forall\, n \in \mathbb{N}$ *and therefore* $\mathbf{E}_2 \in L^2(\mathbb{T}_2)$*. The time scale seems to be quite "innocent" in the beginning, the first* 10 *points all being less than* 5*. However, it can be shown that the following is true:*

$$x_{n+1} \geq x_n + \frac{(1.4)^{n-2}}{n^2} \quad \forall\, n \geq 2.$$

But this condition indicates that there is at least exponential growth in the end — we can proceed in the same way as in Example 14:

$$x_{n+1} \sim \frac{C}{n^2} \prod_{k=N}^{n-1} {x_{k+1}}^{2x_k}.$$

Again, consideration of the same relation for x_n *and resubstitution yield*

$$x_{n+1} \sim \left(\frac{n-1}{n}\right)^2 {x_n}^{2x_{n-1}+1} \sim {x_n}^{2x_{n-1}+1}.$$

Surprisingly, this is the same asymptotic relation as (36), in spite of the totally different behavior in the beginning!

There are time scales $\mathbb{T}_{1,2}$ with different behavior concerning "L^2", but fulfilling the same asymptotic growth relation. This discovery leads us at once to a *necessary* growth condition.

Theorem 16 *If the time scale* $\mathbb{T} \equiv \mathbb{T}_+$ *does not satisfy the growth condition*

$$\sigma(x) = \mathcal{O}(x^{2\varrho(x)+1}) \quad \Longleftrightarrow \quad \frac{\sigma(x)}{x} = \mathcal{O}(x^{2\varrho(x)}), \tag{38}$$

then we have $\mathbf{E} \notin L^2(\mathbb{T})$.

Proof: Let us just show this for isolated time scales like the above. If the condition (38) is not satisfied, then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$E(x_{n_k})^2\mu(x_{n_k}) \geq \text{const} \quad \forall\, k \in \mathbb{N}.$$

(This can be seen from the investigations in Example 14.) The corresponding integral will diverge! □

Remark. After taking logarithms, (38) implies the *logarithmic growth condition*

$$\frac{\log\sigma(x)}{\log x} = \mathcal{O}(\varrho(x)). \tag{39}$$

On the one hand, (39) gives a good hint on what it is all about, since we only need to care about exponents here. On the other hand, it is a lot weaker than (38) — not good for exact calculations!

6 Open Problems

The sufficient condition $\sigma(x)/\varrho(x) = \mathcal{O}(\log\nu(x))$ is much stronger than the necessary one $\sigma(x)/x = \mathcal{O}(x^{2\varrho(x)})$. Indeed (39) tells us that the logarithmic growth in the necessary condition can be larger than the maximal ordinary growth in the sufficient one. That is a huge difference. One of our main tasks in the future — from a theoretical point of view — thus lies in bringing the sufficient condition closer to the necessary one.

Conjecture. Let $\delta > 0$ be arbitrary. Then the condition

$$\frac{\sigma(x)}{x} = \mathcal{O}(x^{2\varrho(x)-\delta}) \tag{40}$$

is a sufficient condition for square integrability!

The condition (40) would be a lot weaker, perhaps the best possible. But why should this hold? Let us consider a time scale, on which the Gaussian bell is not square integrable.

On such a time scale $\mathbb{T}$, w.l.o.g. isolated $(x_n)_{n\in\mathbb{N}}$, we must be able to find a subsequence $(x_{n_k} \equiv y_k)_{k\in\mathbb{N}}$ such that $y_{k+1} - y_k \sim y_{k+1}$ and

$$\sum_{k=1}^{\infty} \mathbf{E}(y_k)^2\mu(y_k) = \infty.$$

Let us now consider the time scale generated by $(y_k)_{k\in\mathbb{N}}$ instead — throwing away the other points. Then certainly this time scale, denoted by $\hat{\mathbb{T}}$, will satisfy $\hat{\mathbf{E}} \notin L^2(\hat{\mathbb{T}})$. Now suppose $\mathbb{T}$ satisfies (40) for some $\delta > 0$. Then also $\hat{\mathbb{T}}$ will satisfy (40), provided the y_k are chosen appropriately. (This has to be specified, of course.) Denote $\gamma_k = \hat{\mathbf{E}}(y_k)^2\hat{\mu}(y_k)$ on $\mathbb{T}$. Since we have $\hat{\mu}(y_k) \sim y_{k+1}$, we directly obtain, similar to Example 15,

$$y_{k+1} \sim \frac{\gamma_k}{\gamma_{k-1}} {y_k}^{2y_{k-1}+1} = \mathcal{O}({y_k}^{2y_{k-1}+1-\delta}).$$

This implies

$$\frac{\gamma_k}{\gamma_{k-1}} \leq \text{const} \cdot {y_k}^{-\delta}$$

for large k, leading to the contradiction $\gamma_k/\gamma_{k-1} \overset{k\to\infty}{\longrightarrow} 0$. (The series $\sum_{k=1}^{\infty} \gamma_k$ would then converge by the ratio test!)

In fact, (40) seems to be a sufficient condition. Another open problem is the one of adapting a Gaussian bell to some ladder formalism as described in section 1. Probably our Gaussian bell will not yield the right operators. The way of succeeding there rather lies in finding some suitable operators first, later on defining the bell as the ground state φ_0. However, it will not be easy to find some adjusted operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ on the one hand, on the other hand a well-defined ground state. (Positive regressivity!) There is still a lot of work to do. For a treatment of such ground states on q-linear grids, see e.g. [6].

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