A SURVEY OF EXPONENTIAL FUNCTIONS ON TIME SCALES

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Abstract. In this paper we will summarize what is known about exponential functions on time scales.

1. Introduction

The theory of dynamic equations on time scales was introduced by Stefan Hilger in his PhD thesis [18] in 1988 in order to unify continuous and discrete analysis. Hilger [19] has defined an exponential function on a time scale. Here we will give several recent results concerning this important exponential function. In Section 2 we give a short introduction to the time scales calculus. In Section 3 we define the exponential function as introduced by Stefan Hilger in [20] and give several important properties of this exponential function. In Section 4 we see that the exponential function is important for solving first order linear dynamic equations. In Section 5 we use exponential functions to solve second order linear dynamic equations with constant coefficients. Using the exponential function, trigonometric functions and hyperbolic functions on time scales are then defined. In Section 6 we solve certain second order linear dynamic equations with variable coefficients using exponential functions. Finally, in Section 7, we discuss Euler-Cauchy dynamic equations on time scales, and present its solutions in terms of the exponential functions.

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2. The Time Scales Calculus

In this section we briefly introduce the time scales calculus. For proofs and further explanations and results we refer to the papers by Hilger [4, 19, 20], to the book by Kaymakçalan, Lakshmikantham, and Sivasundaram [21], and to the more recent papers [1, 2, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16]. We also recommend the book by Bohner and Peterson [10]. A *time scale* $\mathbb{T}$ is a closed subset of $\mathbb{R}$, and the (forward) *jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$), while the *graininess* $\mu : \mathbb{T} \to \mathbb{R}_+$ is

$$\mu(t) := \sigma(t) - t.$$

A point t is called right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$. The notions of left-scattered and left-dense are defined similarly using the backward jump operator. We write $\mathbb{T}^\kappa$ for $\mathbb{T}$ minus a possible left-scattered maximum. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if the left-hand sided limit exists at left-dense points. For a function $f : \mathbb{T} \to \mathbb{R}$ we define the derivative f^Δ as follows: Let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U,$$

then f is said to be (delta) *differentiable* at t, and we call α the derivative of f at t and denote it by $f^\Delta(t)$. Moreover, we denote $f^\sigma = f \circ \sigma$. The following formulas are useful:

- $f^\sigma = f + \mu f^\Delta$;
- $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$ ("Product Rule");
- $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/(g g^\sigma)$ ("Quotient Rule").

A function F with $F^{\Delta} = f$ is called an *antiderivative* of f, and then we define

$$\int_a^b f(t)\Delta t = F(b) - F(a),$$

where $a, b \in \mathbb{T}$. It is well-known [19] that rd-continuous functions possess antiderivatives. A simple consequence of the first formula above is

- $\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$.

3. The Exponential Function

A function $p : \mathbb{T} \to \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

Hilger [20] showed that for $t_0 \in \mathbb{T}$ and rd-continuous and regressive p, the solution of the initial value problem

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1 \tag{3.1}$$

is given by $e_p(\cdot, t_0)$, where

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \quad \text{with} \quad \xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

We now proceed to give some fundamental properties of the exponenetial function. To do so we note that in [10] it is shown that the set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu pq.$$

The additive inverse in this group is denoted by

$$\ominus p := -\frac{p}{1 + \mu p}.$$

We then define subtraction $\ominus$ on the set of regressive functions by

$$p \ominus q := p \oplus (\ominus p).$$

Theorem 3.1 ([9, 10])**.** *If $p, q : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous, then the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$*;*

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$*.*

Proof. So that the reader can get an idea of the proof of this theorem we prove Part (vi). See [9, 10] for a complete proof. Since it can be shown that the dynamic equation in (3.2) is regressive we know that the unique solution of the initial value problem

$$y^\Delta = (p \oplus q)(t)y, \quad y(s) = 1. \tag{3.2}$$

is $e_{p\oplus q}(t,s)$ for each fixed s. To complete the proof it suffices to show that for each fixed s, $y(t) := e_p(t,s)e_q(t,s)$ satisfies (3.2): We have $y(s) = e_p(s,s)e_q(s,s) = 1$, and we use the product rule to calculate

$$\begin{aligned} y^\Delta(t) &= (e_p(\cdot,s)e_q(\cdot,s))^\Delta(t) \\ &= p(t)e_p(t,s)e_q(\sigma(t),s) + e_p(t,s)q(t)e_q(t,s) \\ &= p(t)e_p(t,s)(1+\mu(t)q(t))e_q(t,s) + e_p(t,s)q(t)e_q(t,s) \\ &= (p \oplus q)(t)y(t), \end{aligned}$$

where we have also used part (ii) of this theorem. ☐

Property (v) in Theorem 3.1 is called the *semigroup property* of the exponential function. For time scales with constant graininess, Theorem 3.1 was proved by Hilger [20]. Bohner and Peterson [9] then extended the result to general time scales. The generalization of this result to matrices is given in Bohner and Peterson [10].

The following two theorems are important in determining the sign of the exponential function.

Theorem 3.2 ([9, 10])**.** *Assume $p : \mathbb{T} \to \mathbb{R}$ is regressive and rd-continuous. If $1 + \mu(t)p(t) < 0$ on $\mathbb{T}^\kappa$, then*

$$e_p(t, t_0) = \alpha(t, t_0)(-1)^{n_t}$$

for all $t \in \mathbb{T}$, where

$$\alpha(t, t_0) := \exp\left(\int_{t_0}^{t} \frac{\log|1 + \mu(\tau)p(\tau)|}{\mu(\tau)} \Delta\tau\right) > 0$$

and n_t is one plus the (finite) number of points in $\mathbb{T}$ strictly between t_0 and t.

Theorem 3.3 ([3])**.** *Suppose $p : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is rd-continuous and regressive and suppose further there exists a sequence of distinct points $\{t_n\} \subset \mathbb{T}^\kappa$ such that*

$$1 + \mu(t_n)p(t_n) < 0$$

$n = 1, 2, \cdots$. Then

$$\lim_{n\to\infty} |t_n| = \infty.$$

The following theorem completely characterizes the sign of the exponential function $e_p(t, t_0)$.

Theorem 3.4 ([3])**.** *Assume $p : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is regressive and rd-continuous, then the exponential function $e_p(t, t_0)$ is a nonvanishing, real-valued function satisfying the following properties:*

(i) *If $1 + \mu(t)p(t) > 0$ on $\mathbb{T}^\kappa$, then $e_p(t, t_0)$ is positive on $\mathbb{T}$.*

(ii) *If $1 + \mu(\tau)p(\tau) < 0$ for some $\tau \in \mathbb{T}^\kappa$, then*

$$e_p(\tau, t_0)e_p(\sigma(\tau), t_0) < 0.$$

In this case we say $e_p(t, t_0)$ has a generalized zero (see [17]) at $\sigma(\tau)$.

(iii) *If $1 + \mu(t)p(t) < 0$ on $\mathbb{T}^\kappa$, then $e_p(t, t_0)$ changes sign at every point in $\mathbb{T}$.*

(iv) *Assume there exist a finite or infinite sequence* $\{t_i\} \subset \mathbb{T}^\kappa$ *and a finite or infinite sequence* $\{s_i\} \subset \mathbb{T}^\kappa$ *with*

$$\cdots < s_2 < s_1 < t_0 \le t_1 < t_2 < \cdots$$

such that $1+\mu(t_i)p(t_i) < 0$ *and* $1+\mu(s_i)p(s_i) < 0$ *and* $1+\mu(t)p(t) > 0$ *for* $t \in \mathbb{T}^\kappa - [\{s_i\} \cup \{t_i\}]$. *Furthermore if* $\{t_n\}$ *is infinite, then* $\lim_{n\to\infty} t_n = \infty$ *and if* $\{s_n\}$ *is infinite, then* $\lim_{n\to\infty} s_n = -\infty$. *In this case*

$$e_p(t,t_0) > 0 \quad on \quad [\sigma(s_1), t_1].$$

If $\{t_n\}$ *is infinite, then*

$$(-1)^i e_p(t,t_0) > 0 \quad on \quad [\sigma(t_i), t_{i+1}], \quad i = 1, 2, \cdots.$$

If $\{t_n\}$ *is a finite sequence of* N *points, then*

$$(-1)^i e_p(t,t_0) > 0 \quad on \quad [\sigma(t_i), t_{i+1}], \quad i = 1, 2, \cdots, N-1$$

and

$$(-1)^N e_p(t,t_0) > 0 \quad on \quad [\sigma(t_N), \infty).$$

If $N = 0$, *then*

$$e_p(t,t_0) > 0 \quad on \quad [\sigma(s_1), \infty).$$

If $\{s_n\}$ *is infinite, then*

$$(-1)^i e_p(t,t_0) > 0 \quad on \quad [\sigma(s_{i+1}), s_i], \quad i = 1, 2, \cdots.$$

If $card\{s_n\} = M$, *then*

$$(-1)^i e_p(t,t_0) > 0 \quad on \quad [\sigma(s_{i+1}), s_i], \quad i = 1, 2, \cdots, M-1,$$

and

$$(-1)^M e_p(t,t_0) > 0 \quad on \quad (-\infty, s_M].$$

If $M = 0$, *then*

$$e_p(t,t_0) > 0 \quad on \quad (-\infty, t_1].$$

TABLE 1. Exponential Functions

$\mathbb{T}$	$e_\alpha(t,t_0)$	t_0	$p(t)$	$e_p(t,t_0)$
$\mathbb{R}$	$e^{\alpha(t-t_0)}$	0	1	e^t
$\mathbb{Z}$	$(1+\alpha)^{t-t_0}$	0	1	2^t
$h\mathbb{Z}$	$(1+\alpha h)^{(t-t_0)/h}$	0	1	$(1+h)^{t/h}$
$\frac{1}{n}\mathbb{Z}$	$(1+\frac{\alpha}{n})^{n(t-t_0)}$	0	1	$\left[(1+\frac{1}{n})^n\right]^t$
$q^{\mathbb{N}_0}$	$\cdots$	1	$\frac{1-t}{(q-1)t^2}$	$\sqrt{t}e^{-\frac{\ln^2(t)}{2\ln(q)}}$
$2^{\mathbb{N}_0}$	$\cdots$	1	$\frac{1-t}{t^2}$	$\sqrt{t}e^{-\frac{\ln^2(t)}{\ln(4)}}$
$\mathbb{N}_0^2$	$\cdots$	0	1	$2^{\sqrt{t}}(\sqrt{t})!$
$\left\{\sum_{k=1}^n \frac{1}{k} :\ n\in\mathbb{N}\right\}$	$\binom{n+\alpha-t_0}{n-t_0},\ t=\sum_{k=1}^n \frac{1}{k}$	0	1	$n+1,\ t=\sum_{k=1}^n \frac{1}{k}$

Table 1 appears in [10] and gives several examples of exponential functions for various time scales.

4. First Order Linear Dynamic Equations

In this section we see that exponential functions are important for solving first order linear dynamic equations on a time scale. Bohner and Peterson [10] point out that there are two different forms of first order linear dynamic equations that are important, namely

$$y^\Delta = p(t)y + f(t) \tag{4.1}$$

and

$$x^\Delta = -p(t)x^\sigma + f(t). \tag{4.2}$$

The corresponding homogeneous equations are

$$y^\Delta = p(t)y \tag{4.3}$$

and

$$x^{\Delta} = -p(t)x^{\sigma} \tag{4.4}$$

respectively. Equations (4.1) and (4.2) are said to be *regressive* on $\mathbb{T}$ provided p and f are rd-continuous on $\mathbb{T}$ and p is regressive on $\mathbb{T}$. The next theorem essentially shows that solving the homogeneous equations (4.1) and (4.2) is a matter of finding the exponential functions $e_p(t, t_0)$ and $e_{\ominus p}(t, t_0)$ respectively.

Theorem 4.1 ([10])**.** *If* (4.3) *is regressive, then a general solution of the dynamic equation* (4.3) *is given by*

$$y(t) = e_p(t, t_0)y_0$$

where y_0 is a constant. If (4.4) *is regressive, then a general solution of the dynamic equation* (4.4) *is given by*

$$x(t) = e_{\ominus p}(t, t_0)x_0$$

where x_0 is a constant.

One can then prove the following theorem.

Theorem 4.2 ([3])**.** *Assume equation* (4.3) *is regressive.*

(i) *If*

$$1 + \mu(t)p(t) > 0 \quad \textit{on} \quad \mathbb{T}^{\kappa},$$

then every nontrivial solution of equation (4.3) *is of one sign on $\mathbb{T}$. If, in addition, $\mathbb{T}$ is unbounded, then we say equation* (4.3) *is* nonoscillatory *on $\mathbb{T}$ in this case.*

(ii) *If*

$$1 + \mu(t)p(t) < 0 \quad \textit{on} \quad \mathbb{T}^{\kappa},$$

then every nontrivial solution of equation (4.3) *changes sign at every point in $\mathbb{T}$. If, in addition, $\mathbb{T}$ is unbounded, then we say equation* (4.3) *is* strongly oscillatory *on $\mathbb{T}$ in this case.*

(iii) *If there exists a strictly increasing sequence* $\{s_i\} \subset \mathbb{T}^\kappa$ *such that*

$$1 + \mu(s_i)p(s_i) < 0 \quad \textit{for} \quad i = 1, 2, \cdots,$$

then every nontrivial solution of equation (4.3) *changes sign infinitely often and we say equation* (4.3) *is* oscillatory *on* $\mathbb{T}$ *in this case.*

(iv) *Assume there exists a finite number* N *of points in* $\mathbb{T}^\kappa$ *where* $1+\mu(t)p(t) < 0$, *then every nontrivial solution changes sign exactly* N *times in* $\mathbb{T}$ *and we say equation* (4.3) *is* nonoscillatory *on* $\mathbb{T}$ *in this case.*

The next two results are variation of constants formulas for Equations (4.1) and (4.2) which involve exponential functions.

Theorem 4.3 ([9, 10])**.** *Let* $p : \mathbb{T} \to \mathbb{R}$ *be rd-continuous and regressive. Suppose* $f : \mathbb{T} \to \mathbb{R}$ *is rd-continuous,* $t_0 \in \mathbb{T}$, *and* $y_0 \in \mathbb{R}$. *Then the unique solution of the initial value problem*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0 \tag{4.5}$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Theorem 4.4 ([9, 10])**.** *Assume that* (4.1) *is regressive,* $t_0 \in \mathbb{T}$, *and* $x_0 \in \mathbb{R}$. *Then the unique solution of the initial value problem*

$$x^\Delta = -p(t)x^\sigma + f(t), \quad x(t_0) = x_0 \tag{4.6}$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

5. Second Order Linear Dynamic Equations

The *Wronskian* of two differentiable functions x_1 and x_2 is defined by

$$W(x_1, x_2) = \det \begin{pmatrix} x_1 & x_2 \\ x_1^\Delta & x_2^\Delta \end{pmatrix}.$$

We consider the following two different forms of second order linear differentail equations on a measure chain $\mathbb{T}$:

$$y^{\Delta\Delta} + p(t)y^\Delta + q(t)y = 0 \tag{5.1}$$

and

$$x^{\Delta\Delta} + p(t)x^{\Delta^\sigma} + q(t)x^\sigma = 0. \tag{5.2}$$

We say that equation (5.1) is *regressive* provided $p(t)$ and $q(t)$ are rd-continuous on $\mathbb{T}$ and $1 - \mu(t)p(t) + \mu^2(t)q(t) \neq 0$ on $\mathbb{T}$. We say that equation (5.2) is *regressive* provided $p(t)$ and $q(t)$ are rd-continuous on $\mathbb{T}$ and p is regressive on $\mathbb{T}$.

For each of the equations (5.1) and (5.2), there is an Abel's formula which involves an exponential function which we give in the next two theorems.

Theorem 5.1 (Abel's Formula for (5.1))**.** *Assume that equation* (5.1) *is regressive and let $t_0 \in \mathbb{T}^\kappa$. Suppose that y_1 and y_2 are two solutions of equation* (5.1)*. Then their Wronskian $W(y_1, y_2)$ satifies*

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e_{-p+\mu q}(t, t_0)$$

for $t \in \mathbb{T}^\kappa$.

Theorem 5.2 (Abel's Formula for (5.2))**.** *Assume that equation* (5.2) *is regressive and let $t_0 \in \mathbb{T}^\kappa$. If x_1 and x_2 are solutions of equation* (5.2)*, then*

$$W(x_1, x_2)(t) = W(x_1, x_2)(t_0)e_{\ominus p}(t, t_0)$$

for all $t \in \mathbb{T}^\kappa$.

Corollary 5.1 ([9, 10])**.** *If $q : \mathbb{T} \to \mathbb{R}$, the Wronskian of any two solutions of*

$$x^{\Delta\Delta} + q(t)x^{\sigma} = 0$$

is constant.

Next we will be concerned with solving the second order linear dynamic equation with constant coefficients

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0 \tag{5.3}$$

(where α and β are constants). The characteristic equation associated with (5.3) is

$$\lambda^2 + \alpha\lambda + \beta = 0. \tag{5.4}$$

The next theorem shows that solving equation (5.3) is a matter of finding the appropriate exponential functions.

Theorem 5.3 ([9, 10])**.** *Suppose equation* (5.3) *is regressive and λ_1 and λ_2 (possibly complex) are distinct solutions of the characteristic equation* (5.4)*, then*

$$x(t) = c_1 e_{\lambda_1}(t, t_0) + c_2 e_{\lambda_2}(t, t_0),$$

where $t_0 \in \mathbb{T}$, is a general solution of equation (5.3)

In Theorem 5.3 we gave a form for all complex-valued solutions of (5.3) if the characteristic equation (5.4) has distinct complex roots. We would next like to find a nice form for all real-valued solutions of equation (5.3) for various cases. To do this we now introduce some trigonometric and hyperbolic functions.

Definition 5.1 (Trigonometric Functions)**.** If μp^2 is regressive, we define the trigonometric functions

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} \quad \text{and} \quad \sin_p = \frac{e_{ip} - e_{-ip}}{2i}.$$

In the next theorem we give some properties of these trigonometric functions.

Theorem 5.4 ([9, 10]). *If μp^2 is regressive, then we have*

(i) $\cos_p^\Delta(t, t_0) = -p \sin_p(t, t_0)$;
(ii) $\sin_p^\Delta(t, t_0) = p \cos_p(t, t_0)$;
(iii) $\cos_p^2(t, t_0) + \sin_p^2(t, t_0) = e_{\mu p^2}(t, t_0)$;
(iv) $W[\cos_p(t, t_0), \sin_p(t, t_0)] = p e_{\mu p^2}(t, t_0)$;
(v) $e_{ip}(t, t_0) = \cos_p(t, t_0) + i \sin_p(t, t_0)$.

Definition 5.2 (Hyperbolic Functions). If $-\mu p^2$ is regressive, we define the hyperbolic functions

$$\cosh_p = \frac{e_p + e_{-p}}{2} \quad \text{and} \quad \sinh_p = \frac{e_p - e_{-p}}{2}.$$

Some properties of the hyperbolic functions are given in the next theorem.

Theorem 5.5 ([9, 10]). *If $-\mu p^2$ is regressive, then we have*

(i) $\cosh_p^\Delta(t, t_0) = p \sinh_p(t, t_0)$;
(ii) $\sinh_p^\Delta(t, t_0) = p \cosh_p(t, t_0)$;
(iii) $\cosh_p^2(t, t_0) - \sinh_p^2(t, t_0) = e_{-\mu p^2}(t, t_0)$;
(iv) $W[\cosh_p(t, t_0), \sinh_p(t, t_0)] = p e_{-\mu p^2}(t, t_0)$;
(v) $e_p(t, t_0) = \cosh_p(t, t_0) + \sinh_p(t, t_0)$.

Solving the characteristic equation (5.4) we get

$$\lambda_1 = p + q, \quad \lambda_2 = p - q,$$

where

$$p := -\frac{\alpha}{2} \quad \text{and} \quad q := \frac{\sqrt{4\beta - \alpha^2}}{2}.$$

In the next three theorems we find solutions of (5.3) involving hyperbolic functions if $\alpha^2 - 4\beta > 0$, trigonometric functions if $\alpha^2 - 4\beta < 0$, and we use Theorem 5.2 (i.e., the reduction of order method) to find solutions if $\alpha^2 - 4\beta = 0$.

Theorem 5.6. *Assume $\alpha^2 - 4\beta < 0$, $p, q \in \mathbb{R}$ and p is regressive, then*

$$x(t) = [c_1 \cos_{q/(1+\mu p)}(t, t_0) + c_2 \sin_{q/(1+\mu p)}(t, t_0) e_p(t, t_0)] e_p(t, t_0),$$

where $t_0 \in \mathbb{T}$, is a general solution of equation (5.3)*.*

Theorem 5.7 ([9, 10])**.** *If $\alpha^2 - 4\beta > 0$, then*

$$x(t) = c_1 \cosh_{q/(1+\mu p)}(\cdot, t_0) e_p(t, t_0) + c_2 \sinh_{q/(1+\mu p)}(t, t_0) e_p(t, t_0),$$

where $t_0 \in \mathbb{T}$, is a general solution of equation (5.3)*.*

Theorem 5.8 ([9, 10])**.** *If $\alpha^2 - 4\beta < 0$, then*

$$x(t) = [c_1 \cos_{q/(1+\mu p)}(t, t_0) + c_2 \sin_{q/(1+\mu p)}(t, t_0)] e_p(t, t_0),$$

is a general solution of equation (5.3)*.*

Finally, we state the theorem for the case that the characteristic equation (5.4) has a double zero.

Theorem 5.9 ([9, 10])**.** *If $\alpha^2 - 4\beta = 0$, then a general solution of equation* (5.3) *is given by*

$$x(t) = c_1 e_p(t, t_0) + c_2 e_p(t, t_0) \int_{t_0}^{t} \frac{1}{1 + p\mu(\tau)} \Delta\tau,$$

where $t_0 \in \mathbb{T}$.

6. Nonconstant coefficients

In this section we give some cases where certain second order linear dynamic equations on measure chains with variable coefficients can be solved with the aid of exponential functions.

For regressive p we define the generalized square of p by

$$p^{②} := (-p) \cdot (\ominus p) = \frac{p^2}{1 + \mu p}.$$

We consider the second order linear equation (5.2) and its associated *Riccati equation*

$$z^{\Delta} + z^{②} + p(t)z^{\sigma} + q(t) = 0. \tag{6.1}$$

Definition 6.1. We say that z is a solution of the Riccati equation (6.1) on $\mathbb{T}$ provided z is regressive on $\mathbb{T}$ and

$$z^{\Delta}(t) + z^{②}(t) + p(t)z^{\sigma}(t) + q(t) = 0$$

for $t \in \mathbb{T}^{\kappa}$.

Theorem 6.1 ([9, 10])**.** *If z is a solution of the Riccati equation* (6.1)*, then $e_z(\cdot, t_0)$ is a solution of equation* (5.2)*.*

Using reduction of order one can use Theorem 6.1 to prove the following theorem.

Theorem 6.2 ([9, 10])**.** *If z is a solution of* (6.1)*, then*

$$x(t) = c_1 e_z(t, t_0) + c_2 e_z(t, t_0) \int_{t_0}^{t} \frac{e_{z \oplus z \oplus p}(t_0, \tau)}{1 + \mu(\tau)z(\tau)} \Delta\tau$$

is a general solution of equation (5.2)*.*

In order to apply Theorem 6.1, it is crucial to find a solution of equation (6.1). As is checked readily, this is an easy task if we consider equation (5.3). Another example, in which (6.1) can be solved explicitly, is given in the next example.

Example 6.1 ([9, 10]). Let q be constant and regressive and consider the equation

$$x^{\Delta\Delta} - q^{②}(t)x^{\sigma} = 0. \tag{6.2}$$

Then a solution of equation (6.1) is given by

$$z = q.$$

From Theorem 6.1 we get that

$$x(t) = c_1 e_q(t, t_0) + c_2 e_q(t, t_0) \int_{t_0}^{t} \frac{e_q^2(t_0, \tau)}{[1 + q\mu(\tau)]} \Delta\tau$$

is a general solution of equation (6.2).

Finally we show cases where exponential functions come up when one solves certain self-adjoint equations

$$(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0. \tag{6.3}$$

The p given below in Theorem 6.3 is not necessarily a differentiable function (unless the graininess of the time scale is constant). Hence equation (6.3) can not be rewritten in the form equation (5.1) or equation (5.2).

Theorem 6.3 ([9, 10])**.** *Suppose α is regressive. Let $\tilde{p}$ be rd-continuous and non-vanishing. We put*

$$p = \frac{\tilde{p}}{\ominus\alpha} \quad \text{and} \quad q = \alpha\tilde{p} - \tilde{p}^{\Delta}.$$

Then a general solution of the self-adjoint equation (6.3) *is given by*

$$x(t) = c_1 e_{\ominus\alpha}(t, t_0) + c_2 \int_{t_0}^{t} \frac{\alpha(\tau)}{\tilde{p}(\tau)} e_{\alpha}(\tau, t) e_{\alpha}(\tau, t_0)\Delta\tau.$$

Example 6.2 ([9, 10])*.* Suppose α is regressive. We let $\tilde{p} = e_{\alpha}(\cdot, t_0)$ so that $q = 0$ in the above Theorem 6.3. Hence we consider the equation

$$\left(\frac{e_{\alpha}^{\sigma}(\cdot, t_0)}{\alpha} x^{\Delta}\right)^{\Delta} = 0. \tag{6.4}$$

Using Theorem 6.3, it can be shown that a general solution of (6.4) is given by

$$x(t) = c_1 e_{\ominus\alpha}(t, t_0) + c_2.$$

7. Euler-Cauchy Dynamic Equations

In this section we will show how exponential functions help us solve the *Euler-Cauchy dynamic equation*

$$t\sigma(t)y^{\Delta\Delta} + aty^{\Delta} + by = 0 \quad \text{with} \quad a, b \in \mathbb{R}. \tag{7.1}$$

We will only solve the equation (7.1) for $t \in \mathbb{T}$, $t > 0$. We assume that the regressivity condition

$$1 - \frac{a\mu(t)}{\sigma(t)} + \frac{b\mu^2(t)}{t\sigma(t)} \neq 0 \tag{7.2}$$

for $t \in \mathbb{T}$, $t > 0$ is satisfied. The associated *characteristic equation* of (7.1) is defined by

$$\lambda^2 + (a-1)\lambda + b = 0. \tag{7.3}$$

Theorem 7.1 ([9, 10])**.** *If the regressivity condition* (7.2) *is satisfied and the characteristic equation* (7.3) *has two distinct roots* λ_1 *and* λ_2*, then a general solution of equation* (7.1) *is given by*

$$x(t) = c_1 e_{\lambda_1/t}(\cdot, t_0) + c_2 e_{\lambda_2/t}(\cdot, t_0),$$

where $t_0 \in \mathbb{T}$*,* $t_0 > 0$*. If, in addition,* $1 + \mu(t)\frac{\lambda_i}{t} > 0$*,* $i = 1, 2$*,* $t \in \mathbb{T}$*,* $t > 0$*, then the above exponential functions form a fundamental set of positive solutions of the Euler-Cauchy dynamic equation* (7.1) *on* $\mathbb{T}$*,* $t > 0$*.*

Next we consider the Euler-Cauchy dynamic equation in the double root case.

Theorem 7.2 ([9, 10])**.** *Assume that* $\alpha \in \mathbb{R}$ *and* $t_0 \in \mathbb{T}$ *with* $t_0 > 0$*. If the regressivity condition*

$$1 - \frac{1-2\alpha}{\sigma(t)}\mu(t) + \frac{\alpha^2}{t\sigma(t)}\mu^2(t) \neq 0 \tag{7.4}$$

holds for $t \in \mathbb{T}$*,* $t > 0$*, then a general solution of the Euler-Cauchy dynamic equation*

$$t\sigma(t)y^{\Delta\Delta} + (1-2\alpha)ty^{\Delta} + \alpha^2 y = 0 \tag{7.5}$$

is given by

$$x(t) = c_1 e_{\frac{\alpha}{t}}(t, t_0) + c_2 e_{\frac{\alpha}{t}}(t, t_0) \int_{t_0}^{t} \frac{1}{\tau + \alpha\mu(\tau)} \Delta\tau$$

for $t \in \mathbb{T}$*,* $t > 0$*.*

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