Recent Results Concerning Dynamic Equations on Time Scales

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1. Introduction and Preliminary Results

In this paper we will study certain linear and nonlinear dynamic equations. In sections 2 and 3 we study the second order nonlinear dynamic equation

$$(p(t)x^{\Delta})^{\Delta} + q(t)(f \circ x^{\sigma}) = 0, \tag{1.1}$$

where p and q are real–valued, right–dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$. In sections 2 and 3 we also assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$f'(x) \geq \frac{f(x)}{x} > 0 \quad \text{for} \quad x \neq 0. \tag{1.2}$$

Although in section 2 we shall assume p is a positive function we do not make any explicit sign assumptions on q in contrast to most know results on nonlinear oscillations. In sections 4 and 5 we consider (1.1) under slightly different hypotheses. In section 6 we consider an example with damping and in section 7 we study comparison theorems for linear dynamic equations.

For completeness, we recall the following concepts related to the notion of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and, since boundedness and oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The *forward* and *backward jump operators* are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\},$$

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where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be *left–dense* if $\rho(t) = t$, *right–dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left–scattered* if $\rho(t) < t$ and *right–scattered* if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous* (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

The assumption (1.2) allows f to be of superlinear growth, say

$$f(x) = x^{2n+1}, \quad n \geq 1. \tag{1.3}$$

In sections 4 and 5 we assume the nonlinearity has the property

$$xf(x) > 0 \text{ and } |f(x)| \geq K|x| \text{ for } x \neq 0, \text{ for some } K > 0. \tag{1.4}$$

This essentially says that the equation is, in some sense, not too far from being linear.

We shall see that one may relate oscillation and boundedness of solutions of the nonlinear equation (1.1) to the linear equation

$$\big(p(t)x^\Delta\big)^\Delta + \lambda q(t)x^\sigma = 0, \tag{1.5}$$

where $\lambda > 0$, for which many oscillation criteria are known (see e.g. [1],[3], [4], [6], [8], [11], [16], and [21]). In particular, we will obtain the time scale analogues of the results due to Erbe [10] for the continuous case $\mathbb{T} = \mathbb{R}$. We shall restrict attention to solutions of (1.1) which exist on some interval of the form $[T_x, \infty)$. where $T_x \in \mathbb{T}$ may depend on the particular solution.

On an arbitrary time scale $\mathbb{T}$, the usual chain rule from calculus is no longer valid (see Bohner and Peterson [4], pp 31). One form of the extended chain rule, due to S. Keller [26] and generalized to measure chains by C. Pötzsche [31], is as follows. (See also Bohner and Peterson [4], pp 32.)

Lemma 1.1. *Assume $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and satisfies*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} g^\Delta(t). \tag{1.6}$$

We shall also need the following integration by parts formula (cf. [4]), which is a simple consequence of the product rule and which we formulate as follows:

Lemma 1.2. *Let $a, b \in \mathbb{T}$ and assume $f^\Delta, g^\Delta \in C_{rd}$. Then*

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t. \tag{1.7}$$

2. A Nonlinear Dynamic Equation

Before stating our next results, we recall that a solution of equation (1.1) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. Since $p(t) > 0$ we shall consider both cases

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \infty \tag{2.1}$$

and

$$\int_a^\infty \frac{1}{p(t)}\Delta t < \infty. \tag{2.2}$$

We also introduce the following condition:

$$\liminf_{t\to\infty} \int_T^t q(s)\Delta s \geq 0 \quad \text{and } \not\equiv 0 \tag{2.3}$$

for all large T. It can be shown that (2.3) implies either $\int_a^\infty q(s)\Delta s = +\infty$ or that

$$\int_T^\infty q(s)\Delta s = \lim_{t\to\infty} \int_T^t q(s)\Delta s$$

exists and satisfies $\int_T^\infty q(s)\Delta s \geq 0$ for all large T.

We have the following lemma which describes the behavior of a nonoscillatory solution of (1.1) for the case when (2.1) and (2.3) hold.

Lemma 2.1. *Let x be a nonoscillatory solution of* (1.1) *and assume conditions* (2.1) *and* (2.3) *hold. Then there exists $T_1 \geq T$ such that*

$$x(t)x^\Delta(t) > 0 \quad \textit{for} \quad t \geq T_1.$$

The first result is a boundedness result for (1.1).

Theorem 2.1. *Let $\lambda > 0$ and assume that Equation* (1.5) *is oscillatory. Assume that* (1.2) *holds and let x be a nonoscillatory solution of* (1.1) *with $x(t)x^\Delta(t) > 0$ for all $t \geq T_0$. Then*

$$\lim_{t\to\infty} \frac{f(x(t))}{x(t)} := \gamma \leq \lambda. \tag{2.4}$$

Corollary 2.1. *Let $\lambda > 0$ and assume that Equation* (1.5) *is oscillatory and* (2.1) *and* (2.3) *hold. Suppose that x is a nonoscillatory solution of the generalized Emden–Fowler equation*

$$\left(p(t)x^{\Delta}\right)^{\Delta} + q(t)(x^{\sigma})^{2n+1} = 0. \tag{2.5}$$

Then

$$\lim_{t\to\infty} |x(t)| = \gamma \leq (\lambda)^{\frac{1}{2n}}.$$

Theorem 2.2. *Assume that Equation* (1.5) *is oscillatory for all $\lambda > 0$ and suppose that* (1.2), (2.1), *and* (2.3) *hold. Then all solutions of* (1.1) *oscillate.*

The next theorem deals with the case when (2.2) holds.

Theorem 2.3. *Assume that Equation* (1.5) *is oscillatory for all $\lambda > 0$ and suppose that* (1.2), (2.2), *and* (2.3) *hold. In addition, assume that*

$$\int_T^{\infty} \frac{1}{p(s)} \int_T^s q(\eta)\Delta\eta\Delta s = \infty. \tag{2.6}$$

Then every solution of (1.1) *is either oscillatory or converges to zero on* $[a, \infty)$.

3. Examples

Clearly, equation (1.5) is oscillatory iff equation

$$\left(\frac{1}{\lambda}p(t)x^{\Delta}\right)^{\Delta} + q(t)x^{\sigma} = 0 \tag{3.1}$$

is oscillatory. It was shown in Erbe [11, Corollary 7] (see also Bohner and Peterson [4]) that

$$(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0 \tag{3.2}$$

is oscillatory if there exists a sequence $\{t_k\} \subset \mathbb{T}$ with $\lim_{k\to\infty} t_k = \infty$ and $\mu(t_k) > 0$ such that

$$\limsup_{k\to\infty} \left(Q(t_k) - \frac{p(t_k)}{\mu(t_k)}\right) = \infty, \tag{3.3}$$

where $Q(t) := \int_{\tau}^{t} q(s)\Delta s$. We can therefore conclude that all solutions of (1.1) oscillate in case (1.4), (2.1), and (2.3) hold along with

$$\limsup_{k\to\infty} \left(Q(t_k) - \frac{p(t_k)}{\lambda\mu(t_k)}\right) = \infty, \tag{3.4}$$

for all $\lambda > 0$. We note that there is no assumption on the boundedness of p and μ. If (1.2), (2.2), and (2.3) hold along with (3.4), then every solution on (1.1) oscillates or converges to zero. One may also apply averaging techniques or the telescoping principle to give some more sophistcated results (see Erbe, Kong, and Kong [13] and Erbe [10]).

As a second example, suppose that $\mathbb{T}$ is such that there exists a sequence of points $t_k \in \mathbb{T}$ with $t_k \to \infty$ and positive numbers M, K such that $p(t_k) \leq M$ and $\mu(t_k) \geq K$. Then if (1.2) and (2.3) hold and $\sum_1^\infty \mu(t_k)q(t_k) = \infty$ it follows from results of Erbe, Kong, and Kong [13, Corollary 4.1] that all solutions of (1.5) are oscillatory for $\lambda > 0$. Consequently, all solutions of (1.1) are oscillatory.

As a third example, we consider a particular example for the case when $\mathbb{T} = \mathbb{Z}$. If f has the form of (1.3) (i.e., $f(x) = x^{2n+1}$), $p(t) \equiv 1$, and $q(t) = \frac{\beta}{t\sigma(t)}$, then it is known that equation (3.2) is oscillatory if $\beta > \frac{1}{4}$, and is nonoscillatory if $\beta \leq \frac{1}{4}$. Since in this case (2.3) holds trivially, it follows from Theorem 2.1 that all nonoscillatory solutions of (1.1) satisfy $\lim_{t:\to\infty} |x(t)| \leq (\frac{1}{4})^{\frac{1}{2n}}$.

Remark 3.1. From Theorem 4.64 in [4] (Leighton–Wintner Theorem) it follows that equation (1.5) is oscillatory for all $\lambda > 0$ if

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \int_a^\infty q(t)\Delta t = +\infty. \tag{3.5}$$

Since the second condition in (3.5) implies that (2.3) holds, Theorem 2.2 implies that all solutions of the Emden–Fowler equation (2.5) are oscillatory. That is, the Leighton–Wintner Theorem is valid for (2.5) and more generally for (1.1) if (1.2) holds. We note again that there are no explicit sign conditions on $q(t)$. For the special case when $\mathbb{T} = \mathbb{Z}$ and (1.1) is

$$\Delta^2 x_n + q_n (x_{n+1})^{2m+1} = 0, \tag{3.6}$$

where $m \in \mathbb{N}$, it follows that (3.6) is oscillatory if

$$\sum_{n=1}^{\infty} q_n = +\infty. \tag{3.7}$$

That is (3.7) implies that the linear equation

$$\Delta^2 x_n + \lambda q_n x_{n+1} = 0 \tag{3.8}$$

is oscillatory for all $\lambda > 0$ and so oscillation of (3.6) is a consequence of Theorem 2.2. If we consider equation (3.8) with $\lambda = 1$, then Theorem 4.51 of [4] (see also [17]) implies that (3.6) is oscillatory if for any $k \geq 1$ there exists $k_1 \geq k$ such that

$$\lim_{n\to\infty} \sum_{j=k_1}^{n} q_j \geq 1.$$

Consequently, by Corollary 2.1, all nonoscillatory solutions of (3.6) satisfy

$$\lim_{n\to\infty} |x_n| \le 1.$$

4. Nonlinear Dynamic Equation with Positive $q(t)$

In this section we shall consider the nonlinear dynamic equation (1.1) under some different hypotheses. We assume that both p, q are positive, real–valued right–dense continuous functions, and $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (1.4), and we shall again consider the two cases (2.1) and (2.2).

In Došlý and Hilger [8], the authors consider the second order linear dynamic equation (1.5) $\lambda = 1$ and give necessary and sufficient conditions for oscillation of all solutions on unbounded time scales. Often, however, the oscillation criteria require additional assumptions on the unknown solutions, which may not be easy to check.

In Erbe and Peterson [16], the authors consider the same equation and suppose that there exists $t_0 \in \mathbb{T}$, such that $p(t)$ is bounded above on $[t_0, \infty)$, $h_0 = \inf\{\mu(t) : t \in [t_0, \infty)\} > 0$, and showed via Riccati techniques that

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty.$$

implies that every solution is oscillatory on $[t_0, \infty)$. It is clear that the results given in [16] cannot be applied when p is unbounded, $\mu(t) = 0$ and $q(t) = t^{-\alpha}$ when $\alpha > 1$. The papers [16] and [9] also give additional linear oscillation criteria, and also treat more general situations.

Recently Bohner and Saker [6] considered (1.1) and used Riccati techniques to give some sufficient conditions for oscillation when (2.1) or (2.2) hold. They obtain some sufficient conditions which guarantee that every solution oscillates or converges to zero.

We use a generalized Riccati transformation technique to obtain several oscillation criteria for (1.1) when (2.1) or (2.2) holds. Our results in this section improve the results given in Došlý and Hilger [8] and Erbe and Peterson [16] and complement the results in Bohner and Saker [6]. Applications to equations to which previously known criteria for oscillation are not applicable are given. In section 6, we will apply our results to linear or nonlinear dynamic equations of the form

$$x^{\Delta\Delta}(t) + \alpha(t)x^{\Delta^\sigma}(t) + \beta(t)(f \circ x^\sigma) = 0 \tag{4.1}$$

to give some sufficient conditions for oscillation of all their solutions.

We shall first need to briefly discuss the exponential function $e_p(\cdot, t_0)$, which is defined to be the unique solution of the IVP

$$x^\Delta = p(t)x, \quad x(t_0) = 1,$$

where it is assumed that

$$p \in \mathcal{R} := \{f : \mathbb{T} \to \mathbb{R} \text{ is rd-continuous and regressive}\}.$$

We define

$$\mathcal{R}^+ := \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0, t \in \mathbb{T}\}.$$

For properties of this exponential function, see Bohner and Peterson [4]. One such property that we will use is the formula

$$e_p(\sigma(t), t_0) = [1 + \mu(t)p(t)]e_p(t, t_0).$$

Also if $p \in \mathcal{R}$, then $e_p(t, s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathcal{R}^+$, then $e_p(t, t_0)$ is always positive.

Lemma 4.1. *Assume that* (2.1) *holds, and x solves* (1.1) *with $x(t) > 0$ for all $t > t_0$. Define $y = px^\Delta$. Then we have*

$$y^\Delta(t) < 0 \quad \text{and} \quad 0 \le y(t) \le \frac{x(t)}{\int_{t_0}^t \frac{\Delta s}{p(s)}}, \quad t > t_0 \tag{4.2}$$

and

$$0 \le \frac{x^\Delta(t)}{x(t)} \le \frac{1}{p(t) \int_{t_0}^t \frac{\Delta s}{p(s)}}, \quad t > t_0. \tag{4.3}$$

Next suppose $r \in \mathcal{R}$, assume that $p \cdot r$ is a differentiable function, and define the auxiliary functions

$$\begin{aligned} C(t) &= C(t,t_0) := 1 + \frac{\mu(t)}{p(t)\int_{t_0}^t \frac{\Delta s}{p(s)}}, \quad Q_1(t) = Q_1(t,t_0) := \frac{1+\mu(t)r(t)}{p(t)e_r(t,t_0)}, \\ \psi(t) &= \psi(t,t_0)) := e_r(\sigma(t),t_0)\left[Kq(t) + \frac{1}{2}(p(t)r(t))^\Delta + \frac{r^2(t)p(t)}{4C(t)}\right], \\ Q(t) &= Q(t,t_0) := -\frac{r(t)(1+\mu(t)r(t))}{C(t)} + r(t), \end{aligned}$$

for $t > t_0$. We also introduce the following condition

(A) There exists $M > 0$ such that $r(t)e_r(t, t_0)p(t) \le M$ for all large t.

Our first oscillation result in this section is

Theorem 4.1. *Assume that* (1.4), (2.1), *and (A) hold. Furthermore, assume that there exists $r \in \mathcal{R}^+$ such that $p \cdot r$ is differentiable and such that for any $t_0 \ge a$ there exists a $t_1 > t_0$ so that*

$$\limsup_{t\to\infty} \int_{t_1}^t H(s)\Delta s = \infty, \tag{4.4}$$

where

$$H(t) = H(t,t_0) = \psi(t) - \frac{Q^2(t)C(t)}{4Q_1(t)},$$

for $t > t_0$. *Then equation* (1.1) *is oscillatory on* $[a,\infty)$.

From Theorem 4.1, we can obtain different sufficient conditions for oscillation of all solutions of (1.1) by different choices of $r(t)$. For instance, let $r(t) = 0$, then $Q(t) = 0$, $e_r(t,t_0) = 1$, and $\psi(t) = Kq(t)$ and we get the following well–known result.

Corollary 4.1 (Leighton–Wintner Theorem)**.** *Assume that* (1.4) *and* (2.1) *hold. If*

$$\int_a^\infty q(s)\Delta s = \infty, \tag{4.5}$$

then equation (1.1) *is oscillatory on* $[a,\infty)$.

If $r(t) = \frac{1}{t}$, then $e_r(t,t_0) = \frac{t}{t_0}$ and it follows that condition (A) holds, provided p is bounded above, and so Theorem 4.1 yields the following result:

Corollary 4.2. *Assume p is bounded above, that* (1.4) *and* (2.1) *hold, and for any* $t_0 \geq a$ *there is a* $t_1 > t_0$ *such that*

(4.6)

$$\limsup_{t\to\infty} \int_{t_1}^t \left[\sigma(s)\left[Kq(s) + \left(\frac{p(s)}{2s}\right)^\Delta + \frac{p(s)}{4s^2C(s)}\right] - \frac{A^2(s)C(s)}{4B(s)}\right]\Delta s = \infty,$$

where

$$A(s) := \frac{-1}{sC(s)}\left(1 + \frac{1}{s}\mu(s) - C(s)\right), \quad B(s) := \frac{s+\mu(s)}{s^2p(s)}.$$

Then (1.1) *is oscillatory on* $[a,\infty)$.

If $p(t) = 1$ and $f(x) = x$, then equation (1.1) reduces to the linear dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^\sigma = 0, \tag{4.7}$$

for $t \in [a,\infty)$. From Theorem 4.1 we have the following oscillation criterion for equation (4.7) which improves some of the results in Bohner and Saker [6] and Erbe and Peterson [14].

Corollary 4.3. *Assume that* (1.4) *and* (2.1) *hold and for any* $t_0 \geq a$ *there is a* $t_1 > t_0$ *such that*

(4.8)

$$\limsup_{t\to\infty} \int_{t_1}^t \left[\sigma(s)\left[q(s) - \left(\frac{1}{2s\sigma(s)}\right) + \frac{1}{4s^2C_1(s)}\right] - \frac{A_1^2(s)C_1(s)}{4B_1(s)}\right]\Delta s = \infty,$$

where

$$A_1(s) = \frac{-1}{sC_1(s)}\left(1 + \frac{1}{s}\mu(s) - C_1(s)\right)$$

$$B_1(s) = \frac{s+\mu(s)}{s^2}, \quad C_1(s) = 1 + \frac{\mu(s)}{(s-t_0)}.$$

Then equation (4.7) *is oscillatory on* $[a,\infty)$.

Example 4.1. Consider the Euler–Cauchy dynamic equation

$$x^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)}x^{\sigma} = 0, \tag{4.9}$$

for $t \in [a,\infty)$. Here $q(t) = \frac{\gamma}{t\sigma(t)}$. Then (4.8) in Corollary 4.3 reads

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{\sigma(s)}{4s^2C_1(s)}\right] - \frac{A_1^2(s)C_1(s)}{4B_1(s)}\right]\Delta s = \infty. \tag{4.10}$$

If $\mathbb{T} = \mathbb{R}$, then the dynamic equation (4.9) is the second order Euler–Cauchy differential equation

$$x'' + \frac{\gamma}{t^2}x = 0, \ t \geq 1 \tag{4.11}$$

and in this case $\mu(s) = 0$, $\sigma(s) = s$, $C_1(s) = 1$ and $A_1(s) = 0$. Therefore (4.10) can be rewritten as

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{s}{4s^2}\right]\Delta s = \limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\gamma - \frac{1}{4}}{s}\right]\Delta s = \infty.$$

provided that $\gamma > \frac{1}{4}$. Hence every solution of (4.11) oscillates if $\gamma > \frac{1}{4}$, which agrees with the well–known oscillatory behavior of (4.11), (see Li [29]).

If $\mathbb{T} = \mathbb{Z}$, then (4.11) is the second order discrete Euler–Cauchy difference equation

$$\Delta^2 x_t + \frac{\gamma}{t(t+1)}x_{t+1} = 0, \ t = 1, 2, ... \tag{4.12}$$

and we have $\mu(s) = 1$, $\sigma(s) = s+1$, $C_1(s) = \frac{s-t_0+1}{s-t_0}$,

$$\frac{A_1^2(s)}{B_1(s)} = \frac{t_0^2}{s^2(s+1)(s-t_0+1)^2}.$$

Therefore (4.10) can be rewritten as

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{s^2-1}{4s^3}\right] - \frac{t_0^2}{4s^2(s+1)(s-t_0)(s-t_0+1)}\right]\Delta s$$
$$= \limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\gamma}{s} - \frac{1}{2s} + \frac{1}{4s}\right]\Delta s = \infty.$$

provided that $\gamma > \frac{1}{4}$. Hence every solution of (4.12) oscillates if $\gamma > \frac{1}{4}$, which agrees with the well–known oscillatory behavior of (4.12). It is known in Zhang and Cheng [32] that when $\mu \leq 1/4$, (4.12) has a nonoscillatory solution. Hence, Theorem 4.1 and Corollary 4.3 are sharp. Note that the results in Došlý and Hilger [8] and Erbe and Peterson [16] cannot be applied to (4.12).

Theorem 4.2. *Assume that* (1.4) *and* (2.1) *hold. Furthermore, assume that there exists a function* $r \in \mathcal{R}^+$ *such that* $p \cdot r$ *is differentiable and given any* $t_0 \geq a$ *there is a* $t_1 > t_0$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \left[\psi(s) - \frac{Q^2(s)C(s)}{4Q_1(s)} \right] \Delta s = \infty, \tag{4.13}$$

where m *is a positive integer. Assume further that*

$$\left(\frac{1}{t^m}\right) \int_{t_1}^{t} e_r^{\sigma}(s,t_0) p^{\sigma}(s) r^{\sigma}(s) \sum_{\nu=0}^{m-1} (\sigma(s)-t)^{\nu} (s-t)^{m-\nu-1} \Delta s \tag{4.14}$$

is bounded above. Then every solution of equation (1.1) *is oscillatory on* $[a,\infty)$.

Note that if $r \in \mathcal{R}^+$ and $r(t) \leq 0$, then (4.14) holds. When $r(t) = 0$, then (4.13) reduces to

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m q(s) \Delta s = \infty, \tag{4.15}$$

which can be considered as an extension of Kamenev type oscillation criteria for second order differential equations, (see Kamenev [24]).

When $\mathbb{T} = \mathbb{R}$, then (4.15) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m q(s) ds = \infty, \tag{4.16}$$

and when $\mathbb{T} = \mathbb{Z}$, then (4.15) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \sum_{s=t_1}^{t-1} (t-s)^m q(s) = \infty, \tag{4.17}$$

We next give some sufficient conditions for the case when (2.2) holds, which guarantee that every solution of the dynamic equation (1.1) oscillates or converges to zero on $[a,\infty)$. The next result removes a monotonicity assumption on f in Bohner and Saker [6].

Theorem 4.3. *Assume that* (1.4) *and* (2.2) *hold and assume there exists* $r \in \mathcal{R}^+$ *such that* $p \cdot r$ *is differentiable and such that* (4.4) *holds. Furthermore, assume*

$$\int_a^\infty \frac{1}{p(t)} \int_a^t q(s)\Delta s \Delta t = \infty. \tag{4.18}$$

and let (A) hold. Then every solution of equation (1.1) *is either oscillatory or converges to zero on* $[a, \infty)$.

In a similar manner, one may establish the following theorem.

Theorem 4.4. *Let all of the conditions of Theorem 4.3 hold with condition* (4.13) *replacing* (4.4). *Then every solution of equation* (1.1) *is oscillatory or converges to zero on* $[a, \infty)$.

5. Application to equations with damping

Our aim is to apply the results in section 4, to give some sufficient conditions for oscillation of all solutions of the dynamic equation (4.1) with damping terms. We note that all of the results in section 4, are true in the linear case. Before stating our main results in this section we will need the following Lemmas, (see Bohner and Peterson [4]).

Lemma 5.1. *If* $\alpha, \beta \in C_{rd}$ *and*

$$1 - \mu(t)\alpha(t) + \mu^2(t)\beta(t) \neq 0, \quad t \in \mathbb{T}, \tag{5.1}$$

then the second order dynamic equation (4.1) *with* $f(x) = x$ *can be written in the self-adjoint form* (1.5), *where*

$$p(t) = e_\gamma(t, t_0), \quad q(t) = [1 + \mu(t)\gamma(t)]p(t)\beta(t) \tag{5.2}$$

$$\gamma(t) = \frac{\alpha(t) - \mu(t)\beta(t)}{1 - \mu(t)\alpha(t) + \mu^2(t)\beta(t)}. \tag{5.3}$$

Lemma 5.2. *If* α *is a regressive function, then the second order dynamic equation* (4.1) *with* $f(x) = x$) *can be written in the self-adjoint form*

$$(p(t)x^\Delta(t))^\Delta + q(t) f \circ x^\sigma = 0, \tag{5.4}$$

where

$$p(t) = e_\alpha(t, t_0) \quad \textit{and} \quad q(t) = \beta(t)p(t) \tag{5.5}$$

Now, by using the results in section 3 and Lemma 5.1 we have the following results immediately.

Theorem 5.1. *Let p, q be defined as in* (5.5) *and ssume that* (2.1) *holds. Furthermore, assume that there exists a $r \in \mathcal{R}$ with r differentiable such that* (4.4) *holds with*

$$\psi(t) = e_r(\sigma(t), t_0)\left[q(t) + \frac{1}{2}(p(t)r(t))^{\Delta} + \frac{r^2(t)p(t)}{4C(t)}\right]. \tag{5.6}$$

Then equation (4.1) *with $f(x) = x$ is oscillatory on $[a, \infty)$.*

Corollary 5.1. *Assume that* (2.1) *and* (3.5) *hold, where p and q are as defined in* (5.5)*. Then equation* (4.1) *with $f(x) = x$ is oscillatory on $[a, \infty)$.*

Corollary 5.2. *Assume that* (2.1) *and* (4.6) *hold except that the term $Kq(t)$ is replaced by $q(t)$, where p and q are as defined in* (5.5)*. Then equation* (4.1) *with $f(x) = x$ is oscillatory on $[a, \infty)$.*

Theorem 5.2. *Assume that* (2.1) *holds. Furthermore, assume that there exists $r \in \mathcal{R}$ with r differentiable such that* (4.10) *holds, where p, q and ψ are as defined by* (5.5) *and* (5.6) *respectively, and m is odd integer. Then* (4.1) *with $f(x) = x$ is oscillatory on $[a, \infty)$.*

Theorem 5.3. *Assume that all the assumption of Theorem 5.1 hold except that the condition* (2.1) *is replaced by* (2.2)*. If* (4.18) *holds, then every solution of equation* (4.1) *with $f(x) = x$ is oscillatory or converges to zero on $[a, \infty)$.*

Theorem 5.4. *Assume that all the assumption of Theorem 5.2 hold except that the condition* (1.3) *is replaced by* (2.2)*. If* (4.18) *holds, then every solution of equation* (4.1) *with $f(x) = x$ is oscillatory or converges to zero on $[a, \infty)$.*

Oscillation criteria for equation (4.1) are now elementary consequences of the oscillation results in Theorems 5.1-5.4. The details are left to the reader.

6. Linear Second Order Dynamic Equations

In this section we shall be interested in obtaining comparison theorems for the second order linear equations

$$[p(t)x^{\Delta}(t)]^{\Delta} + q(t)x^{\sigma}(t) = 0, \tag{6.1}$$

$$[p(t)y^{\Delta}(t)]^{\Delta} + a^{\sigma}(t)q(t)y^{\sigma}(t) = 0, \tag{6.2}$$

$$[p(t)z^{\Delta}(t)]^{\Delta} + a(t)q(t)z^{\sigma}(t) = 0, \tag{6.3}$$

where $p(t) > 0$ and p, q, a are right-dense continuous on $\mathbb{T}$.

Definition 6.1. We say that a solution x of (6.1) has a *generalized zero* at t in case $x(t) = 0$. We say x has a *generalized zero* in $(t, \sigma(t))$ in case $x(t)x(\sigma(t)) < 0$ and $\mu(t) > 0$. We say that (6.1) is *disconjugate* on the interval $[c, d]$, if there is no nontrivial solution of (6.1) with two (or more) generalized zeros in $[c, d]$.

Definition 6.2. Equation (6.1) is said to be *nonoscillatory* on $[\tau, \infty)$ if there exists $c \in [\tau, \infty)$ such that this equation is disconjugate on $[c, d]$ for every $d > c$. In the opposite case (6.1) is said to be *oscillatory* on $[\tau, \infty)$. Oscillation of (6.1) may equivalently be defined as follows. A nontrivial solution y of (6.1) is called *oscillatory* if it has infinitely many (isolated) generalized zeros in $[\tau, \infty)$. By the Sturm type separation theorem, one solution of (6.1) is (non)oscillatory iff every solution of (6.1) is (non)oscillatory. Hence we can speak about *oscillation* or *nonoscillation of equation* (6.1).

Basic oscillatory properties of (6.1) are described by the so-called Reid Roundabout Theorem which is proved e.g. in [4, Theorem 4.53, Theorem 4.57].

Theorem 6.1 (Reid Roundabout Theorem)**.** *The following statements are equivalent:*

(i) *Equation* (6.1) *is disconjugate on* $[c, d]$*.*
(ii) *Equation* (6.1) *has a solution without generalized zeros on* $[c, d]$*.*
(iii) *The Riccati dynamic equation*

$$u^{\Delta}(t) + q(t) + \frac{u^2(t)}{p(t) + \mu(t)u(t)} = 0 \tag{6.4}$$

has a solution u *with* $p(t) + \mu(t)u(t) > 0$ *for* $t \in [c, d]^{\kappa}$ *(except for the case when* d *is left-dense and right-scattered at which* $p + \mu u$ *may be nonpositive).*

(iv) *The quadratic functional*

$$\mathcal{F}(\xi; c, d) = \int_c^d \left\{ p(t) \left(\xi^{\Delta}(t)\right)^2 - q(t)(\xi^{\sigma}(t))^2 \right\} \Delta t$$

is positive definite for $\xi \in U(c, d)$*, where*

$$U(c, d) = \{\xi \in C_p^1[c, d] \, : \, \xi(c) = \xi(d) = 0\}.$$

This result makes it therefore clear that there are at least two methods of investigation of (non)oscillation of (6.1). The first one – the *variational method* – is based on the equivalence of (i) and (iv) and its basic statement can be reformulated as follows:

Lemma 6.1 (Variational method)**.** *If for any* $T \in [\tau, \infty)$ *there exists* $0 \not\equiv \xi \in U(T)$*, where*

$$U(T) = \{\xi \in C_p^1[T,\infty) \,:\, \xi(t) = 0 \text{ for } t \in [\tau, T] \text{ and } \exists \hat{T}, \hat{T} > \sigma(T), \\ \text{such that } \xi(t) = 0 \text{ for } t \in [\sigma(\hat{T}), \infty)\},$$

such that $\mathcal{F}(\xi; T, \infty) = \mathcal{F}(\xi, T, \sigma(\hat{T})) \leq 0$*, then* (6.1) *is oscillatory.*

Another method of investigation for the oscillation theory of (6.1) is based on the equivalence of (i) and (iii) in Proposition 6.1. This is usually referred to as the *Riccati technique* and by virtue of the Sturm Comparison Theorem implies that for nonoscillation of (6.1), it is sufficient to find a solution of the Riccati-type inequality as given in the next lemma. A proof may be found in [12] or [4].

Lemma 6.2 (Riccati technique)**.** *Equation* (6.1) *is nonoscillatory if and only if there exists* $T \in [\tau, \infty)$ *and a function* u *satisfying the Riccati dynamic inequality*

$$u^{\Delta}(t) + q(t) + \frac{u^2(t)}{p(t) + \mu(t)u(t)} \leq 0$$

with $p(t) + \mu(t)u(t) > 0$ *for* $t \in [T, \infty)$.

For completeness, we recall the following

Lemma 6.3 (Sturm-Picone Comparison Theorem)**.** *Consider the equation*

$$[\tilde{p}(t)x^{\Delta}(t)]^{\Delta} + \tilde{q}(t)x^{\sigma}(t) = 0, \tag{6.5}$$

where $\tilde{p}$ *and* $\tilde{q}$ *satisfy the same assumptions as* p *and* q*. Suppose that* $\tilde{p}(t) \leq p(t)$ *and* $q(t) \leq \tilde{q}(t)$ *on* $[T, \infty)$ *for all large* T*. Then* (6.5) *is nonoscillatory on* $[\tau, \infty)$ *implies* (6.1) *is nonoscillatory on* $[\tau, \infty)$.

We mention first a few background details which serve to motivate the results in this section. Suppose that $\mathbb{T}$ is the real interval $[0, +\infty)$ so that (6.1) becomes

$$[p(t)x'(t)]' + q(t)x(t) = 0, \tag{6.6}$$

where $p(t)$ is continuous and positive and $q(t)$ is continuous on $[0, +\infty)$. It was shown in [9] that if (6.6) is oscillatory, then multiplying the coefficient $q(t)$ by a function $a(t)$ where $a(t) \geq 1$ and $p(t)a'(t)$ is nonincreasing preserves oscillation; i.e.,

$$[p(t)x'(t)]' + a(t)q(t)x(t) = 0, \tag{6.7}$$

is also oscillatory. Of course, if $q(t)$ is nonnegative, these results follow immediately from the usual Sturm-Picone Comparison Theorem, but when $q(t)$ changes sign on each half line, oscillation of (6.7) is not obvious if (6.6) is oscillatory. One may also notice that if (6.6) is oscillatory and if $a(t) = const = \lambda \geq 1$ then oscillation of (6.7) follows immediately from the Sturm-Picone Theorem by dividing the equation by λ (for the case when $q(t)$ may change sign). This result, i.e., the statement that says that if (6.6) is oscillatory, then so is

$$(p(t)y')' + \lambda q(t)y = 0$$

for any constant $\lambda \geq 1$, was also observed by Fink and St.Mary [22]. Kwong in [26] then showed that the result of [9] may be strengthened to a larger class of functions $a(t)$ by relaxing somewhat the monotonicity assumption on $p(t)a'(t)$ as given in [9]. We present below three different comparison theorems along with their corresponding corollaries, and show by examples, that they are all independent. In addition to extending the results of [26] and [9] in the case of equations (6.6) and (6.7) in the continuous case, the results we obtain are new in the discrete case and the more general time scales case. It should also be noted that because of the techniques of proof used, both (6.2) and (6.3) may be viewed as the time-scales extensions of (6.1), obtained when multiplying $q(t)$ by $a(t)$ (which is the same as $a^\sigma(t)$ when $\mathbb{T} = \mathbb{R}$.)

Our first result shows that if, "on average", $q(t)$ is more positive than negative, then the assumptions on $a(t)$ are quite mild. To be precise, we have

Theorem 6.2. *Assume $a \in C^1_{rd}$ and*

(i) $\liminf_{t\to\infty} \int_T^t q(s)\Delta s \geq 0$ *but* $\not\equiv 0$ *for all large* T,

(ii) $\int_\tau^\infty \frac{1}{p(s)}\Delta s = \infty$,

(iii) $0 < a(t) \leq 1, \quad a^\Delta(t) \leq 0.$

Then (6.1) *is nonoscillatory implies* (6.3) *is nonoscillatory.*

The corresponding "oscillation" result is

Corollary 6.1. *Assume $a \in C^1_{rd}$ and*

(i) $\liminf_{t\to\infty} \int_T^t a(s)q(s)\Delta s \geq 0$ *but* $\not\equiv 0$ *for all large* T,

(ii) $\int_\tau^\infty \frac{1}{p(s)}\Delta s = \infty$,

(iii) $a(t) \geq 1, \quad a^\Delta(t) \geq 0.$

Then (6.1) *is oscillatory implies* (6.3) *is oscillatory.*

If we strengthen the assumptions on $a(t)$ somewhat, then we may relax the assumptions on $q(t)$ and in this case, we consider the relation between (6.1) and (6.2). For convenience, we state first the "oscillation" result.

Theorem 6.3. *Assume $pa^\Delta \in C^1_{rd}$ and*

(i) $a(t) \geq 1$,

(ii) $\mu(t)a^{\Delta}(t) \geq 0$,
(iii) $\left(p(t)a^{\Delta}(t)\right)^{\Delta} \leq 0$.

Then (6.1) *is oscillatory implies* (6.2) *is oscillatory.*

In this case, the analogous "nonoscillation" result becomes

Corollary 6.2. *If* $p(\frac{1}{a})^{\Delta} \in C^1_{rd}$,
(i) $0 < a(t) \leq 1$,
(ii) $\mu(t)a^{\Delta}(t) \leq 0$,
(iii) $\left(p(t)\left(\frac{1}{a(t)}\right)^{\Delta}\right)^{\Delta} \leq 0$.

Then (6.1) *is nonoscillatory on implies* (6.2) *is nonoscillatory.*

In the following theorem we let χ denote the characteristic function of the set of right-scattered points $\hat{\mathbb{T}}$ defined by

$$\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}.$$

That is,

$$\chi(t) := \begin{cases} 1, & t \in \hat{\mathbb{T}} \\ 0, & t \notin \hat{\mathbb{T}}. \end{cases}$$

Theorem 6.4. *Assume* $pa^{\Delta} \in C^1_{rd}$, *and that the following conditions hold:*
(*i*) $a(t) > 0$ *and* $2a(t) + \mu(t)a^{\Delta}(t) \leq 2$
(ii) $p(t) > \epsilon_1\mu(t)$ *for some* $\epsilon_1 > 0$ *and for all* $t \in \mathbb{T}$,
(iii) *there is an* $\epsilon_0 > 0$ *such that the function*

$$G_{\epsilon_0}(t) := 2\left(a^{\Delta}(t)p(t)\right)^{\Delta} - \frac{\left(a^{\Delta}(t)p(t)\right)^2}{a(t)\left(p(t) - \mu(t)\epsilon_0\right)} \geq 0$$

for all large t, *where* $p(t) - \mu(t)\epsilon_0 > 0$,
(iv) $\limsup_{t\to\infty} \chi(t)\int_{\tau}^{t} q(s)\Delta s > -\infty$,
(v) *there exists a constant* $M > 0$ *such that* $\chi(t)p(t) \leq M\mu(t)$, *for* $t \in \mathbb{T}$.

Then (6.1) *is nonoscillatory implies* (6.2) *is nonoscillatory.*

Again, we have a corresponding "oscillation" result:

Corollary 6.3. *Assume* $p\left(\frac{1}{a}\right)^{\Delta} \in C^1_{rd}$ *and that the following conditions hold:*
(i) $a(t) + \frac{\mu(t)a^{\Delta}(t)}{2a^{\sigma}(t)} \geq 1$ *for all large* t,
(ii) *there is an* $\epsilon_1 > 0$ *such that* $p(t) > \mu(t)\epsilon_1$ *for* $t \in \mathbb{T}$,
(iii) *there exists* $\epsilon_0 > 0$ *such that the function*

$$H_{\epsilon_0}(t) := 2\left(\delta(t)\right)^{\Delta} + \frac{a(t)\delta^2(t)}{p(t) - \mu(t)\epsilon_0} \leq 0,$$

for all large t, where $p(t)-\mu(t)\epsilon_0>0$, and where

$$\delta(t):=\frac{p(t)a^{\Delta}(t)}{a(t)a^{\sigma}(t)}=-p(t)\left(\frac{1}{a(t)}\right)^{\Delta},$$

(iv) $\limsup_{t\to\infty}\chi(t)\int_\tau^t a^\sigma(s)q(s)\Delta s>-\infty$,

(v) *there is an $M>0$ such that $\chi(t)p(t)\le M\mu(t)$ for all large t.*

Then (6.1) *is oscillatory, implies that* (6.2) *is oscillatory.*

We notice in the last two results how the graininess function is involved in the criteria for oscillation/nonoscillation. In particular, for the case when $\mathbb{T}=[0,+\infty)$, then $\mu(t)\equiv 0$, so conditions (ii), (iv), and (v) of Corollary 6.3 hold trivially, and it may be shown that (iii) reduces to the condition of Kwong in [26]. The nonoscillation result Theorem 6.4 is new in all cases. It turns out that Lemma 2.1 is useful in proving the above results.

7. Examples and Remarks

We begin this section with several examples showing the independence of the above criteria.

Example 7.1. Let $r>1$. Consider the time scale

$$\mathbb{T}=r^{\mathbb{N}_0}:=\left\{r^k\,:\,k\in\mathbb{N}_0\right\}.$$

In this case, $\sigma(t)=rt$, $\mu(t)=(r-1)t$ for all $t\in\mathbb{T}$, and any dynamic equation on the time scale $r^{\mathbb{N}_0}$ is called an *r-difference equation*. Let

$$a(t)=\frac{1}{t^2},\quad p(t)=t\quad\text{and}\quad q(t)=\frac{\gamma\ln r}{(r-1)t\ln t\ln(rt)}+\frac{\lambda(-1)^{N(t)}}{t\ln t},$$

where γ,λ are real constants and $N(t):=\ln t/\ln r\in\mathbb{N}_0$. Observe that $q(t)$ is not eventually of one sign for $\lambda\neq 0$. Since $(\ln t)^{\Delta}=\frac{\ln r}{(r-1)t}$, it follows that we have

$$\int_1^t\frac{1}{p(s)}\Delta s=\int_1^t\frac{1}{s}\Delta s=\frac{(r-1)\ln t}{\ln r}\to\infty$$

as $t\to\infty$ and so (2.1) holds. Further,

$$\left(p(t)\left(\frac{1}{a(t)}\right)^{\Delta}\right)^{\Delta}=(r+1)^2t>0$$

for $t\in\mathbb{T}$, so condition (iii) of Corollary 6.2 fails to hold. Note that this condition is not even satisfied for any $a(t)=t^{-\omega}$, $\omega>0$. On the other hand, we have

$$2a(t)\left(a^{\Delta}(t)\right)^{\Delta}(p(t)-\mu(t)\epsilon_0)-\left(a^{\Delta}(t)p(t)\right)^2=\frac{(r+1)^2}{r^4t^4}[1-2(r-1)\epsilon_0]\ge 0$$

for $0 < \epsilon_0 \leq 1/[2(r-1)]$, and condition (iii) of Theorem 6.4 is satisfied, and

$$a^{\Delta}(t) = -\frac{r+1}{r^2t^3} < 0,$$

so (iii) of Theorem 6.2 holds. If $0 < \epsilon_1 < 1/(r-1) \leq M$, then $\epsilon_1\mu(t) < p(t) \leq M\mu(t)$ for all $t \in \mathbb{T}$, so conditions (ii) and (v) of Theorem 6.4 hold. Breaking up the integral and using the identity $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t)$ we get

$$\int_t^{\infty} q(s)\Delta s = \frac{\gamma}{\ln t} + (r-1)\lambda(-1)^{N(t)}\left[\frac{1}{\ln t} - \frac{1}{\ln(rt)} + \frac{1}{\ln(r^2t)} - \dots\right].$$

Hence

$$\gamma - (r-1)\lambda < \ln t \int_t^{\infty} q(s)\Delta s < \gamma + (r-1)\lambda.$$

In [30] it was proved that equation (6.1) is nonoscillatory provided

$$\lim_{t\to\infty} \frac{\mu(t)\frac{1}{p(t)}}{\int_\tau^t \frac{1}{p(s)}\Delta s} = 0 \tag{7.1}$$

and

$$-\frac{3}{4} < \liminf_{t\to\infty} \mathcal{A}(t) \leq \limsup_{t\to\infty} \mathcal{A}(t) < \frac{1}{4},$$

where

$$\mathcal{A}(t) := \left(\int_1^t \frac{1}{p(s)}\Delta s\right)\left(\int_t^{\infty} q(s)\Delta s\right).$$

We have

$$\frac{\mu(t)\frac{1}{p(t)}}{\int_1^t \frac{1}{p(s)}\Delta s} = \frac{\ln r}{\ln t}$$

and so condition (7.1) is satisfied. Further,

$$\frac{r-1}{\ln r}[\gamma - (r-1)\lambda] < \mathcal{A}(t) < \frac{r-1}{\ln r}[\gamma + (r-1)\lambda].$$

Set

$$\alpha = \frac{\gamma(r-1)}{\ln r} \quad \text{and} \quad \beta = \frac{\lambda(r-1)^2}{\ln r}.$$

If $\alpha \geq \beta > 0$ and $\alpha + \beta < 1/4$, then (6.1) is nonoscillatory and Theorem 6.2 or Theorem 6.4 can be applied to show that (6.3) and (6.2), respectively, are nonoscillatory. If $0 < \beta < -\alpha$ and $\alpha - \beta > -3/4$, then (2.3) fails to hold, equation (6.1) is nonoscillatory and in this case, only Theorem 6.4 can be applied.

Example 7.2. (i) Let $\mathbb{T} = \mathbb{Z}$, $a(t) = 1/\sqrt{t}$ and $p(t) = \sqrt{t} + \sqrt{t+1}$. Then condition (v) of Theorem 6.4 fails to hold since $p(t)$ is unbounded. Theorem 6.2 (for $q(t)$ satisfying (2.3)) or Corollary 6.3 can be applied since

$$\sum_{t=\tau}^{\infty} \frac{1}{p(t)} = \sum_{t=\tau}^{\infty} \frac{1}{\sqrt{t} + \sqrt{t+1}} = \infty,$$

$$\Delta a(t) = \frac{\sqrt{t} - \sqrt{t+1}}{\sqrt{t(t+1)}} < 0$$

and

$$\Delta \left(p(t) \Delta \left(\frac{1}{a(t)} \right) \right) = \Delta \left[\left(\sqrt{t+1} + \sqrt{t} \right) \left(\sqrt{t+1} - \sqrt{t} \right) \right] = 0.$$

(ii) Let $\mathbb{T} = \mathbb{Z}$, $a(t) = t^{-2}$ and $p(t) = (2t+1)^{-1}$. Then condition (ii) of Theorem 6.4 fails to hold since $p(t) \to 0$ as $t \to \infty$. Theorem 6.2 (for $q(t)$ satisfying (2.3)) or Corollary 6.3 can be applied since

$$\sum_{t=\tau}^{\infty} \frac{1}{p(t)} = \sum_{t=\tau}^{\infty} (2t+1) = \infty,$$

$$\Delta a(t) = \frac{-1-2t}{t^2(t+1)^2} < 0$$

and

$$\Delta \left(p(t) \Delta \left(\frac{1}{a(t)} \right) \right) = \Delta \left((2t+1)^{-1}(2t+1) \right) = 0.$$

(iii) Let $\mathbb{T} = \mathbb{Z}$, $a(t) = \gamma^{-t}$, $\gamma > 1$ and $p(t) = \lambda^t$, $\lambda \in (0,1)$. Then condition (ii) of Theorem 6.4 and condition (ii) of Theorem 6.2 fail to hold since $p(t) \to 0$ as $t \to \infty$ and

$$\sum_{t=\tau}^{\infty} \frac{1}{p(t)} = \sum_{t=\tau}^{\infty} \lambda^{-t} = \infty,$$

respectively. On the other hand, the assumptions of Corollary 6.3 are satisfied provided $\gamma\lambda \in (0,1]$ since we have

$$\Delta a(t) = (1-\gamma)\gamma^{-t-1} < 0$$

and

$$\Delta \left(p(t) \Delta \left(\frac{1}{a(t)} \right) \right) = (\gamma - 1)(\gamma\lambda - 1)(\gamma\lambda)^t < 0.$$

Notice that only Corollary 6.3 can be applied in this case.

Following the idea of the above examples, it is not difficult to find examples showing the independence of Theorem 6.3 and Corollary 6.3.

Example 7.3. Let $\mathbb{T} = \mathbb{Z}$, $p(t) = 1$,

$$a(t) = \frac{1}{2t + (-1)^t} \quad \text{and} \quad q(t) = \frac{\gamma}{t(t+1)} + \frac{\lambda(-1)^t}{t}.$$

It is easy to see that $q(t)$ changes sign for $\lambda \neq 0$ and

$$\Delta a(t) = \frac{-2 + 2(-1)^t}{(2t + 2 - (-1)^t)(2t + (-1)^t)} \leq 0.$$

It can also be shown that conditions (iii) from Corollary 6.3 and (iii) from Theorem 6.4 fail to hold since

$$\Delta \left(p(t) \Delta \left(\frac{1}{a(t)} \right) \right) = 4(-1)^t$$

and $2a(t)(1 - \epsilon_0)\Delta^2 a(t) - (\Delta a(t))^2$ is equal to a fraction with a positive denominator and a numerator such that the coefficient in the numerator of the highest power t^2 changes sign. Further we have

$$\gamma - \lambda < t \sum_{s=t}^{\infty} q(s) < \gamma + \lambda.$$

Hence, if $\gamma \geq \lambda > 0$ and $\gamma + \lambda < 1/4$, then equation (6.1) is nonoscillatory and (2.3) holds, so only Theorem 6.2 can be applied. We may obtain the same conclusion for the corresponding oscillatory counterparts provided $a(t) = 2t + (-1)^t$ and $\gamma - \lambda > 1/4$ with $\lambda > 0$.

Remark 7.1. (Case $\mathbb{T} = \mathbb{R}$) (i) In this case, with the assumption that the expression $p(t)a'(t)$ is differentiable, condition (iii) of Theorem 6.4 is equivalent to

$$2a(t)(p(t)a'(t))' - p(t)(a'(t))^2 \geq 0,$$

while condition (iii) of Corollary 6.3 takes the form

$$a(t)(p(t)a'(t))' - 2p(t)(a'(t))^2 \geq 0.$$

This shows that Corollary 6.3 is a consequence of Theorem 6.4 in this case. This remark holds also for the oscillatory counterparts if $\mathbb{T} = \mathbb{R}$, see [26].

(ii) Theorem 6.3 (and Corollary 6.3) do not require $a(t)$ to be nondecreasing (resp. nonincreasing) on $\mathbb{T} = \mathbb{R}$. Indeed, with $a(t) = 1 - 1/t$ and $p(t) = (t-1)^2$ we have an example of an increasing $a(t)$, where Corollary 6.3 can be applied. This, however, has no "discrete" counterpart since conditions (ii) from Corollary 6.3 and (v) from Theorem 6.4 fail to hold when $\mathbb{T} = \mathbb{Z}$. Note that in Theorem 6.2 (and Corollary 6.1) the function $a(t)$ is required to be nonincreasing (resp. nondecreasing) on any time scale.

Remark 7.2. (Repeated application)

(i) A repeated application of Theorem 6.3 (resp. Corollary 6.3) gives the following more general result: Let equation (6.1) be oscillatory (resp. nonoscillatory), and let the functions $a_1(t), a_2(t), \ldots, a_n(t)$ satisfy the assumptions of Theorem 6.3 (resp. of Corollary 6.3) and let $a(t) = \prod_{i=1}^{n} a_i(t)$. Then equation (6.2) is oscillatory (resp. nonscillatory). It is easy to see that this result is indeed more general; e.g., let $\mathbb{T} = \mathbb{Z}$, $a_1(t) = a_2(t) = t^{-1}$ and $p(t) = 1$. The functions $a_1(t), a_2(t)$ satisfy all the assumptions of Corollary 6.3, but condition (iii) fails to hold for $a(t) = t^{-2}$. Therefore, an iteration (repeated application) gives a better result. Note that the (weaker) assumption (iii) of Theorem 6.4 is satisfied directly for $a(t) = t^{-2}$, however, but to apply this theorem directly, one needs an additional restriction on $q(t)$.

(ii) Theorem 6.4 can also be applied repeatedly for monotonic functions $a(t)$ but we must show that

$$\limsup_{t\to\infty} \chi(t) \int_\tau^t q(s)\Delta s > -\infty \quad \text{implies} \quad \limsup_{t\to\infty} \chi(t) \int_\tau^t a^\sigma(t) q(s)\Delta s > -\infty.$$

By the time scale version of the second mean value theorem of integral calculus, see [30], there exists $T = T(t) \in \mathbb{T}$ such that

$$\begin{aligned} &\limsup_{t\to\infty} \chi(t) \int_\tau^t a^\sigma(t) q(s)\Delta s \\ &\geq \limsup_{t\to\infty} \chi(t) \left[a(\tau) \int_\tau^{T(t)} q(s)\Delta s + a(t) \int_{T(t)}^t q(s)\Delta s \right]. \end{aligned}$$

The expression on the right-hand side is greater than $-\infty$ since $a(t)$ is bounded and both integrals are of the same type as that in the assumptions.

Additional comments may be made to extend the results to dynamic equations of the form

$$\begin{aligned} \left[a(t)p(t)z^\Delta(t)\right]^\Delta + a(t)q(t)z^\sigma(t) &= 0, && (7.2) \\ \left[a(t)p(t)y^\Delta(t)\right]^\Delta + a^\sigma(t)q(t)y^\sigma(t) &= 0, && (7.3) \\ \left[p(t)y^\Delta(t)\right]^\Delta + a^\sigma(t)q(t)y^\sigma(t) &= 0. && (7.4) \end{aligned}$$

We leave this to the interested reader.

8. Euler–Cauchy Dynamic Equation

In this section we are concerned with the so-called Euler–Cauchy dynamic equation

$$\sigma(t) t x^{\Delta\Delta} + a t x^\Delta + b x = 0, \tag{8.1}$$

on a time scale $\mathbb{T}$ (closed subset of the reals $\mathbb{R}$), where we assume $t_0 = \inf \mathbb{T} > 0$. We will assume throughout the regressivity condition

$$\sigma(t)t - at\mu(t) + b\mu^2(t) \neq 0 \tag{8.2}$$

for $t \in \mathbb{T}^\kappa$. The equation

$$\lambda^2 + a\lambda + b = 0 \tag{8.3}$$

is called the characteristic equation of the Euler–Cauchy dynamic equation (8.1) and the roots of (8.3) are called the characteristic roots of (8.1).

We have then

Theorem 8.1. *Assume λ_1, λ_2 are solutions of the chacteristic equation* (8.3)*. If $\lambda_1 \neq \lambda_2$, then*

$$x(t) = c_1 e_{\frac{\lambda_1}{t}}(t, t_0) + c_2 e_{\frac{\lambda_2}{t}}(t, t_0)$$

is a general solution of (8.1)*. If $\lambda_1 = \lambda_2$, then*

$$x(t) = c_1 e_{\frac{\lambda_1}{t}}(t, t_0) + c_2 e_{\frac{\lambda_1}{t}}(t, t_0) \int_{t_0}^{t} \frac{1}{s + \lambda_1 \mu(s)} \Delta s$$

is a general solution of (8.1).

Next we would like to show that if our characteristic roots are complex, then there is a nice form for all real-valued solutions of the Euler–Cauchy dynamic equation in terms of the generalized exponential and trigonometric functions. Even in the difference equations case the complex roots are not considered (see Kelley and Peterson [27]).

Theorem 8.2. *Assume that the characteristic roots of* (8.1) *are complex, that is $\lambda_{1,2} = \alpha \pm i\beta$, where $\beta > 0$, and $\frac{\alpha}{t}, \frac{\beta}{t + \alpha\mu(t)} \in \mathcal{R}$. Then*

$$x(t) = c_1 e_{\frac{\alpha}{t}}(t, t_0) \cos_{\frac{\beta}{t+\alpha\mu(t)}}(t, t_0) + c_2 e_{\frac{\alpha}{t}}(t, t_0) \sin_{\frac{\beta}{t+\alpha\mu(t)}}(t, t_0)$$

is a general solution of the Euler–Cauchy dynamic equation (8.1).

In the remainder of this section we will be concerned with the oscillation of the Euler–Cauchy dynamic equation (8.1). We assume throughout this section that $\mathbb{T}$ is now unbounded above. We now show if the characteristic roots of (8.1) are complex how a general solution can be written in terms of the classical exponential function and classical trigonometric functions.

Lemma 8.1. *If the characteristic roots are complex, that is $\lambda_{1,2} = \alpha \pm i\beta$, where $\beta > 0$, then*

$$x(t) = A(t)\left(c_1 \cos B(t) + c_2 \sin B(t)\right),$$

where

$$\begin{aligned} A(t) &= e^{\int_{t_0}^{t} \Re\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau}, \\ B(t) &= \int_{t_0}^{t} \Im\left(\xi_{\mu(\tau)}\left(\frac{\alpha+i\beta}{\tau}\right)\right)\Delta\tau \end{aligned}$$

is a general solution of the Euler–Cauchy dynamic equation (8.1)*. If, in addition, every point in* $\mathbb{T}$ *is isolated, then for* $t \in \mathbb{T}$*,*

$$A(t) = \prod_{\tau=t_0}^{\rho(t)} \frac{1}{\tau}\sqrt{((\tau+\mu(\tau)\alpha)^2 + \beta^2\mu^2(t))},$$

$$B(t) = \sum_{\tau=t_0}^{\rho(t)} Arctan\left(\frac{\beta\mu(\tau)}{\tau+\alpha\mu(\tau)}\right).$$

Definition 8.1. If the characteristic roots of (8.1) are complex, then we say the Euler–Cauchy dynamic equation (8.1) is oscillatory iff $B(t)$ is unbounded.

As a well-known example note that if $\mathbb{T}$ is the real interval $[1, \infty)$ and the Euler–Cauchy equation has complex roots, then the Euler–Cauchy equation is oscillatory. This follows from what we said here because in this case by (1.1)

$$B(t) = \beta \int_1^t \frac{1}{\tau} d\tau = \beta \log t$$

which is unbounded. If $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$, then one can again show that $B(t)$ is unbounded and hence the Euler–Cauchy dynamic equation on $\mathbb{T} = q^{\mathbb{N}_0}$ is oscillatory when the characteristuc roots are complex. If $\mathbb{T} = \mathbb{N}$, then

$$B(t) = \sum_{k=1}^{t-1} Arctan\left(\frac{\beta}{k+\alpha}\right),$$

which can be shown to be unbounded and hence the Euler–Cauchy dynamic equation on $\mathbb{T} = \mathbb{N}$ is oscillatory when the characteristuc roots are complex. These last two examples were shown in Bohner and Saker [6], Erbe, Peterson, and Saker [20], and Erbe and Peterson [14], but here we established these results directly.

Theorem 8.3 (Comparison Theorem)**.** *Let* $\mathbb{T}_1 := \{t_0, t_1, \cdots\}$ *and* $\mathbb{T}_2 := \{s_0, s_1, \cdots\}$*, where* $\{t_n\}$ *and* $\{s_n\}$ *are strictly increasing sequences of positive numbers with limit* ∞*. If the Euler–Cauchy equation* (8.1) *on* $\mathbb{T}_1$ *is oscillatory and* $-\alpha < \frac{s_n}{\mu(s_n)} \leq \frac{t_n}{\mu(t_n)}$*, for* $n \geq 0$*, then the Euler–Cauchy equation* (8.1) *on* $\mathbb{T}_2$ *is oscillatory.*

Theorem 8.4. *Assume every point in the time scale $\mathbb{T}$ is isolated and $\lim_{t\to\infty} \frac{t}{\mu(t)}$ exists as a finite number, then the Euler–Cauchy equation in the complex characteristic roots case is oscillatory on $\mathbb{T}$.*

Theorem 8.4 does not cover the case when $\mathbb{T}$ is a time scale where $\lim_{t\to\infty} \frac{t}{\mu(t)} = \infty$. The next theorem considers a time scale where $\lim_{t\to\infty} \frac{t}{\mu(t)} = \infty$.

Theorem 8.5. *Let $p \geq 0$ and let $\mathbb{T}_p := \{t_n : t_0 = 1, t_{n+1} = t_n + \frac{1}{t_n^p}, n \in \mathbb{N}_0\}$. In the complex characteristic roots case, the Euler–Cauchy dynamic equation* (8.1) *is oscillatory on $\mathbb{T}_p$.*

One might think that one could use the argument in the proof of Theorem 8.5 to show that if there is an increasing unbounded sequence of points $\{t_j\}$ in $\mathbb{T}$ with $\mu(t_j) = \frac{1}{t_j^p}$, then the Euler–Cauchy equation (8.1) is oscillatory on $\mathbb{T}$ in the complex characteristic roots case. The following example shows that the same type of argument does not work.

Example 8.1. Assume that the Euler–Cauchy dynamic equation (8.1) has complex characteristic roots $\alpha \pm i\beta$, $\beta > 0$ and $\mathbb{T} := \cup_{n=1}^{\infty}[(n-1)^2 + 1, n^2]$, then (8.1) is oscillatory.

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