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Oscillation criteria for second-order nonlinear delay dynamic equations

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This paper is dedicated to Professor William F. Ames

Abstract

In this paper, we consider the second-order nonlinear delay dynamic equation

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)f\left(x\left(\tau(t)\right)\right) = 0,$$

on a time scale $\mathbb{T}$. By employing the generalized Riccati technique we will establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. The obtained results improve the well-known oscillation results for dynamic equations and include as special cases the oscillation results for differential equations. Some applications and examples are considered to illustrate the main results.

Keywords: Oscillation; Generalized Riccati technique; Dynamic equations; Time scales; Second-order nonlinear dynamic equation; Riccati transformation technique

1. Introduction

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on time scales which attempts to harmonize the oscillation

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theory for the continuous and the discrete, to include them in one comprehensive theory, and to eliminate obscurity from both. We refer the reader to the papers [2–6,8–15,18–24] and the references cited therein. For oscillation of nonlinear delay dynamic equations, Zhang and Shanliang [24] considered the equation

$$x^{\Delta\Delta}(t)+p(t)f\big(x(t-\tau)\big)=0, \quad t\in\mathbb{T}, \tag{1.1}$$

where $\tau\in\mathbb{R}$ and $t-\tau\in\mathbb{T}$, $f:\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing ($f'(u)>k>0$), and $uf(u)>0$ for $u\neq 0$. By using comparison theorems they proved that the oscillation of (1.1) is equivalent to the oscillation of the nonlinear dynamic equation

$$x^{\Delta\Delta}(t)+p(t)f\big(x\big(\sigma(t)\big)\big)=0, \quad t\in\mathbb{T} \tag{1.2}$$

and established some sufficient conditions for oscillation by applying the results established in [13] for (1.2). However, the results established in [24] are valid only when the graininess function $\mu(t)$ is bounded which is a restrictive condition (for example the results cannot be applied on $\mathbb{T}=q^{N_0}:=\{q^k\colon k\in\mathbb{N}_0\}$, $q>1$, where $\mu(t)=(q-1)t$ is unbounded). Also the restriction $f'(x)\geqslant k>0$ is required. This condition does not hold and cannot be applied in the case when

$$f(x)=x\left(\frac{1}{9}+\frac{1}{1+x^2}\right),$$

since

$$f'(x)=\frac{(x^2-2)(x^2-5)}{9(1+x^2)^2},$$

changes sign four times. Sahiner [23] considered the nonlinear delay dynamic equation

$$x^{\Delta\Delta}(t)+p(t)f\big(x\big(\tau(t)\big)\big)=0, \quad \text{for } t\in\mathbb{T}, \tag{1.3}$$

where $f:\mathbb{R}\to\mathbb{R}$ is continuous, $uf(u)>0$ for $u\neq 0$ and $|f(u)|\geqslant L|u|$, $\tau:\mathbb{T}\to\mathbb{T}$, $\tau(t)\leqslant t$ and $\lim_{t\to\infty}\tau(t)=\infty$ and proved that if there exists a Δ-differentiable function $\delta(t)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[p(s)\delta^{\sigma}(s)\frac{\tau(s)}{\sigma(s)}-\frac{\sigma(s)}{4Lk^2\tau(s)}\frac{(\delta^{\Delta}(s))^2}{\delta^{\sigma}(s)}\right]\Delta s=\infty, \tag{1.4}$$

then every solution of (1.3) oscillates. We observe that the condition (1.4) depends on an additional constant $k\in(0,1)$ which implies that the results are not sharp. As a special case he deduced that if

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[p(s)\tau(s)-\frac{1}{4Lk^2\tau(s)}\right]\Delta s=\infty, \tag{1.5}$$

then every solution of (1.3) oscillates. In Example 2 we will show that the dynamic equation

$$x^{\Delta\Delta}(t)+\frac{\beta}{t\tau(t)}x\big(\tau(t)\big)=0,$$

is oscillatory if $\beta>\frac{1}{4}$, but (1.5) does not give this result.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume throughout this paper that our time scale is unbounded above. We assume $t_0\in\mathbb{T}$

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and it is convenient to assume $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Our main interest is to consider the general nonlinear delay dynamic equation

$$\big(r(t)x^{\Delta}(t)\big)^{\Delta} + p(t)f\big(x\big(\tau(t)\big)\big) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.6}$$

where r and p are real rd-continuous positive functions defined on $\mathbb{T}$, the so-called delay function τ satisfies $\tau : [t_0, \infty)_{\mathbb{T}} \to \mathbb{T}$ is rd-continuous, $\tau(t) \leqslant t$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and $\lim_{t\to\infty} \tau(t) = \infty$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $uf(u) > 0$ for all $u \neq 0$ and $|f(u)| \geqslant K|u|$, and established some new oscillation criteria which improve the results established by Zhang and Shanliang [24] and Sahiner [23].

Our attention is restricted to those solutions $x(t)$ of (1.6) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)|: t > t_0\} > 0$ for any $t_0 \geqslant t_x$. A solution $x(t)$ of (1.6) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis (see [16]). Not only can this theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between," e.g., to the so-called q-difference equations and can be applied on other different types of time scales. Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of the new theory, see the paper by Agarwal et al. [1] and the references cited therein. A book on the subject of time scales by Bohner and Peterson [7] summarizes and organizes much of time scale calculus.

We note that (1.6) in its general form involves different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt$, and (1.6) becomes the linear delay differential equation

$$\big(r(t)x'(t)\big)' + p(t)f\big(x\big(\tau(t)\big)\big) = 0. \tag{1.7}$$

When $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.6) becomes the linear delay difference equation

$$\Delta\big(r(t)\Delta x(t)\big) + p(t)f\big(x\big(\tau(t)\big)\big) = 0. \tag{1.8}$$

When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $x^{\Delta}(t) = \Delta_h x(t) = (x(t+h) - x(t))/h$, $\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a + kh)h$, and (1.6) becomes the second-order delay difference equation with constant step size

$$\Delta_h\big(r(t)\Delta_h x(t)\big) + p(t)f\big(x\big(\tau(t)\big)\big) = 0. \tag{1.9}$$

When $\mathbb{T} = \{t: t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x^{\Delta}(t) = \Delta_q x(t) = (x(qt) - x(t))/(q-1)t$, $\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} f(q^k)\mu(q^k)$, and (1.6) becomes the second-order q-difference equation with variable step size

$$\Delta_q\big(r(t)\Delta_q x(t)\big) + p(t)f\big(x\big(\tau(t)\big)\big) = 0. \tag{1.10}$$

Of course many more examples may be given. A well-known integration formula (see [7]) on an isolated time scale is given by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)_{\mathbb{T}}} f(t)\mu(t).$$

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In the study of oscillation of differential equations, there are two techniques which are used to reduce the higher order equations to the first order Riccati equation (or inequality). One of them is the Riccati transformation technique which has been recently extended to dynamic equations. The other one is called the generalized Riccati technique. This technique can introduce some new sufficient conditions for oscillation and can be applied to different equations which cannot be covered by the results established by the Riccati technique. Li [17] considered the equation

$$\left(r(t)x'\right)' + p(t)x = 0, \tag{1.11}$$

and used the generalized Riccati substitution and established some new sufficient conditions for oscillation. Li utilized the class of functions as follows: Suppose there exist continuous functions $H, h : \mathbb{D} \equiv \{(t,s)\colon\ t \geqslant s \geqslant t_0\} \to \mathbb{R}$ such that $H(t,t) = 0$, $t \geqslant t_0$, $H(t,s) > 0$, $t > s \geqslant t_0$, and H has a continuous and nonpositive partial derivative on $\mathbb{D}$ with respect to the second variable. Moreover, let $h : \mathbb{D} \to \mathbb{R}$ be a continuous function with

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)}, \quad t, s \in \mathbb{D}.$$

He then proved that if there exists a positive function $g \in C^1[t_0,\infty), \mathbb{R}^+)$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)r(s)h(t,s)\,ds < \infty, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.12}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)\left[H(t,s)\psi(s) - \frac{1}{4}r(s)h^2(t,s)\right]\Delta s = \infty, \tag{1.13}$$

where $a(s) = \exp\{-\int_0^s g(u)\,du\}$ and $\psi(s) = \{p(s) + r(s)g^2(s) - (r(s)g(s)f(s))'\}$ then every solution of (1.43) oscillates. Li [17] applied the condition (1.13) to the equation

$$\left(\frac{1}{t}x'\right)' + \frac{1}{t^3}x = 0, \tag{1.14}$$

and proved that this equation is oscillatory and showed that the results that had been established by the Riccati technique cannot be applied.

So the following question arises. Can we obtain oscillation criteria on time scales which improve the results established in [23,24] and from which we are able to deduce the corresponding results for differential and difference equations and as a special case, cover criteria of the type established by Li and others?

The aim of this paper is to give a positive answer to this question by extending the generalized Riccati transformation techniques in the time scales setting to obtain some new oscillation criteria of Li-type for Eq. (1.6) when $\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty$. Also we consider the case when $\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} < \infty$ and establish some conditions that ensure that all solutions are either oscillatory or converge to zero. Our results in this paper improve the results established by Zhang and Shanliang [24] and Sahiner [23] and can be applied to arbitrary time scales. Some applications and examples are considered to illustrate the main results.

2. Main results

In this section, we will employ the generalized Riccati substitution on time scales and establish new oscillation criteria for (1.6). We define the function space $\Re$ as follows: $H \in \Re$ provided H

is defined for $t_0 \leqslant s \leqslant \sigma(t)$, $t, s \in [t_0, \infty)_{\mathbb{T}}$, $H(t,s) \geqslant 0$, $H(\sigma(t), t) = 0$, $H^{\Delta_s}(t,s) \leqslant 0$ for $t \geqslant s \geqslant t_0$, and for each fixed t, $H^{\Delta_s}(t,s)$ is delta integrable with respect to s. Important examples of H when $\mathbb{T} = \mathbb{R}$ are $H(t,s) = (t-s)^m$ for $m \geqslant 1$. When $\mathbb{T} = \mathbb{Z}$, $H(t,s) = (t-s)^{\underline{k}}$, $k \in \mathbb{N}$, where $t^{\underline{k}} = t(t-1)\cdots(t-k+1)$.

Before we state and prove our main oscillation results we prove the following lemma which is important in the proof of the main results.

Lemma 1. *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty, \tag{2.1}$$

and

$$\int_{t_0}^{\infty} \tau(t) p(t) \Delta t = \infty, \tag{2.2}$$

and assume that (1.6) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

(1) $x^\Delta(t) > 0$, $x(t) > t x^\Delta(t)$ *for* $t \in [T, \infty)_{\mathbb{T}}$;
(2) x *is strictly increasing and* $x(t)/t$ *is strictly decreasing on* $[T, \infty)_{\mathbb{T}}$.

Proof. Assume x is a positive solution of (1.6) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > 0$ and so that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Then from (1.6), we have

$$\left(r(t)x^\Delta(t)\right)^\Delta = -p(t) f\left(x\left(\tau(t)\right)\right) < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

This implies that $r(t)x^\Delta(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. We claim that $r(t)x^\Delta(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not, then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r(t_2)x^\Delta(t_2) =: c < 0$. Then

$$r(t)x^\Delta(t) \leqslant r(t_2)x^\Delta(t_2) = c, \quad t \in [t_2, \infty)_{\mathbb{T}},$$

and therefore

$$x^\Delta(t) \leqslant \frac{c}{r(t)}, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Integrating, we get by (2.2)

$$x(t) = x(t_2) + \int_{t_2}^{t} x^\Delta(s)\Delta s \leqslant x(t_2) + c \int_{t_2}^{t} \frac{\Delta s}{r(s)} \to -\infty \quad \text{as } t \to \infty, \tag{2.4}$$

which implies $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)x^\Delta(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ and so $x^\Delta(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Let

$$X(t) := x(t) - t x^\Delta(t). \tag{2.5}$$

We claim that there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $X(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Assume not. Then $X(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^\Delta = \frac{t x^\Delta(t) - x(t)}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)} > 0, \quad t \in [t_2, \infty)_{\mathbb{T}},$$

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which implies that $x(t)/t$ is strictly increasing on $[t_2,\infty)_{\mathbb{T}}$. Pick $t_3\in[t_2,\infty)_{\mathbb{T}}$ so that $\tau(t)\geqslant\tau(t_3)$, for $t\geqslant t_3$. Then

$$x\big(\tau(t)\big)/\tau(t)\geqslant x\big(\tau(t_3)\big)/\tau(t_3)=:d>0,$$

so that $x(\tau(t))\geqslant d\tau(t)$ for $t\geqslant t_3$. Now by integrating both sides of (1.4) from t_3 to t, we have

$$r(t)x^{\Delta}(t)-r(t_3)x^{\Delta}(t_3)+K\int_{t_3}^{t}p(s)x\big(\tau(s)\big)\Delta s\leqslant 0,$$

which implies that

$$r(t_3)x^{\Delta}(t_3)\geqslant r(t)x^{\Delta}(t)+K\int_{t_3}^{t}p(s)x\big(\tau(s)\big)\Delta s$$

$$\geqslant K\int_{t_3}^{t}p(s)x\big(\tau(s)\big)\Delta s>dK\int_{t_3}^{t}p(s)\tau(s)\Delta s,$$

which contradicts (2.2). Hence there is a $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $X(t)>0$ on $[t_2,\infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta}=\frac{tx^{\Delta}(t)-x(t)}{t\sigma(t)}=-\frac{X(t)}{t\sigma(t)}<0,\quad t\in[t_2,\infty)_{\mathbb{T}}$$

and we have that $\frac{x(t)}{t}$ is strictly decreasing on $[t_2,\infty)_{\mathbb{T}}$. □

Theorem 1. *Assume that* (2.1) *and* (2.2) *hold,* $H\in\Re$, *and there is a function* $a(t)$ *and a positive, differentiable function* $\delta(t)$, *such that for sufficiently large* t_1

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_1)}\int_{t_1}^{t}\bar H(t,s)\delta^{\sigma}(s)\big[\psi(s)-\phi(t,s)\big]\Delta s=\infty,\tag{2.6}$$

where $\bar H(t,s):=H(\sigma(t),\sigma(s))$, $\phi(t,s):=\frac14(\frac{\delta(s)}{\delta^{\sigma}(s)})^2\frac{r(s)A^2(t,s)}{C(s)}$, $C(t):=\frac{t}{\sigma(t)}$

$$\psi(s):=\frac{Kp(s)\tau(s)}{\sigma(s)}-\big(a(s)r(s)\big)^{\Delta}+\frac{sr(s)a^2(s)}{\sigma(s)}$$

and

$$A(t,s):=\frac{\delta^{\sigma}(s)C_1(s)}{\delta(s)}+\frac{H^{\Delta_s}(\sigma(t),s)}{\bar H(t,s)},\quad C_1(s):=\frac{\delta^{\Delta}(s)}{\delta^{\sigma}(s)}+2\frac{sa(s)}{\sigma(s)}.$$

Then every solution of (1.6) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.6). Then there is a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(\tau(t))\neq 0$ on $[t_1,\infty)_{\mathbb{T}}$. We will only consider the case where $x((\tau(t)))>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ as the proof in the other case is similar. In view of Lemma 1, there is some $t_2\geqslant t_1$ such that

$$x^{\Delta}(t)>0,\quad\big(r(t)x^{\Delta}(t)\big)^{\Delta}<0,\quad\text{for }t\geqslant t_2.\tag{2.7}$$

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Define the function $w(t)$ by the generalized Riccati substitution

$$w(t) := \delta(t)\left[\frac{r(t)x^\Delta(t)}{x(t)} + r(t)a(t)\right], \quad \text{for } t \geqslant t_2. \tag{2.8}$$

Hence

$$\begin{aligned} w^\Delta &= \delta^\Delta\left[\frac{rx^\Delta}{x} + ra\right] + \delta^\sigma\left[\frac{rx^\Delta}{x} + ra\right]^\Delta \\ &= \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta + \delta^\sigma\left[\frac{rx^\Delta}{x}\right]^\Delta \\ &= \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta + \delta^\sigma\left[\frac{x(rx^\Delta)^\Delta - r(x^\Delta)^2}{xx^\sigma}\right] \\ &= \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta + \delta^\sigma\frac{(rx^\Delta)^\Delta}{x^\sigma} - \delta^\sigma\frac{r(x^\Delta)^2}{xx^\sigma}. \end{aligned} \tag{2.9}$$

Then from (1.6), (2.8) and (2.9), we have

$$\begin{aligned} w^\Delta &= -\delta^\sigma\frac{pf\circ x\circ\tau}{x^\sigma} + \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta - \delta^\sigma r\frac{(x^\Delta)^2}{xx^\sigma} \\ &\leqslant -\delta^\sigma\frac{Kpx\circ\tau}{x^\sigma} + \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta - \delta^\sigma r\frac{x}{x^\sigma}\left(\frac{x^\Delta}{x}\right)^2. \end{aligned} \tag{2.10}$$

From the definition of $w(t)$, we see that

$$\left(\frac{x^\Delta}{x}\right)^2 = \left[\frac{w}{r\delta} - a\right]^2 = \left[\frac{w}{r\delta}\right]^2 + a^2 - 2\frac{wa}{r\delta}. \tag{2.11}$$

Also from Lemma 1, since $x(t)/t$ is strictly decreasing, we have

$$x\big(\tau(t)\big)/x^\sigma(t) \geqslant \tau(t)/\sigma(t) \quad \text{and} \quad x(t)/x^\sigma(t) \geqslant C(t), \tag{2.12}$$

where $C(t) := t/\sigma(t)$. Substituting from (2.11) and (2.12) into (2.10), we obtain

$$\begin{aligned} w^\Delta &\leqslant -\delta^\sigma\frac{Kp\tau}{\sigma} + \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta - \delta^\sigma rC\left[\left[\frac{w}{r\delta}\right]^2 + a^2 - 2\frac{aw}{r\delta}\right] \\ &= -\delta^\sigma\frac{Kp\tau}{\sigma} + \frac{\delta^\Delta}{\delta}w + \delta^\sigma(ra)^\Delta - \frac{C\delta^\sigma w^2}{r\delta^2} - \delta^\sigma rCa^2 + 2\frac{\delta^\sigma Ca}{\delta}w \\ &= -\delta^\sigma\psi + \frac{\delta^\sigma}{\delta}\left[\frac{\delta^\Delta}{\delta^\sigma} + 2aC\right]w - \frac{C\delta^\sigma}{r\delta^2}w^2, \end{aligned}$$

where $\psi = \frac{Kp\tau}{\sigma} - (ar)^\Delta + ra^2C$. Then, we have

$$w^\Delta \leqslant -\delta^\sigma\psi + \frac{\delta^\sigma C_1}{\delta}w - \frac{\delta^\sigma C}{r\delta^2}w^2. \tag{2.13}$$

Evaluating both sides of (2.13) at s, multiplying by $H(\sigma(t),\sigma(s))$ and integrating we get

$$\int_{t_2}^{t} H\big(\sigma(t),\sigma(s)\big)\delta^\sigma(s)\psi(s)\Delta s$$

$$\leqslant -\int_{t_2}^{t} H\big(\sigma(t),\sigma(s)\big)w^{\Delta}(s)\Delta s + \int_{t_2}^{t} \frac{\delta^{\sigma}(s)H(\sigma(t),\sigma(s))C_1(s)}{\delta(s)}w(s)\Delta s - \int_{t_2}^{t} \frac{C(s)H(\sigma(t),\sigma(s))\delta^{\sigma}(s)}{r(s)\delta^2(s)}w^2(s)\Delta s. \tag{2.14}$$

Integrating by parts and using the fact that $H(\sigma(t),t)=0$, we get

$$\int_{t_2}^{t} H\big(\sigma(t),\sigma(s)\big)w^{\Delta}(s)\Delta s = -H\big(\sigma(t),t_2\big)w(t_2) - \int_{t_2}^{t} H^{\Delta_s}\big(\sigma(t),s\big)w(s)\Delta s.$$

Substituting this into (2.14), we have

$$\int_{t_2}^{t} H\big(\sigma(t),\sigma(s)\big)\delta^{\sigma}(s)\psi(s)\Delta s \leqslant H\big(\sigma(t),t_2\big)w(t_2) - \int_{t_2}^{t} H\big(\sigma(t),\sigma(s)\big)\frac{C(s)\delta^{\sigma}(s)}{r(s)\delta^2(s)}w^2(s)\Delta s + \int_{t_2}^{t} H\big(\sigma(t),\sigma(s)\big)A(t,s)w(s)\Delta s. \tag{2.15}$$

This implies, after completing the square, that

$$\int_{t_2}^{t} \bar{H}(t,s)\delta^{\sigma}(s)\psi(s)\Delta s \leqslant H\big(\sigma(t),t_2\big)w(t_2) + \int_{t_2}^{t} \bar{H}(t,s)\frac{r(s)\delta^2(s)A^2(t,s)}{4C(s)\delta^{\sigma}(s)}\Delta s.$$

But, then

$$\frac{1}{H(\sigma(t),t_2)}\int_{t_2}^{t} \bar{H}(t,s)\delta^{\sigma}(s)\left[\psi(s) - \frac{1}{4}\left(\frac{\delta(s)}{\delta^{\sigma}(s)}\right)^2\frac{r(s)A^2(t,s)}{4C(s)}\right]\Delta s \leqslant w(t_2),$$

which contradicts (2.6). □

From Theorem 1 by choosing the function $H(t,s)$, appropriately, we can obtain different sufficient conditions for oscillation of (1.6). For instance, if we define a function $h(t,s)$ by

$$H^{\Delta_s}\big(\sigma(t),s\big) = -h(t,s)\sqrt{H\big(\sigma(t),\sigma(s)\big)}, \tag{2.16}$$

we have the following oscillation result. Note that when $\mathbb{T}=\mathbb{R}$, we have $H(t,\sigma(s))=H(t,s)$ and when $\mathbb{T}=\mathbb{N}$, we have $H(t,\sigma(s))=H(t,s+1)$.

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Corollary 1. *Assume that* (2.1) *and* (2.2) *hold, $H \in \Re$, and there is a differentiable function $\delta(t)$ such that for sufficiently large t_1*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_1)} \int_{t_1}^{t} \bar{H}(t,s)\delta^{\sigma}(s)\big[\psi(s)-\phi(t,s)\big]\Delta s = \infty, \tag{2.17}$$

where $\bar{H}(t,s)$, $\psi(s)$, $C(s)$ and $\phi(t,s)$ are as in Theorem 1*, and $A(t,s)$ simplifies to*

$$A(t,s) = \frac{\delta^{\sigma}(s)C_1(s)}{\delta(s)} - \frac{h(t,s)}{\sqrt{\bar{H}(t,s)}},$$

then every solution of (1.6) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

From Theorem 1 and Corollary 1, we can establish different sufficient conditions for the oscillation of (1.6) by using different choices of $\delta(t)$ and $a(t)$. For instance, if we consider $\delta(t) = t$, $a(t) = \frac{1}{t}$ and define $H(t,s)$ for $t_0 \leqslant s \leqslant \sigma(t)$ by $H(\sigma(t),t) = 0$ and $H(t,s) = 1$ otherwise, then we get the following oscillation result.

Corollary 2. *Assume that* (2.1) *and* (2.2) *hold, and for sufficiently large t_1*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[Kp(s)\tau(s) - \sigma(s)\left(\frac{r(s)}{s}\right)^{\Delta} - \frac{5r(s)}{4s}\right]\Delta s = \infty, \tag{2.18}$$

then every solution of (1.6) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

If in Theorem 4 we choose $a(t)$ and $\delta(t)$ such that

$$a(t) = -\frac{\sigma(t)\delta^{\Delta}(t)}{2t\delta^{\sigma}(t)}, \tag{2.19}$$

we have $C_1(t) = 0$ and from Corollary 1 we have the following oscillation result for (1.6).

Corollary 3. *Assume that* (2.1) *and* (2.2) *hold, $H \in \Re$, and h is defined by* (2.16)*, and there is a positive differentiable function $\delta(t)$ such that for t_1 sufficiently large*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_1)} \int_{t_1}^{t} \left[\bar{H}(t,s)\delta^{\sigma}(s)\psi(s) - \frac{\sigma(s)\delta^2(s)r(s)h^2(t,s)}{4s\delta^{\sigma}(s)}\right]\Delta s = \infty, \tag{2.20}$$

then every solution of (1.6) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

If we define $H(t,s)$ for $t_0 \leqslant s \leqslant \sigma(t)$ by $H(\sigma(t),t) = 0$ and $H(t,s) = 1$ otherwise, $H(t,s) = 1$; and $a(t)$ and $\delta(t)$ such that (2.19) holds, we have $C_1(t) = 0$, $h(t,s) = 0$ and from Corollary 3 we have the following oscillation result for (1.6).

Corollary 4. *Assume that* (2.1), (2.2), *and* (2.19) *hold. Furthermore assume that for sufficiently large t_1*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \delta^{\sigma}(s)\psi(s)\Delta s = \infty,$$

where here $\psi(s)$ *reduces to*

$$\psi(s)=\frac{Kp(s)\tau(s)}{\sigma(s)}-\big(r(s)a(s)\big)^{\Delta}+\frac{sr(s)a^2(s)}{\sigma(s)},$$

then every solution of (1.6) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

From Corollary 4, we can also establish different sufficient conditions for the oscillation of (1.6) by using different choices of $\delta(t)$. For instance, if $\delta(t)=t$ then $a(t)=-\frac{1}{2t}$ and if $\delta(t)=t^2$, then $a(t)=-\frac{t+\sigma(t)}{2t\sigma(t)}$, and from Corollary 4 we have the following oscillation results respectively.

Corollary 5. *Assume that* (2.1) *and* (2.2) *hold. If for sufficiently large* t_1

$$\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\psi(s)\Delta s=\infty, \tag{2.21}$$

where in this case,

$$\psi(s)=\frac{Kp(s)\tau(s)}{\sigma(s)}+\frac{1}{2}\left(\frac{r(s)}{s}\right)^{\Delta}+\frac{r(s)}{4\sigma(s)s},$$

then every solution of (1.6) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Corollary 6. *Assume that* (2.1) *and* (2.2) *hold. If for sufficiently large* t_1

$$\limsup_{t\to\infty}\int_{t_1}^{t}\big(\sigma(s)\big)^2\psi(s)\Delta s=\infty, \tag{2.22}$$

where, in this case,

$$\psi(s):=\frac{Kp(s)\tau(s)}{\sigma(s)}+\left(\frac{(s+\sigma(s))r(s)}{2s\sigma(s)}\right)^{\Delta}+\frac{r(s)(s+\sigma(s))^2}{4s(\sigma(s))^3},$$

then every solution of (1.6) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

In the following we consider (1.6), when r does not satisfy (2.1), i.e., when

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\Delta t<\infty. \tag{2.23}$$

We start with the following auxiliary result, whose proof is similar to that which can be found in [13], and so is omitted.

Lemma 2. *Assume* (2.23) *holds, and*

$$\int_{t_0}^{\infty}\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\Delta t=\infty. \tag{2.24}$$

Suppose that x *is a nonoscillatory solution of* (1.6) *such that there exists* $t_1\in T$ *with*

$$x(t)x^{\Delta}(t)<0\quad \textit{for all } t\geqslant t_1.$$

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Then

$$\lim_{t\to\infty} x(t) = 0.$$

Using Lemma 2, we can derive the following criterion.

Theorem 2. *Let the assumptions* (2.23) *and* (2.24) *hold, and assume* $H \in \Re$. *If there exists a function* $a(t)$ *and a positive differentiable function* $\delta(t)$ *such that* (2.6) *holds, then every solution of* (1.6) *is oscillatory or converges to zero as* $t \to \infty$.

Following Corollaries 1–6 we can obtain more examples similar to those above. The details are left to the reader.

3. Applications

In this section we apply the oscillation results to different types of time scales and establish some oscillation criteria for Eqs. (1.7)–(1.10). We start with the case when $\mathbb{T} = \mathbb{R}$, then we have from Theorem 1 the following oscillation result for the differential equation (1.7).

Theorem 3. *Assume that*

$$\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty, \tag{3.1}$$

and

$$\int_{t_0}^{\infty} \tau(s)p(s)\,ds = \infty. \tag{3.2}$$

Furthermore, assume that there exist a differentiable function $a(t)$ *and a positive differentiable function* $\delta(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)\delta(s)\left[\psi(s) - \frac{1}{4}r(s)A^2(t,s)\right] ds = \infty, \tag{3.3}$$

where

$$A(t,s) = C_1(s) + \frac{\frac{\partial H(t,s)}{\partial s}}{H(t,s)}, \quad C_1(s) = \frac{\delta'(s)}{\delta(s)} + 2a(s), \tag{3.4}$$

and

$$\psi(s) = \frac{Kp(s)\tau(s)}{s} - (ar)'(s) + r(s)a^2(s).$$

Then every solution of the differential equation (1.7) *is oscillatory.*

From Theorem 3 by choosing $h(t,s)$ so that $\frac{\partial}{\partial s}H(t,s) = -h(t,s)\sqrt{H(t,s)}$, and taking $a(t) = -\frac{\delta'(t)}{2\delta(t)}$ we get the following result.

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Corollary 7. *Assume that* (3.1) *and* (3.2) *hold and let* $H \in \Re$ *and define* $h(t,s)$ *by* $\frac{\partial}{\partial s}H(t,s) = -h(t,s)\sqrt{H(t,s)}$. *Assume there is a differentiable function* $\delta(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)\delta(s)\psi(s) - \frac{1}{4}\delta(s)r(s)h^2(t,s) \right] ds = \infty, \tag{3.5}$$

where

$$\psi(s) = \frac{Kp(s)\tau(s)}{s} + \frac{1}{2}\left(\frac{r(s)\delta'(s)}{\delta(s)}\right)' + r(s)\left(\frac{\delta'(s)}{2\delta(s)}\right)^2,$$

then every solution of the delay differential equation (1.7) *oscillates.*

If we let $\tau(t) = t$ in Corollary 7 we get a result by Li [17].

Letting $H(t,s) = 1$ for $t_0 \leqslant s < t$ and $H(t,t) = 0$ in Corollary 7 we get the following result.

Corollary 8. *Assume that* (2.1) *and* (2.2) *hold and there exists a differentiable function* $\delta(t)$ *such that for sufficiently large* t_1

$$\limsup_{t\to\infty} \int_{t_1}^{t} \delta(s)\left[\frac{Kp(s)\tau(s)}{s} + \frac{1}{2}\left(\frac{r(s)\delta'(s)}{\delta(s)}\right)' + r(s)\left(\frac{\delta'(s)}{2\delta(s)}\right)^2 \right] ds = \infty,$$

then every solution of the delay differential equation (1.7) *is oscillatory on* $[t_0,\infty)$.

Letting $\delta(t) = t$ in Corollary 8 we get the following result.

Corollary 9. *Assume that* (2.1) *and* (2.2) *hold. If for sufficiently large* t_1

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[Kp(s)\tau(s) + \frac{s}{2}\left(\frac{r(s)}{s}\right)' + \frac{r(s)}{4s} \right] ds = \infty, \tag{3.6}$$

then every solution of the delay differential equation (1.7) *is oscillatory on* $[t_0,\infty)$.

Letting $\delta(t) = t^2$ in Corollary 8 we get the following result.

Corollary 10. *Assume that* (2.1) *and* (2.2) *hold. If for sufficiently large* t_1

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[Ksp(s)\tau(s) + s^2\left(\frac{r(s)}{s}\right)' + r(s) \right] ds = \infty, \tag{3.7}$$

then every solution of the delay differential equation (1.7) *is oscillatory on* $[t_0,\infty)$.

Now we apply our results in Section 2 to the time scale $\mathbb{T} = \mathbb{N}$ and establish some oscillation criteria for the delay difference equation (1.8). From Theorem 1 we get the following result.

Theorem 4. *Assume that*

$$\sum_{t=t_0}^{\infty} \frac{1}{r(t)} = \infty, \tag{3.8}$$

and

$$\sum_{t=t_0}^{\infty} \tau(t)p(t) = \infty. \tag{3.9}$$

Furthermore, assume that there exists a sequence $a(t)$ and a positive sequence $\delta(t)$ such that for sufficiently large integers t_1

$$\limsup_{t\to\infty} \frac{1}{H(t+1,t_1)} \sum_{s=t_1}^{t-1} H(t+1,s+1)\delta(s+1)\big[\psi(s)-\phi(t,s)\big] = \infty, \tag{3.10}$$

where

$$\psi(s) = \frac{Kp(s)\tau(s)}{s+1} - \Delta\big(a(s)r(s)\big) + \frac{sr(s)a^2(s)}{s+1},$$
$$\phi(t,s) = \frac{(s+1)r(s)\delta^2(s)A^2(t,s)}{4s\delta^2(s+1)},$$

and

$$A(t,s) = \frac{\delta(s+1)C_1(s)}{\delta(s)} + \frac{H(t+1,s+1)-H(t+1,s)}{H(t+1,s+1)},$$

where

$$C_1(s) := \frac{\Delta\delta(s)}{\delta(s+1)} + 2\frac{sa(s)}{s+1}.$$

Then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

From Theorem 4 by choosing $h(t,s)$ so that

$$\Delta_s H(t+1,s) = -h(t,s)\sqrt{H(t+1,s+1)}, \tag{3.11}$$

we have the following result.

Corollary 11. *Assume that* (3.8), (3.9) *hold and let $H \in \Re$ such that* (3.11) *holds. If there exist a sequence $a(t)$ and a positive sequence $\delta(t)$ such that for sufficiently large integers t_1,* (3.10) *holds where ψ, ϕ, and C_1 are as in Theorem* 4*, where*

$$A(t,s) := \frac{\delta(s+1)C_1(s)}{\delta(s)} - \frac{h(t,s)}{\sqrt{H(t+1,s+1)}},$$

then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

From Theorem 4 and Corollary 11, we can establish different sufficient conditions for the oscillation of (1.7) by using different choices of $H(t,s)$, $\delta(t)$ and $a(t)$. For instance, if we let $\delta(t)=t$, $a(t)=\frac{1}{t}$ and $H(t,s)=1$ for $t>s\geqslant t_0$, and $H(t+1,t)=0$ in Theorem 4, we get the following oscillation result.

Corollary 12. *Assume that* (3.8) *and* (3.9) *hold. Furthermore, assume that for sufficiently large integers t_1*

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1} \left[Kp(s)\tau(s) - (s+1)\Delta\left(\frac{r(s)}{s}\right) - \frac{5}{4}\frac{r(s)}{s}\right]\Delta s = \infty, \tag{3.12}$$

then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

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If we choose $a(t)$ and $\delta(t)$ such that

$$a(t) = -\frac{(t+1)\Delta\delta(t)}{2t\delta(t+1)}, \tag{3.13}$$

we have $C_1(t) = 0$ and we obtain the following result from Corollary 11.

Corollary 13. *Assume that* (3.8) *and* (3.9) *hold and let* $H \in \Re$ *such that* (3.11) *and* (3.13) *hold. If for* t_1 *sufficiently large*

$$\limsup_{t\to\infty} \frac{1}{H(t+1,t_1)} \sum_{s=t_1}^{t-1} H(t+1,s+1)\delta(s+1)\big[\psi(s) - \phi(t,s)\big] = \infty, \tag{3.14}$$

where

$$\phi(t,s) = \frac{1}{4}\left(\frac{\delta(s)}{\delta(s+1)}\right)^2 \frac{(s+1)r(s)h^2(t,s)}{sH(t+1,s+1)},$$

then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

From Corollary 13 we have, taking $H(t,s) = 1$ for $t_0 \leqslant s \leqslant t$ and $H(t+1,t) = 0$, the following oscillation result for the delay difference equation (1.8).

Corollary 14. *Assume that* (3.8) *and* (3.9) *hold and there is a sequence* $a(t)$ *and a positive sequence* $\delta(t)$ *such that* (3.13) *holds. If for sufficiently large integers* t_1

$$\lim_{t\to\infty} \sup \sum_{s=t_0}^{t-1} \delta(s+1)\left[\frac{Kp(s)\tau(s)}{s+1} - \Delta\big(r(s)a(s)\big) + \frac{sr(s)a^2(s)}{s+1}\right] = \infty,$$

then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

Letting $\delta(t) = t$ and $a(t) = -\frac{1}{2t}$ in Corollary 14 we get the following result.

Corollary 15. *Assume that* (3.8) *and* (3.9) *hold. Furthermore assume that for sufficiently large integers* t_1

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1}\left[Kp(s)\tau(s) + \frac{s+1}{2}\Delta\left(\frac{r(s)}{s}\right) + \frac{r(s)}{4s}\right] = \infty, \tag{3.15}$$

then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

Letting $\delta(t) = t^2$ and $a(t) = -\frac{2t+1}{2t(t+1)}$ in Corollary 14 we get the following result.

Corollary 16. *Assume that* (3.8) *and* (3.9) *hold. Furthermore assume that for sufficiently large integers* t_1

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t-1}(s+1)^2\left[\frac{Kp(s)\tau(s)}{s+1} + \Delta\left(\frac{(2s+1)r(s)}{2s(s+1)}\right) + \frac{r(s)(2s+1)^2}{4s(s+1)^3}\right] = \infty, \tag{3.16}$$

then every solution of the delay difference equation (1.8) *is oscillatory on* $\mathbb{N}$.

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Next we give an oscillation result for the delay q-difference equation (1.10). This result follows easily by applying Theorem 1 for the time scale $\mathbb{T} = [q^{n_0}, \infty)_{q^{\mathbb{N}_0}}$, $q > 1$.

Theorem 5. *Assume that*

$$\sum_{k=n_0}^{\infty} \mu(q^k) \frac{1}{r(q^k)} = \infty, \tag{3.17}$$

and

$$\sum_{k=k_0}^{\infty} \mu(q^k) \tau(q^k) p(q^k) = \infty. \tag{3.18}$$

Furthermore, assume that there exists a positive sequence $\delta(t)$ *such that for sufficiently large integers* n_1

$$\limsup_{n\to\infty} \frac{1}{H(q^{n+1}, q^{n_1})} \sum_{k=n_1}^{n-1} q^n H(q^{n+1}, q^{k+1}) \delta(q^{k+1}) \left[\psi(q^k) - \phi(q^n, q^k)\right] = \infty, \tag{3.19}$$

where

$$\phi(t,s) = \frac{q r(s) \delta^2(s) A^2(t,s)}{4\delta^2(qs)},$$

$$A(t,s) = \frac{\delta(qs) C_1(s)}{\delta(s)} + \frac{H^{\Delta_s}(qt,s)}{H(t,qs)}, \quad C_1(s) := \frac{\delta^{\Delta_s}(s)}{\delta(qs)} + 2\frac{a(s)}{q},$$

and

$$\psi(s) := \frac{K p(s) \tau(s)}{qs} - \left(r(s)a(s)\right)^{\Delta_s} + \frac{r(s)}{q} a^2(s).$$

Then every solution of the delay q-difference equation (1.10) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

As we did above one can deduce several new oscillation criteria for the delay q-difference equation (1.10) from Theorem 5, which we leave to the interested reader.

4. Examples

In this section, we give some examples to illustrate the main results.

Example 1. If

$$\beta \geqslant k \limsup_{t\to\infty} \frac{\sigma(t)}{\tau(t)}, \quad \text{where } k > \frac{1}{4},$$

then the Euler–Cauchy delay dynamic equation

$$x^{\Delta\Delta}(t) + \frac{\beta}{t\sigma(t)} x(\tau(t)) = 0, \tag{4.1}$$

is oscillatory on $[t_0, \infty)_{\mathbb{T}}$. To verify this result we will apply Corollary 5. Here $r(t) = 1$, $p(t) = \frac{\beta}{t\sigma(t)}$. It is clear that (2.1) and (2.2) hold and $f(u) = u$, so that $K = 1$. It remains to prove that the condition (2.21) holds. In this case (2.21) reads

$$\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\left[\frac{\beta}{s\sigma(s)}\frac{\tau(s)}{\sigma(s)}+\frac{1}{2}\left(\frac{1}{s}\right)^{\Delta}+\frac{1}{4\sigma(s)s}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\left[\frac{\beta}{s\sigma(s)}\frac{\tau(s)}{\sigma(s)}-\frac{1}{4}\frac{1}{\sigma(s)s}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{\beta\frac{\tau(s)}{\sigma(s)}-\frac{1}{4}}{s}\right]\Delta s=\infty,$$

provided that $\beta \geqslant k \limsup_{t\to\infty}\frac{\sigma(t)}{\tau(t)}$, where $k>\frac{1}{4}$, and hence Eq. (4.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$. In particular, if $\mathbb{T}=\mathbb{R}$, $t_0>0$, and the delay is the constant delay $\tau(t)=t-\delta$, where $\delta>0$, then if $\beta>\frac{1}{4}$, Eq. (4.1) is oscillatory; if $\mathbb{T}=\mathbb{Z}$, $t_0=1$, and the delay is the constant delay $\tau(t)=t-n$, n a positive integer, then if $\beta>\frac{1}{4}$, Eq. (4.1) is oscillatory; and finally if $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$, $t_0=1$, and the delay function is $\tau(t)=\rho^n(t)=\frac{t}{q^n}$, where n is a positive integer, then if $\beta>\frac{1}{4}q^{n+1}$, Eq. (4.1) is oscillatory.

Example 2. Consider the dynamic delay equation

$$x^{\Delta\Delta}(t)+\frac{\beta}{t\tau(t)}x\big(\tau(t)\big)=0, \tag{4.2}$$

for $t\in[1,\infty)_{\mathbb{T}}$. Here $r(t)=1$, $p(t)=\frac{\beta}{\tau(t)t}$. We will apply Corollary 5. It is clear that (2.1) and (2.2) hold and $f(u)=u$, so that $K=1$. It remains to prove that the condition (2.21) holds. In this case (2.21) reads

$$\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\left[\frac{\beta}{\tau(s)s}\frac{\tau(s)}{\sigma(s)}+\frac{1}{2}\left(\frac{1}{s}\right)^{\Delta}+\frac{1}{4\sigma(s)s}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_{t_1}^{t}\sigma(s)\left[\frac{\beta}{\tau(s)s}\frac{\tau(s)}{\sigma(s)}-\frac{1}{4}\frac{1}{\sigma(s)s}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_{t_1}^{t}\left[\frac{\beta-\frac{1}{4}}{s}\right]\Delta s=\infty,$$

provided that $\beta>\frac{1}{4}$.

Example 3. Consider the delay differential equation

$$\big(t^{\alpha}x'(t)\big)'+\frac{\beta}{t^{\eta}\tau(t)}x\big(\tau(t)\big)=0, \tag{4.3}$$

for $t\in[1,\infty)_{\mathbb{R}}$, $\beta>0$. If $0<\alpha\leqslant 1$, $0\leqslant\eta\leqslant 1$, and if either $\alpha+\eta<1$ or $\alpha+\eta\geqslant 1$ and $\frac{1}{2}<\alpha\leqslant 1$, then by Corollary 9, every solution of the delay differential equation (4.3) is oscillatory on $[1,\infty)$. Using Corollary 5 it is easy to see that the corresponding delay dynamic equation

$$\big(t^{\alpha}x^{\Delta}(t)\big)^{\Delta}+\frac{\beta}{t^{\eta}\tau(t)}x\big(\tau(t)\big)=0$$

is oscillatory on $[t_0,\infty)_{\mathbb{T}}$, if $0<\alpha<1$, $0\leqslant\eta<1$, and $\alpha+\eta<1$.

Example 4. Consider the delay differential equation

$$\big(t^{\alpha} x'(t)\big)' + \frac{\beta}{t^{\alpha+\eta}} x\big(\tau(t)\big) = 0, \tag{4.4}$$

for $t \in [1,\infty)_{\mathbb{R}}$, where $\beta > 0$ and $0 < \eta \leqslant 1$. Using Theorem 2 it is easy to show that if $\gamma < 0$, $\alpha + \gamma < 1$, $0 < \gamma + \eta < 1$, $\alpha > 1$, and $\alpha + \gamma + \eta < 2$, then every solution of the delay differential equation (4.4) is oscillatory on $[1,\infty)$ or converges to zero. In particular if $\beta > 0$, $\gamma = -\frac{1}{2}$, $\eta = \frac{3}{4}$ and $1 < \alpha < \frac{7}{4}$, then every solution of the delay differential equation (4.4) is oscillatory on $[1,\infty)$ or converges to zero.

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