BOUNDEDNESS AND UNIQUENESS OF SOLUTIONS TO DYNAMIC EQUATIONS ON TIME SCALES

ALLAN C. PETERSON* AND CHRISTOPHER C. TISDELL**

*Department of Mathematics
University of Nebraska - Lincoln
Lincoln NE 68588-0323
USA
apeterso@math.unl.edu

**School of Mathematics
The University of New South Wales
Sydney NSW 2052
AUSTRALIA
cct@maths.unsw.edu.au

[1]

This paper is dedicated to Saber Elaydi

Abstract. In this work we investigate the boundedness and uniqueness of solutions to systems of dynamic equations on time scales. We define suitable Lyapunov-type functions and then formulate appropriate inequalities on these functions that guarantee all solutions to first-order initial value problems are uniformly bounded and / or unique. Several examples are given.

1. Introduction

This paper considers the boundedness and uniqueness of solutions to the first-order dynamic equation

$$x^{\Delta} = f(t, x), \quad t \geq 0, \tag{1}$$

subject to the initial condition

$$x(t_0) = x_0, \quad t_0 \geq 0, \quad x_0 \in \mathbb{R}, \tag{2}$$

where $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and t is from a so-called "time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$). Equation (1) subject to (2) is known as an initial value problem (IVP) on time scales.

[1]AMS Subject Classification: 39A12

Keywords: time scale, Lyapunov function, system of equations

If $\mathbb{T} = \mathbb{R}$ then $x^\Delta = x'$ and (1), (2) become the following IVP for ordinary differential equations

$$x' = f(t, x), \quad t \geq 0, \tag{3}$$

$$x(t_0) = x_0, \quad t_0 \geq 0. \tag{4}$$

Recently, Raffoul [8] used Lyapunov-type functions to formulate some sufficient conditions that ensure all solutions to (3), (4) are bounded, while in a more classical setting, Hartman [3, Chapter 3] employed Lyapunov-type functions to prove that solutions to (3), (4) are unique. Motivated by [8] and [3] (see also references therein), we investigate the boundedness and uniqueness of solutions to systems of dynamic equations in the more general time scale setting. We define suitable Lyapunov-type functions on time scales and then formulate appropriate inequalities on these functions that guarantee solutions to (1), (2) are uniformly bounded and / or unique. In fact, our theory generalizes some of the results in [8] and [3] for the special case $\mathbb{T} = \mathbb{R}$.

To understand the notation used above and the idea of time scales some preliminary definitions are needed.

Definition A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. We assume that $0 \in \mathbb{T}$ (for convenience) and $\mathbb{T}$ is unbounded above.

Since a time scale may or may not be connected, the concept of the jump operator is useful.

Definition Define the forward jump operator $\sigma(t)$ at t by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \text{ for all } t \in \mathbb{T},$$

and define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ as $\mu(t) = \sigma(t) - t$. Also let $x^\sigma(t) = x(\sigma(t))$, that is x^σ is the composite function $x \circ \sigma$. The jump operator σ then allows the classification of points in a time scale in the following way: If $\sigma(t) > t$ then call the point t right-scattered; while $\sigma(t) = t$ then call the point t right-dense.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}^n$. Define $x^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t with

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U \text{ and each } i = 1, \dots, n.$$

Call $x^\Delta(t)$ the (delta) derivative of $x(t)$ and say that x is (delta) differentiable.

Definition If $G^\Delta(t) = g(t)$ then define the Cauchy integral by

$$\int_a^t g(s)\Delta s = G(t) - G(a).$$

For a more general definition of the delta integral see [1], [2].

The following theorem is due to Hilger [4].

Theorem 1. *Assume that $g : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}$.*
(i) If g is differentiable at g then g is continuous at t.
(ii) If g is continuous at t and t is right-scattered then g is differentiable at t with

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t}.$$

(iii) If g is differentiable and t is right-dense then

$$g^{\Delta}(t) = \lim_{s\to t} \frac{g(t)-g(s)}{t-s}.$$

(iv) If g is differentiable at t then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.

We assume throughout that $t_0 \geq 0$ and $t_0 \in \mathbb{T}$. By the interval $[t_0, \infty)$ we mean the set $[t_0, \infty) \cap \mathbb{T}$.

Definition Define S to be the set of all functions $x : \mathbb{T} \to \mathbb{R}^n$ such that

$$S = \{x : x \in C([t_0, \infty); \mathbb{R}^n)\}.$$

A solution to (1) is a function $x \in S$ which satisfies (1) for each $t \geq t_0$.

The theory of time scales dates back to Hilger [4]. The monographs [1], [2] and [5] also provide an excellent introduction.

2. Lyapunov Functions

The following Chain Rule shall be very useful throughout the remainder of the paper.

Theorem 2. *Let $p : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $q : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $p \circ q$ is delta differentiable and*

$$[p(q(t))]^{\Delta} = \left(\int_0^1 p'(q(t) + h\mu(t)q^{\Delta}(t))dh \right) q^{\Delta}(t),$$

Proof Keller [6] and Potzsche [7]. See also Bohner and Peterson [1], Theorem 1.90.

Definition Call $V : \mathbb{R}^n \to \mathbb{R}$ a "type I" function when

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \ldots + V_n(x_n),$$

where each $V_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Now assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a type I function and x is a solution to (1). Consider

$$\begin{aligned}
[V(x(t))]^{\Delta} &= \left[\sum_{i=1}^{n} V_i(x_i(t)) \right]^{\Delta} = \sum_{i=1}^{n} [V_i(x_i(t))]^{\Delta}, \\
&= \sum_{i=1}^{n} \int_0^1 V_i'(x_i(t) + h\mu(t)x_i^{\Delta}(t))dh \; x_i^{\Delta}(t), \\
&= \sum_{i=1}^{n} \int_0^1 V_i'(x_i(t) + h\mu(t)f_i(t, x(t)))f_i(t, x(t)) \, dh, \\
&= \int_0^1 \nabla V(x(t) + h\mu(t)f(t, x(t))) \cdot f(t, x(t)) \, dh,
\end{aligned}$$

where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is the gradient operator.

This motivates us to define $\dot{V} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\dot{V}(t, x) = \int_0^1 \nabla V(x + h\mu(t)f(t, x)) \cdot f(t, x)dh. \tag{5}$$

Next we find another formula for $\dot{V}(t,x)$. If $\mu(t)=0$, then we simply get

$$\dot{V}(t,x) = \nabla V(x)\cdot f(t,x).$$

On the other hand if $\mu(t)\neq 0$, then

$$\begin{aligned}
\dot{V}(t,x) &= \int_0^1 \nabla V(x+h\mu(t)f(t,x))\cdot f(t,x)\,dh\\
&= \sum_{i=1}^{n} V_i'(x_i+h\mu(t)f_i(t,x))f_i(t,x)\,dh\\
&= \sum_{i=1}^{n} \frac{1}{\mu(t)}\int_0^1 V_i'(x_i+h\mu(t)f_i(t,x))\mu(t)f_i(t,x)\,dh\\
&= \sum_{i=1}^{n} \frac{V_i(x_i+\mu(t)f_i(t,x))-V_i(x_i)}{\mu(t)}.
\end{aligned}$$

Summarizing, we get that

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^{n}[V_i(x_i+\mu(t)f_i(t,x))-V_i(x_i)]/\mu(t), & \text{when } \mu(t)\neq 0,\\ \nabla V(x)\cdot f(t,x), & \text{when } \mu(t)=0. \end{cases} \tag{6}$$

If, in addition to the above, $V:\mathbb{R}^n\to[0,\infty)$ then we call V a type I Lyapunov function. Sometimes the domain of V will be a subset D of $\mathbb{R}^n$.

Note that $V=V(x)$ and even if the vector field associated with the dynamic equation is autonomous then $\dot{V}$ still depends on t (and x of course) when the graininess function of $\mathbb{T}$ is nonconstant.

Using formulas (5) and (6) we can easily calculate $\dot{V}(t,x)$ for each of the following examples:

Example 1 Let $V(x)=\sum_{i=1}^{n} x_i^{1/2}$, for $x\in D$, where

$$D=\left\{x\in\mathbb{R}^n : x_i>0,\ x_i+\mu(t)f_i(t,x)\geq 0,\ i=1,2,\cdots,n\right\}.$$

Then

$$\dot{V}(t,x)=\sum_{i=1}^{n}\frac{f_i(t,x)}{\sqrt{x_i+\mu(t)f_i(t,x)}+\sqrt{x_i}}.$$

Example 2 Let $V(x)=\sum_{i=1}^{n} a_i x_i^2$, for $x\in\mathbb{R}^n$ and $a_i>0$, $i=1,2,\cdots,n$. For $x\in\mathbb{R}^n$, we define the associated weighted vector by

$$w(x):=(a_1x_1,a_2x_2,\cdots,a_nx_n).$$

Then

$$\dot{V}(t,x)=2w(x)\cdot f(t,x)+\mu(t)w(f(t,x))\cdot f(t,x).$$

In particular, if $V(x)=\|x\|^2=\sum_{i=1}^{n}x_i^2$, then

$$\dot{V}(t,x)=2x\cdot f(t,x)+\mu(t)\|f(t,x)\|^2. \tag{7}$$

Example 3 Let $V(x) = \sum_{i=1}^n a_i x_i^4$ for $x \in \mathbb{R}^n$ and $a_i > 0$, $i = 1, 2, \cdots, n$. Then

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^n a_i[(x_i + \mu(t) f_i(t,x))^4 - x_i^4]/\mu(t), & \text{when } \mu(t) \neq 0, \\ \sum_{i=1}^n 4a_i x_i^3 f_i(t,x), & \text{when } \mu(t) = 0. \end{cases}$$

Example 4 Let $V(x) = \sum_{i=1}^n a_i x_i^{4/3}$ for $x \in \mathbb{R}^n$ and $a_i > 0$, $i = 1, 2, \cdots, n$. Then

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^n a_i[(x_i + \mu(t) f_i(t,x))^{4/3} - x_i^{4/3}]/\mu(t), & \text{when } \mu(t) \neq 0, \\ \sum_{i=1}^n 4a_i x_i^{1/3} f_i(t,x)/3, & \text{when } \mu(t) = 0. \end{cases}$$

Example 5 For Lyapunov functions which may not be power functions, let

$$V(x) = \sum_{i=1}^n \int_0^{x_i} p_i(u) du,$$

where each $p_i : \mathbb{R} \to \mathbb{R}^+$ is continuous. Then

$$\begin{aligned} \dot{V}(t,x) &= \sum_{i=1}^n \int_0^1 p_i(x_i + h\mu(t) f_i(t,x)) f_i(t,x) dh. \\ &= P(t,x) \cdot f(t,x), \end{aligned}$$

where

$$P(t,x) := \left(\int_0^1 p_1(x_1 + h\mu(t) f_1(t,x)) dh, \ldots, \int_0^1 p_n(x_n + h\mu(t) f_n(t,x)) dh \right).$$

Note that if $\mathbb{T} = \mathbb{R}$, then

$$P(t,x) = P(x) = (p_1(x_1), \ldots, p_n(x_n)).$$

3. Boundedness of Solutions

In this section we present some results on the boundedness of solutions to (1), (2).

Definition We say solutions x of the IVP (1), (2) $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ are uniformly bounded provided there is a constant $C = C(x_0)$ which may depend on x_0 but not on t_0 such that

$$\|x(t)\| \leq C$$

for all $t \in [t_0, \infty)$.

First a few more preliminaries.

Definition Assume $g : \mathbb{T} \to \mathbb{R}$. Define and denote $g \in C_{rd}(\mathbb{T}; \mathbb{R})$ as right-dense continuous (rd-continuous) if g is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} g(s)$ exists and is finite, at every left-dense point $t \in \mathbb{T}$.

Now define the so-called set of regressive functions, $\mathcal{R}$, by

$$\mathcal{R} = \{p : \mathbb{T} \to \mathbb{R};\ p \in C_{rd}(\mathbb{T}; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0 \text{ on } \mathbb{T}\}$$

and define the set of positively regressive functions by

$$\mathcal{R}^+ = \{p \in \mathcal{R} :\ 1 + p(t)\mu(t) > 0 \text{ on } \mathbb{T}\}.$$

For $p \in \mathcal{R}$, we define (see Theorem 2.35, [1]) the exponential function $e_p(\cdot, t_0)$ on the time scale $\mathbb{T}$ as the unique solution to the IVP

$$x^\Delta = p(t)x, \quad x(t_0) = x_0.$$

If $p \in \mathcal{R}^+$, then (see Theorem 2.48, [1]) $e_p(t, t_0) > 0$ for $t \in \mathbb{T}$.

We are now ready to present some results.

Theorem 3. *Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$:*

$$V(x) \to \infty, \quad \textit{as} \quad \|x\| \to \infty; \tag{8}$$

$$V(x) \leq \phi(\|x\|); \tag{9}$$

$$\dot{V}(t, x) \leq \frac{\psi(\|x\|) + L}{1 + \mu(t)}; \tag{10}$$

$$\psi(\phi^{-1}(V(x))) + V(x) \leq \gamma; \tag{11}$$

where ϕ, ψ are functions such that $\phi : [0, \infty) \to [0, \infty)$, $\psi : [0, \infty) \to (-\infty, 0]$, ψ is nonincreasing, and ϕ^{-1} exists; L and γ are nonnegative constants. Then all solutions of (1), (2) *that stay in D are uniformly bounded.*

Proof Let x be a solution to (1), (2) that stays in D for all $t \geq t_0 \geq 0$. Consider $V(x(t))e_1(t, t_0)$ ($e_1(t, t_0)$ is the unique solution to the IVP $x^\Delta = x$, $x(t_0) = 1$). Since $p = 1 \in \mathcal{R}^+$, $e_1(t, t_0)$ is well defined and positive on $\mathbb{T}$. Now consider

$$\begin{aligned}
[V(x(t))e_1(t, t_0)]^\Delta &= \dot{V}(t, x(t))e_1^\sigma(t, t_0) + V(x(t))e_1^\Delta(t, t_0), \\
&\leq \frac{(\psi(\|x(t)\|) + L)}{(1 + \mu(t))}(1 + \mu(t))e_1(t, t_0) + V(x(t))e_1(t, t_0), \quad \text{by (10)}, \\
&= (\psi(\|x(t)\|) + L + V(x(t)))e_1(t, t_0), \\
&\leq (\psi(\phi^{-1}(V(x(t))) + V(x(t)) + L)e_1(t, t_0), \quad \text{by (9)}, \\
&\leq (\gamma + L)e_1(t, t_0), \quad \text{by (11)}.
\end{aligned}$$

Integrating both sides from t_0 to t we obtain

$$V(x(t))e_1(t, t_0) \leq V(x_0) + (\gamma + L)(e_1(t, t_0) - 1),$$

where $x_0 = x(t_0)$. It follows that

$$\begin{aligned}
V(x(t)) &\leq V(x_0)/e_1(t, t_0) + (\gamma + L), \\
&\leq V(x_0) + (\gamma + L), \quad \text{for all } t \in [t_0, \infty). \qquad (12)
\end{aligned}$$

Thus by (8),

$$V(x(t)) \leq V(x_0) + \gamma + L \text{ implies that } \|x(t)\| \leq R \text{ for some } R > 0,$$

and R will depend on $V(x_0)$ and γ and L. Hence all solutions of (1), (2) that stay in D are uniformly bounded. This concludes the proof.

We now provide a special case of Theorem 3 for certain functions ϕ and ψ.

Theorem 4. *Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$:*

$$V(x) \to \infty, \quad \text{as} \quad \|x\| \to \infty; \tag{13}$$

$$V(x) \leq \lambda_2 \|x\|^q; \tag{14}$$

$$\dot{V}(t, x) \leq \frac{-\lambda_3 \|x\|^r + L}{1 + M\mu(t)}; \tag{15}$$

$$V(x) - V^{r/q}(x) \leq \gamma; \tag{16}$$

where λ_2, λ_3, q, r are positive constants; L and γ are nonnegative constants, and $M := \lambda_3/\lambda_2^{r/q}$. Then all solutions of (1), (2) *that stay in D are uniformly bounded.*

Proof Note that $M := \lambda_3/\lambda_2^{r/q} \in \mathcal{R}^+$, so $e_M(t, t_0)$ is well defined and positive. Consider

$$[V(x(t))e_M(t, t_0)]^\Delta.$$

Following the steps in the proof of Theorem 3 we obtain

$$V(x(t)) \leq V(x_0) + (\gamma + L)/M, \quad \text{for all } t \in [t_0, \infty), \tag{17}$$

with a bound on solutions following from (13).

Corollary 1. *Assume that the conditions of Theorem 4 are satisfied with* (13) *replaced with*

$$\lambda_1 \|x\|^p \leq V(x), \tag{18}$$

where λ_1 and p are positive constants. Then all solutions of (1), (2) *that stay in D satisfy*

$$\|x(t)\| \leq \lambda_1^{-1/p} \left(V(x_0) + (\gamma + L)/M\right)^{1/p}, \quad \textit{for all } t \in [t_0, \infty). \tag{19}$$

Proof Let x be a solution of (1), (2) that stays in D. Then (17) and (18) imply that

$$\lambda_1 \|x(t)\|^p \leq V(x(t)) \leq V(x_0) + (\gamma + L)/M,$$

and (19) follows. This concludes the proof.

Theorem 5. *Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0, \infty)$ such that for all $(t, x) \in [0, \infty) \times D$:*

$$V(x) \to \infty, \quad \text{as} \quad \|x\| \to \infty; \tag{20}$$

$$\dot{V}(t, x) \leq \frac{-\lambda_3 V(x) + L}{1 + \lambda_3 \mu(t)}; \tag{21}$$

where $\lambda_3 > 0$ and $L \geq 0$ are constants. Then all solutions of (1), (2) *that stay in D are uniformly bounded.*

Proof Let x be a solution to (1), (2) that stays in D for all $t \in [t_0, \infty)$. Since $\lambda_3 \in \mathcal{R}^+$, $e_{\lambda_3}(t, t_0)$ is well defined and positive. Now consider

$$\begin{aligned}
[V(x(t))e_{\lambda_3}(t, t_0)]^\Delta &= \dot{V}(t, x(t))e^\sigma_{\lambda_3}(t, t_0) + V(x(t))\lambda_3 e_{\lambda_3}(t, t_0), \\
&= \dot{V}(t, x(t))(1 + \lambda_3\mu(t))e_{\lambda_3}(t, t_0) + V(x(t))\lambda_3 e_{\lambda_3}(t, t_0), \\
&\leq (-\lambda_3 V(x(t)) + L)e_{\lambda_3}(t, t_0) + V(x(t))\lambda_3 e_{\lambda_3}(t, t_0), \quad \text{by (21)}, \\
&= L e_{\lambda_3}(t, t_0).
\end{aligned}$$

Integrating both sides from t_0 to t we obtain

$$V(x(t))e_{\lambda_3}(t,t_0) \leq V(x_0) + Le_{\lambda_3}(t,t_0)/\lambda_3,$$

and therefore

$$\begin{aligned} V(x(t)) &\leq V(x_0)/e_{\lambda_3}(t,t_0) + L/\lambda_3, \\ &\leq V(x_0) + L/\lambda_3. \end{aligned} \tag{22}$$

Thus by (20),

$$V(x(t)) \leq V(x_0) + L/\lambda_3 \text{ implies that } \|x(t)\| \leq R \text{ for some } R > 0,$$

and R will depend on $V(x_0)$, L and λ_3. Hence all solutions of (1), (2) that stay in D are uniformly bounded. This concludes the proof.

Corollary 2. *Assume that the conditions of Theorem 5 are satisfied with (20) replaced with*

$$\lambda_1\|x\|^p \leq V(x), \tag{23}$$

where λ_1 and p are positive constants. Then all solutions of (1), (2) *that stay in D satisfy*

$$\|x(t)\| \leq \lambda_1^{-1/p}\left(V(x_0) + L/\lambda_2\right)^{1/p}. \tag{24}$$

Proof Let x be a solution of (1), (2) that stays in D. Then (22) and (23) imply that

$$\lambda_1\|x(t)\|^p \leq V(x(t)) \leq V(x_0) + L/\lambda_2,$$

and (24) follows. This concludes the proof.

4. Examples

We now present some examples to illustrate the theory developed in Section 3.

Example Consider the IVP

$$x^\Delta = ax + bx^{1/3}, \quad x(t_0) = x_0, \tag{25}$$

where a, b are constants, $x_0 \in \mathbb{R}^n$, and $t_0 \in [0,\infty)$. If there is a constant $\lambda_3 > 0$ such that

$$(2a + a^2\mu(t) + \frac{4}{3} + \frac{1}{3}\mu(t))(1 + \lambda_3\mu(t)) \leq -\lambda_3, \tag{26}$$

and

$$\left(|b + ab\mu(t)|^3 + \mu(t)|b|^3\right)(1 + \lambda_3\mu(t)) \leq M,$$

for some constant $M \geq 0$ and all $t \in [0,\infty)$, then all solutions to (25) are uniformly bounded.

Proof We shall show that under the above assumptions, the conditions of Theorem 4 are satisfied. Choose $D = \mathbb{R}$ and $V(x) = x^2$ so $q = 2$, $\lambda_2 = 1$ and (13) holds. Now from (7)

$$\begin{aligned}
&\dot{V}(t,x)\\
=\;& 2x\cdot f(t,x)+\mu(t)\|f(t,x)\|^2,\\
=\;& 2x(ax+bx^{1/3})+\mu(t)(ax+bx^{1/3})^2,\\
=\;& (2a+a^2\mu(t))x^2+2(b+ab\mu(t))x^{4/3}+b^2\mu(t)x^{2/3},\\
\le\;& (2a+a^2\mu(t))x^2+2\left[\frac{(x^{4/3})^{3/2}}{3/2}+\frac{|b+ab\mu(t)|^3}{3}\right]+\mu(t)\left[\frac{(x^{2/3})^3}{3}+\frac{(b^2)^{\frac{3}{2}}}{\frac{3}{2}}\right],\\
\le\;& (2a+a^2\mu(t)+\frac{4}{3}+\frac{\mu(t)}{3})x^2+\frac{2}{3}|b+ab\mu(t)|^3+\frac{2}{3}\mu(t)|b|^3,
\end{aligned}$$

where we have made use of Young's inequality twice. Dividing and multiplying the right hand side by $(1+\lambda_3\mu(t))$ we see that (15) holds under the above assumptions with $r=2$ and $\gamma=0$. Therefore all the conditions of Theorem 4 are satisfied and we conclude that all solutions to (25) are uniformly bounded.

Case 1: If $\mathbb{T}=\mathbb{R}$ (Raffoul, [8]) then $\mu(t)=0$ and (26) reduces to $2a+4/3\le-\lambda_3$. If $a\le-2/3$ then we take $\lambda_3=-(2a+4/3)>0$ and we can choose $M=|b|^3$, concluding that all solutions to (25) are uniformly bounded.

Case 2: If $\mathbb{T}=h\mathbb{N}_0=\{0,h,2h,\ldots\}$ then $\mu(t)=h$ and (26) reduces to

$$(2a+a^2h+4/3+h/3)\le-\lambda_3/(1+\lambda_3h).$$

Therefore we want to find those $h>0$ such that

$$ha^2+2a+(4+h)/3<0.$$

Now the polynomial

$$p(a):=ha^2+2a+(4+h)/3,$$

will have distinct real roots

$$a_1(h)=(-\sqrt{3}-\sqrt{3-4h-h^2})/(\sqrt{3}h)$$
$$a_2(h)=(-\sqrt{3}+\sqrt{3-4h-h^2})/(\sqrt{3}h)$$

if $0<h<\sqrt{7}-2$. Therefore if $0<h<\sqrt{7}-2$ and $a_1(h)<a<a_2(h)$, then

$$A:=ha^2+2a+(4+h)/3<0.$$

Now, for such an h, let λ_3 be defined by

$$-\lambda_3(1+\lambda_3h)=A<0,$$

that is

$$\lambda_3:=-A/(1+hA).$$

Therefore if $0<h<\sqrt{7}-2$ then for $a_1(h)<a<a_2(h)$ all solutions are uniformly bounded by Theorem 4.

Remark 1. *It is interesting to note that*

$$\lim_{h\to0+}a_2(h)=\lim_{h\to0+}(-\sqrt{3}+\sqrt{3-4h-h^2})/(\sqrt{3}h)=-2/3,$$

and

$$\lim_{h\to0+}a_1(h)=\lim_{h\to0+}(-\sqrt{3}-\sqrt{3-4h-h^2})/(\sqrt{3}h)=-\infty,$$

recalling that if $\mathbb{T} = \mathbb{R}$ *then for* $-\infty < a \le -2/3$ *then all solutions are uniformly bounded.*

Example Consider the the following system of IVPs for $t \ge t_0 \ge 0$

$$x_1^\Delta = -ax_1 + ax_2, \tag{27}$$

$$x_2^\Delta = -ax_1 - ax_2, \tag{28}$$

$$(x_1(t_0), x_2(t_0)) = (c, d), \tag{29}$$

for certain constants $a > 0$; c and d. If there is a constant $\lambda_3 > 0$ such that for all $t \in [0, \infty)$

$$\lambda_3/(1 + \lambda_3\mu(t)) \le 2a(1 - a\mu(t)), \tag{30}$$

then all solutions to (27) - (29) are uniformly bounded.
Proof We will show that, under the above assumptions, the conditions of Theorem 5 are satisfied. Choose $D = \mathbb{R}^2$ and $V(x) = \|x\|^2 = x_1^2 + x_2^2$ so (20) holds. From (7) we see that

$$\begin{aligned} \dot{V}(t,x) &= 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2, \\ &= -2a(1 - a\mu(t))\|x\|^2, \\ &\le -\lambda_3\|x\|^2/(1 + \lambda_3\mu(t)), \quad \text{by (30)}, \\ &= -\lambda_3 V(x)/(1 + \lambda_3\mu(t)). \end{aligned}$$

Hence (21) holds under the above assumptions with $L = 0$. Therefore all the conditions of Theorem 5 are satisfied and we conclude that all solutions to (27) - (29) are uniformly bounded.

In fact, if there is a constant K such that

$$0 \le a\mu(t) \le K < 1 \tag{31}$$

for all $t \in [0, \infty)$ then (30) will hold.
Case 1: If $\mathbb{T} = \mathbb{R}$ then $\mu(t) = 0$ and (31) will hold for any $0 \le K < 1$ which, in turn, will make (30) hold and we conclude that all solutions are uniformly bounded.
Case 2: If $\mathbb{T} = \{H_n\}_0^\infty$ defined by

$$H_0 = 0, \quad H_n = \sum_{r=1}^{n} 1/r, \quad n \in \mathbb{N},$$

then $\mu(t) = 1/(n+1)$ and (31) will hold when $a < 1$ which, in turn, will make (30) hold and we conclude that all solutions are uniformly bounded.
Case 3: If $\mathbb{T} = h\mathbb{N}_0$ then $\mu(t) = h$ and (31) will hold when $ah < 1$ which, in turn, will make (30) hold and we conclude that all solutions are uniformly bounded.

Remark 2. *By using standard methods* [1], *the system (27) - (29) has solutions*

$$\begin{aligned} x_1(t) &= c_1 e_{-a+ia}(t,t_0) + c_2 e_{-a-ia}(t,t_0), \\ x_2(t) &= ac_1(-1+i)e_{-a+ia}(t,t_0) - ac_2(1+i)e_{-a-ia}(t,t_0), \end{aligned}$$

and for $\mathbb{T} = h\mathbb{N}_0$ *we see by closely investigating these exponentials that when* $h < 1/a$ *all solutions are uniformly bounded and when* $h > 1/a$ *all nontrivial solutions are unbounded.*

It is interesting to note that even though the eigenvalues of the coefficient matrix

$$A = \begin{pmatrix} -a & a \\ -a & -a \end{pmatrix}, \tag{32}$$

are complex with negative real parts, our system is not stable when $ah > 1$.

5. Uniqueness of Solutions

In this brief section we present a result on the uniqueness of solutions of the IVP (1), (2).

Theorem 6. *Assume that* f *satisfies*

$$(x_2 - x_1) \cdot (f(t, x_2) - f(t, x_1)) + \mu(t)\|f(t, x_2) - f(t, x_1)\|^2 \leq 0,$$

for $t \in [t_0, \infty)$ *and* x_1, $x_2 \in \mathbb{R}^n$. *Then there is, at most, one solution to the IVP* (1), (2).

Proof Let x_1, x_2 be two solutions to (1), (2) and let

$$x(t) = x_2(t) - x_1(t).$$

Choose $V(x) = \|x\|^2$ and note that

$$\begin{aligned} [V(x(t))]^{\Delta} &= 2x(t) \cdot x^{\Delta}(t) + \mu(t)\|x^{\Delta}(t)\|^2, \\ &= (x_2(t) - x_1(t)) \cdot (f(t, x_2(t)) - f(t, x_1(t))) \\ &\quad + \mu(t)\|f(t, x_2(t)) - f(t, x_1(t))\|^2 \leq 0. \end{aligned}$$

Hence $V(x(t))$ is nonincreasing and since $V(x(t_0)) = V(0) = 0$ we conclude that V is identically equal to 0 along $x(t)$. This implies that $x(t) = x_2(t) - x_1(t) = 0$ for $t \in [t_0, \infty)$ and solutions of (1), (2) are unique and this concludes the proof.

Acknowledgement This work was undertaken while the first author was visiting UNSW. The first author would like to thank the NSF for their support under NSF Grant 0072505. The second author gratefully acknowledges the support of the Faculty Research Grants Program, UNSW.

References

1. Martin Bohner and Allan Peterson, Dynamic Equations on Time Scales, Birkhäuser, Boston, 2001.
2. Martin Bohner and Allan Peterson, (Eds), Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
3. P. Hartman, Ordinary Differential Equations, Classics in Applied Mathematics, 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
4. Stefan Hilger, Analysis on Measure Chains - A Unified Approach to Continuous and Discrete Calculus, Res. Math., 18 (1990), 18– 56.
5. B. Kaymakçalan, V. Lakshmikantham and S. Sivasudaram, Dynamical Systems on Measure Chains, Kluwer Academic Publishers, Boston, 1996.
6. S. Keller, Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen, PhD thesis, Universität Augsburg, 1999.
7. C. Pötzsche, Chain Rule and Invariance Principle on Measure Chains, Dynamic Equations on Time Scales, J. Comput. Appl. Math. 141 (2002), 249–254.
8. Youseff N. Raffoul, Boundedness in Nonlinear Differential Equations, (to appear).