Oscillation criteria for a forced second order nonlinear dynamic equation

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This paper is dedicated to Gerry Ladas.

Abstract. In this paper, we will establish some new interval oscillation criteria for forced second-order nonlinear dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)|x^{\sigma}(t)|^{\gamma}\text{sgn } x^{\sigma}(t) = f(t), \quad t \in [a, b],$$

on a time scale $\mathbb{T}$ where $\gamma \geq 1$. As a special case when $\mathbb{T} = \mathbb{R}$ our results not only include the oscillation results for second-order differential equations established by Wong (J. Math. Anal. Appl., 231 (1999) 233-240) and Nasr (Proc. Amer. math. Soc., 126 (1998) 123-125) but also improve these results. When $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = h\mathbb{N}$ or $\mathbb{T} = q^{N}$, i.e., for difference equations, generalized difference equations or q-difference equations our results are essentially new and also can be applied on different types of time scales, as illustrated in several examples.
Keywords and Phrases: Oscillation, Second order dynamic equations, Forced Emden-Fowler type, time scale.
2000 AMS Subject Classification: 34K11, 34C10, 39A13.

1. Introduction

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on time scales which seeks to harmonize the oscillation of the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both. For the convenience we refer the reader to the results given [**2**]-[**4**], [**8**]-[**18**], [**22**], [**24**]-[**25**], [**30**]-[**31**]. In this paper, we are concerned with oscillation of the second-order nonlinear dynamic equation of Emden-Fowler type with forcing term

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)|x^{\sigma}(t)|^{\gamma}\text{sgn } x^{\sigma}(t) = f(t) \tag{1.1}$$

on a time scale $\mathbb{T}$, where $\gamma \geq 1$, $p(t) > 0$, $q(t)$, and $f(t)$ are $rd-$continuous functions.

A special case of (1.2) is the formally self-adjoint linear dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)x^{\sigma} = f(t) \tag{1.2}$$

which we will study in Section 2.

A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many potential applications, among them the study of population dynamic models (see [**7**]).

A number of authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [**1**] and the references cited therein. Two books on the subject of time scales by Bohner and Peterson [**6**], [**7**] summarize and organize much of the time scale calculus.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1), we mean a nontrivial real–valued function $x \in C^2_{rd}[t_0, \infty)$ which satisfies equation (1.1) for $t \geq t_0$. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy

$$\sup\{|x(t)| : t > t_1\} > 0,$$

for any $t_1 \geq t_x$. We say that a solution x of (1.1) has a generalized zero at t if $x(t) = 0$, and has a generalized zero in $(t, \sigma(t))$ in case $x(t)x^{\sigma}(t) < 0$ and $\mu(t) > 0$. Equation (1.1) is disconjugate on the interval $[a,\ b]_{\mathbb{T}}$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[a,\ b]_{\mathbb{T}}$. Equation (1.1) is said to be nonosillatory on $[a,\ \infty)_{\mathbb{T}}$ if there exists $c \in [a,\ \infty)_{\mathbb{T}}$ such that this equation is disconjugate on $[c,\ d]_{\mathbb{T}}$ for every $d > c$. In the opposite case (1.1) is said to be oscillatory on $[a,\ \infty)_{\mathbb{T}}$. The oscillation of solutions of equation (1.1) may equivalently be defined as follows: A nontrivial solution $x(t)$ of (1.1) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[a,\ \infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. We illustrate these comments by means of several examples.

By the Sturm Separation Theorem, we see that oscillation is an interval property, i.e., if there exists a sequence of subintervals $[a_i,\ b_i]_{\mathbb{T}}$ of $[t_0,\ \infty)_{\mathbb{T}}$, as $i \to \infty$, such that for every i there exists a solution of (1.1) that has at least two generalized zeros in $[a_i,\ \sigma(b_i)]_{\mathbb{T}}$, then every solution of (1.1) is oscillatory in $[t_0,\ \infty)_{\mathbb{T}}$. Hence we can speak about oscillation and nonoscillation of (1.1).

In this paper, we intend to use the Riccati substitution to obtain some interval oscillation criteria for (1.1). Our results do not require that q be of definite sign. To the best of our knowledge nothing is known regarding the qualitative behavior of (1.1) on time scales until now, so our results initiates the study. In the case, when $\mathbb{T} = \mathbb{R}$ and $\gamma = 1$ our results reduce to the oscillation results of (1.2) established by Wong [**31**] and also improve the results established by Nasr [**21**] when $p(t) \equiv 1$, $\mathbb{T} = \mathbb{R}$ and $\gamma > 1$. The oscillation results for the equations (1.1)-(1.2) are essentially new and also can be applied on different types of time scales.

2. Main Results

In this section, by using the Riccati substitution we will establish some interval oscillation criteria, that is, criteria given by the behavior of p and q only on a sequence of subintervals of $[t_0, \infty)_{\mathbb{T}}$.

THEOREM 2.1. *Assume that given any $T \in [a, \infty)_{\mathbb{T}}$, there exists points $T \leq s_1 < t_1 \leq s_2 < t_2$ such that*

$$f(t) \begin{cases} \leq 0, & t \in [s_1, t_1]_{\mathbb{T}} \\ \geq 0, & t \in [s_2, t_2]_{\mathbb{T}}. \end{cases} \tag{2.1}$$

Further assume there exists a function $u \in C^1_{rd}(\mathbb{T})$ such that for $i = 1, 2$ we have

$$Q_i[u] := \int_{s_i}^{t_i} \left[p(t)(u^{\Delta}(t))^2 - q(t)(u^{\sigma}(t))^2\right] \Delta t$$

satisfies $Q_i[u] \leq 0$, and $u(t) \not\equiv 0$ on $[s_i, t_i]_{\mathbb{T}}$, with $u(s_i) = 0 = u(t_i)$. Then the dynamic equation (1.2) is oscillatory on $[a, \infty)_{\mathbb{T}}$.

PROOF. Assume (1.2) is nonoscillatory on $[a, \infty)_{\mathbb{T}}$. Then there is a solution x of (1.2) and a $T \in [a, \infty)_{\mathbb{T}}$ such that $x(t)$ is of one sign on $[T, \infty)_{\mathbb{T}}$. We consider the case $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. We make the Riccati substitution

$$z(t) = \frac{p(t)x^{\Delta}(t)}{x(t)}, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Then

$$\begin{aligned} z^{\Delta}(t) &= \frac{x(t)(p(t)x^{\Delta}(t))^{\Delta} - p(t)(x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)} \\ &= \frac{x(t)(f(t) - q(t)x^{\sigma}(t)) - p(t)(x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)} \\ &= -q(t) + \frac{f(t)}{x^{\sigma}(t)} - \frac{x(t)}{p(t)x^{\sigma}(t)}z^2(t). \end{aligned} \tag{2.2}$$

But

$$\begin{aligned} 0 &< \frac{x(t)}{p(t)x^{\sigma}(t)} \\ &= \frac{x(t)}{p(t)[x(t) + \mu(t)x^{\Delta}(t)]} \\ &= \frac{1}{p(t) + \mu(t)\frac{p(t)x^{\Delta}(t)}{x(t)}} \\ &= \frac{1}{p(t) + \mu(t)z(t)}. \end{aligned} \tag{2.3}$$

Hence we get from (2.2) and (2.3) that $z(t)$ satisfies the equation

$$z^{\Delta} = -q + \frac{f}{y^{\sigma}} - \frac{z^2}{p + \mu z} \tag{2.4}$$

for $t \in [T, \infty)_{\mathbb{T}}$ and, by (2.3),

$$p(t) + \mu(t)z(t) > 0, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Let $T < s_1 < t_1 < s_2 < t_2$ and $u(t)$ be as in the statement of this theorem. Since $f(t) \leq 0$ and $y(t) > 0$ on $[s_1, t_1]_{\mathbb{T}}$, z solves the inequality

$$z^{\Delta} \leq -q - \frac{z^2}{p + \mu z}, \quad t \in [s_1, t_1]_{\mathbb{T}}.$$

Multiplying by $(u^{\sigma}(t))^2$ and integrating we have (suppressing arguments)

$$\int_{s_1}^{t_1} (u^{\sigma})^2 z^{\Delta} \Delta t \leq -\int_{s_1}^{t_1} q(u^{\sigma})^2 \Delta t - \int_{s_1}^{t_1} \frac{z^2 (u^{\sigma})^2}{p + \mu z} \Delta t.$$

Using integration by parts ([**6**, Theorem 1.77 v]) on the first integral we get

$$\begin{aligned} & u^2(t) z(t)]_{s_1}^{t_1} - \int_{s_1}^{t_1} [u(t) + u^{\sigma}(t)] u^{\Delta}(t) z(t) \Delta t \\ \leq \; & -\int_{s_1}^{t_1} q(t)(u^{\sigma}(t))^2 \Delta t - \int_{s_1}^{t_1} \frac{z^2(t)(u^{\sigma}(t))^2}{p(t) + \mu(t) z(t)} \Delta t. \end{aligned}$$

Rearranging and using $u(s_1) = 0 = u(t_1)$ we get

$$\begin{aligned} 0 \;\geq\; & \int_{s_1}^{t_1} \frac{z^2 (u^{\sigma})^2}{p + \mu z} \Delta t \\ - & \int_{s_1}^{t_1} [u + u^{\sigma}] u^{\Delta} z \Delta t + \int_{s_1}^{t_1} q(u^{\sigma})^2 \Delta t \\ = & \int_{s_1}^{t_1} \frac{z^2 (u^{\sigma})^2}{p + \mu z} \Delta t \\ - & \int_{s_1}^{t_1} [2u^{\sigma} u^{\Delta} z - \mu (u^{\Delta})^2 z] \Delta t + \int_{s_1}^{t_1} q(u^{\sigma})^2 \Delta t. \end{aligned}$$

Adding and subtracting the term $\int_{s_1}^{t_1} p(u^{\Delta})^2 \Delta t$, we obtain

$$\begin{aligned} 0 \;\geq\; & \int_{s_1}^{t_1} \left(\frac{z^2 (u^{\sigma})^2}{p + \mu z} - 2u^{\sigma} u^{\Delta} z + [p + \mu z](u^{\Delta})^2 \right) \Delta t \\ & - \int_{s_1}^{t_1} [p(u^{\Delta})^2 - q(u^{\sigma})^2] \Delta t \\ = & \int_{s_1}^{t_1} \left(\frac{z u^{\sigma}}{\sqrt{p + \mu z}} - \sqrt{p + \mu z}\, u^{\Delta} \right)^2 \Delta t - Q_1[u] \\ \geq & \int_{s_1}^{t_1} \left(\frac{z u^{\sigma}}{\sqrt{p + \mu z}} - \sqrt{p + \mu z}\, u^{\Delta} \right)^2 \Delta t, \end{aligned}$$

since $Q_1[u] \leq 0$. It follows that

$$\int_{s_1}^{t_1} \left(\frac{z u^{\sigma}}{\sqrt{p + \mu z}} - \sqrt{p + \mu z}\, u^{\Delta} \right)^2 \Delta t = 0.$$

This implies that

$$\frac{z(t) u^{\sigma}(t)}{\sqrt{p(t) + \mu(t) z(t)}} - \sqrt{p(t) + \mu(t) z(t)}\, u^{\Delta}(t) = 0$$

for $t \in [s_1, t_1)_{\mathbb{T}}$. Solving for u^{Δ}, we get that u solves the IVP

$$u^{\Delta} = \frac{z(t)}{p(t) + \mu(t)z(t)} u^{\sigma}, \quad u(s_1) = 0$$

for $t \in [s_1, t_1)_{\mathbb{T}}$. But this implies that $u(t) \equiv 0$ on $[s_1, t_1]_{\mathbb{T}}$, which is a contradiction. The case $x(t) < 0$ on $[T, \infty)_{\mathbb{T}}$ is similar, where we use $f(t) \geq 0$ on $[s_2, t_2]_{\mathbb{T}}$ and $Q_2[u] \leq 0$. □

The following lemma will be used in the proof of our next main result. We let $\mathbb{D}$ denote the set of all $u : \mathbb{T} \to \mathbb{R}$ such that u is differentiable and $(pu)^{\Delta} \in C^1_{rd}(\mathbb{T})$.

Lemma 2.2. *Assume $u, v \in \mathbb{D}$ and $v = uw$. Then*

$$(puu^{\sigma} w^{\Delta})^{\Delta} + (Lu)v^{\sigma} = u^{\sigma}(Lv).$$

Proof. Let $u, v \in \mathbb{D}$ and let $v = uw$. By the product rule,

$$v^{\Delta} = u^{\sigma} w^{\Delta} + u^{\Delta} w,$$

which implies that

$$u^{\sigma} w^{\Delta} = v^{\Delta} - u^{\Delta} w.$$

Multiplying both sides by pu we get

$$puu^{\sigma} w^{\Delta} = puv^{\Delta} - puwu^{\Delta} = u(pv^{\Delta}) - v(pu^{\Delta}).$$

Differentiating both sides leads to

$$\begin{aligned}
(puu^{\sigma} w^{\Delta})^{\Delta} &= u^{\sigma}(pv^{\Delta})^{\Delta} + u^{\Delta} pv^{\Delta} - v^{\sigma}(pu^{\Delta})^{\Delta} - v^{\Delta} pu^{\Delta} \\
&= u^{\sigma}(pv^{\Delta})^{\Delta} - v^{\sigma}(pu^{\Delta})^{\Delta} \\
&= u^{\sigma}[(pv^{\Delta})^{\Delta} + qv^{\sigma}] - v^{\sigma}[(pu^{\Delta})^{\Delta} + qu^{\sigma}] \\
&= u^{\sigma}(Lv) - v^{\sigma}(Lu)
\end{aligned}$$

which is the desired result. □

By Theorem 4.61 in [**6**] if $Lu = 0$ is nonoscillatory on $[a, \infty)_{\mathbb{T}}$, then there exists a solution $u(t)$ of $Lu = 0$, called a dominant solution at ∞, with the property that there is a $t_0 \in [a, \infty)_{\mathbb{T}}$ such that $u(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$ and

$$\int_{t_0}^{\infty} \frac{1}{p(t)u(t)u^{\sigma}(t)} \Delta t < \infty.$$

If $u(t)$ is a dominant solution of Lu=0, then we define

$$H(t) := \int_{t_0}^{t} \frac{1}{p(s)u(s)u^{\sigma}(s)} \int_{t_0}^{s} u^{\sigma}(\tau) f(\tau) \Delta\tau \Delta s.$$

These comments will be used in the following theorem.

Theorem 2.3. *Assume $Lu = 0$ is nonoscillatory on $[a, \infty)_{\mathbb{T}}$ and $u(t)$ is a dominant solution of $Lu = 0$ at ∞. If*

$$\limsup_{t \to \infty} H(t) = \infty = -\liminf_{t \to \infty} H(t),$$

then the forced equation (1.2) *is oscillatory.*

PROOF. Let $u(t)$ be a dominant solution of $Lu = 0$ at ∞, let $y(t)$ be a solution of the forced equation (1.2), and let

$$y(t) = u(t)w(t).$$

By Lemma 2.2

$$(puu^\sigma w^\Delta)^\Delta + (Lu)y^\sigma = u^\sigma(Ly),$$

which simplifies to

$$(puu^\sigma w^\Delta)^\Delta = u^\sigma f.$$

After two integrations we obtain

$$w(t) = A + B\int_{t_0}^t \frac{1}{p(s)u(s)u^\sigma(s)}\Delta s + H(t).$$

It follows from this that

$$\limsup_{t\to\infty} w(t) = \infty = -\liminf_{t\to\infty} w(t).$$

Hence $w(t)$ is oscillatory and this implies that $y(t)$ is oscillatory. □

3. The Nonlinear Case

In this section we consider the nonlinear forced dynamic equation (1.2) when $\gamma > 1$. In addition, to our earlier assumptions, we assume throughout this section that

$$p(t) > 0, \quad q(t) > 0, \quad \text{on} \quad [a,\infty)_{\mathbb{T}}.$$

We shall be interested in establishing interval oscillation criteria similar to those in the previous section but for the nonlinear case. We first establish a lemma which will be used in the proof of the main result in this section (Theorem 3.2). We introduce the auxiliary functions

$$\hat{q}(t) := h(\gamma)(q(t))^{1/\gamma}|f(t)|^{1-\frac{1}{\gamma}}, \quad \text{where} \quad h(\gamma) := \frac{\gamma}{(\gamma-1)^{1-1/\gamma}}.$$

LEMMA 3.1. *If $a > 0$, $b > 0$, and $\gamma > 1$, then $f(y) := ay^{1-\gamma} + \frac{b}{y}$ satisfies*

$$f(y) \geq h(\gamma)a^{1/\gamma}b^{1-1/\gamma}, \quad y > 0.$$

PROOF. This can be established by calculus methods or as an application of Young's inequality (cf. Beckenbach and Bellman [**5**, p. 15]), which says that if $\alpha, \beta \geq 0$ and $\gamma > 1$, then

$$\alpha^{1/\gamma}\beta^{(1-1/\gamma)} \leq \frac{\alpha}{\gamma} + \frac{\beta}{\gamma/(1-\gamma)}.$$

Thus if we set

$$\alpha = \gamma a y^{\gamma-1} = \left(\gamma^{1/\gamma}a^{1/\gamma}y^{(\gamma-1)/\gamma}\right)^\gamma$$

and

$$\beta = \frac{\gamma}{\gamma-1}\frac{b}{y} = \left[\left(\frac{\gamma}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{b}{y}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\gamma/(\gamma-1)},$$

then we get

$$\alpha^{1/\gamma}\beta^{(1-\gamma)/\gamma} = \gamma^{1/\gamma}\left(\frac{\gamma}{\gamma-1}\right)^{(\gamma-1)/\gamma}\alpha^{1/\gamma}b^{(\gamma-1)/\gamma} = h(\gamma)\alpha^{1/\gamma}b^{(\gamma-1)/\gamma}.$$

□

THEOREM 3.2. *Assume for any $T \in [a,\infty)_{\mathbb{T}}$ there exist points $T \le s_1 < t_1 \le s_2 < t_2$ in the time scale such that (2.1) holds and there exists a $u \in C^1_{rd}$ such that for $i = 1, 2$, $u(s_i) = 0 = u(t_i)$ and*

$$Q_i(u) := \int_{s_i}^{t_i} [p(t)(u^\Delta(t))^2 - \hat{q}(t)(u^\sigma(t))^2]\Delta t,$$

satisfies $Q_i(u) \le 0$ and $u(t) \not\equiv 0$ on $[s_i, t_i]$. Then (1.2) is oscillatory.

PROOF. Assume there is a nonoscillatory solution y which we assume satisfies $y(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. We then make the Riccati substitution

$$z(t) := -\frac{p(t)y^\Delta(t)}{y(t)}, \quad t \in [T,\infty)_{\mathbb{T}}.$$

Therefore,

$$\begin{aligned} z^\Delta &= \frac{y(-py^\Delta)^\Delta + p(y^\Delta)^2}{yy^\sigma} \\ &= \frac{q(y^\sigma)^\gamma - f}{y^\sigma} + \frac{p(y^\Delta)^2}{yy^\sigma} \\ &= q(y^\sigma)^{\gamma-1} - \frac{f}{y^\sigma} + \frac{z^2 y}{py^\sigma} \\ &= q(y^\sigma)^{\gamma-1} - \frac{f}{y^\sigma} + \frac{z^2 y}{p(y+\mu y^\Delta)} \\ &= q(y^\sigma)^{\gamma-1} - \frac{f}{y^\sigma} + \frac{z^2}{p(1+\frac{\mu y^\Delta}{y})} \\ &= q(y^\sigma)^{\gamma-1} - \frac{f}{y^\sigma} + \frac{z^2}{p-\mu z}. \end{aligned}$$

Since

$$0 < \frac{y}{py^\sigma} = \frac{y}{p(y+\mu y^\Delta)} = \frac{1}{p-\mu z}$$

we see that $p - \mu z > 0$. This implies

$$z^\Delta = q(y^\sigma)^{\gamma-1} - \frac{f}{y^\sigma} + \frac{z^2}{p-\mu z}.$$

Now on $[s_1, t_1]_{\mathbb{T}}$, $f(t) \le 0$, so $-f = |f|$ and thus we have

$$z^\Delta = q(y^\sigma)^{\gamma-1} + \frac{|f|}{y^\sigma} + \frac{z^2}{p-\mu z}.$$

Now from Lemma 3.1 we have for $t \in [s_1, t_1]_{\mathbb{T}}$

$$q(t)y^{\gamma-1}(t) + \frac{|f(t)|}{y(t)} \ge \hat{q}(t)$$

and so we get

$$z^\Delta \ge \hat{q}(t) + \frac{z^2}{p-\mu z}.$$

Now multiply by $(u^\sigma)^2$, where $u \in \mathbb{D}$, and integrate to get

$$\int_{s_1}^{t_1} (u^\sigma)^2 z^\Delta \,\Delta t \ge \int_{s_1}^{t_1} \hat{q}(u^\sigma)^2 \Delta t + \int_{s_1}^{t_1} \frac{z^2(u^\sigma)^2}{p-\mu z}\Delta t.$$

The rest of the argument proceeds as in Theorem 2.1 (using $u(s_1) = u(t_1) = 0$) to get

$$\begin{aligned} 0 &\geq \int_{s_1}^{t_1} \left[\frac{zu^\sigma}{\sqrt{p-\mu z}} + \sqrt{p-\mu z}\, u^\Delta\right]^2 \Delta t \\ &+ \int_{s_1}^{t_1} \hat{q}(u^\sigma)^2 \Delta t - \int_{s_1}^{t_1} p(u^\Delta)^2 \Delta t \\ &\geq -Q_1(u) = \int_{s_1}^{t_1} [p(u^\Delta)^2 - \hat{q}(u^\sigma)^2] \Delta t \geq 0, \end{aligned}$$

which leads to a contradiction as in Theorem 2.1. □

4. Examples

EXAMPLE 4.1. Let $\mathbb{T} = \{t_0, t_1, t_2, \cdots\}$, where $t_0 < t_1 < t_2 < \cdots$ and $\lim_{n\to\infty} t_n = \infty$. Define

$$f(t) = \begin{cases} 0\ , & t = t_{2n}, \quad n \in \mathbb{N}_0 \\ (-1)^n, & t = t_{2n+1},\ n \in \mathbb{N}_0. \end{cases}$$

Let $u(t) = f(t)$, $t \in \mathbb{T}$, and first note that

$$f(t) = \begin{cases} \geq 0, & t \in [t_{4n}, t_{4n+2}]_{\mathbb{T}}, \quad n \in \mathbb{N}_0 \\ \leq 0, & t \in [t_{4n+2}, t_{4n+4}]_{\mathbb{T}}, \quad n \in \mathbb{N}_0. \end{cases}$$

Furthermore,

$$\begin{aligned} Q[u] &:= \int_{t_{2n}}^{t_{2n+2}} \{p(t)(u^\Delta(t))^2 - q(t)(u^\sigma(t))^2\} \Delta t \\ &= \left\{p(t_{2n})(u^\Delta(t_{2n}))^2 - q(t_{2n})u^2(t_{2n+1})\right\} \mu(t_{2n}) \\ &+ \left\{p(t_{2n+1})(u^\Delta(t_{2n+1}))^2 - q(t_{2n+1})u^2(t_{2n+2})\right\} \mu(t_{2n+1}) \\ &= \left\{\frac{p(t_{2n})}{\mu^2(t_{2n})} - q(t_{2n})\right\} \mu(t_{2n}) + \frac{p(t_{2n+1})}{\mu(t_{2n+1})} \\ &= \frac{p(t_{2n})}{\mu(t_{2n})} - \mu(t_{2n})q(t_{2n}) + \frac{p(t_{2n+1})}{\mu(t_{2n+1})}. \end{aligned}$$

Hence we see that if we assume

$$q(t_{2n})\mu(t_{2n}) \geq \frac{p(t_{2n})}{\mu(t_{2n})} + \frac{p(t_{2n+1})}{\mu(t_{2n+1})} \tag{4.1}$$

then $Q[u] \leq 0$. When n is even this implies that $Q_1[u] \leq 0$ and when n is odd this gives us that $Q_2[u] \leq 0$. Therefore, from Theorem 2.1 we get that the dynamic equation (1.2) is oscillatory on $\mathbb{T}$. As a special case we note that if $\mathbb{T} = \mathbb{N}_0$ and

$$q(2n) \geq p(2n) + p(2n+1), \quad n \in \mathbb{N}_0,$$

then (1.2) is oscillatory on $\mathbb{T} = \mathbb{N}_0$. It is interesting to compare this result with [**20**, Corollary 8.3]. Another interesting special case is when our time scale is $\mathbb{T} = r^{\mathbb{N}_0}$, $r > 1$, in which case we get from (4.1) that if

$$(r-1)r^{4n+1}q(r^{2n}) \geq rp(r^{2n}) + p(r^{2n+1}), \quad n \in \mathbb{N}_0,$$

then (1.2) is oscillatory on $\mathbb{T} = r^{\mathbb{N}_0}$.

EXAMPLE 4.2. We consider in this example the same time scales as in Example 4.1, but for the case $\gamma > 1$ in equation (1.1). By virtue of Theorem 3.2, we see that if

$$\hat{q}(t) = h(\gamma) q^{1/\gamma}(t) |f(t)|^{1-1/\gamma},$$

where $h(\gamma) = \frac{\gamma}{(\gamma-1)^{1-1/\gamma}}$, and if (4.1) holds with $q(t)$ replaced by $\hat{q}(t)$, then it follows that the forced nonlinear equation (1.1) is oscillatory. We remark that since $\lim_{\gamma\to 1} h(\gamma) = 1$, Example 4.1 may be viewed as a limiting case of this example for all the time scales considered in Example 4.1.

EXAMPLE 4.3. Following Nasr [**21**], we consider the superlinear equation

$$x'' + \beta \ \sin t \ |x(t)|^\gamma \ \mathrm{sgn}\ x(t) = \cos t \tag{4.2}$$

for $t \in \mathbb{R} = \mathbb{T}$, where $\beta > 0$, $\gamma > 1$. We let $s_n := 2n\pi$, $t_n := (2n+1)\pi$, $\tau_n = 2n\pi + \frac{\pi}{2}$ and define

$$u_n(t) := \begin{cases} \sin 2t, & t \in [s_n, \tau_n] \\ -\sin 2t, & t \in [\tau_n, t_n]. \end{cases}$$

We have

$$\hat{q}(t) = h(\gamma)\beta^{\frac{1}{\gamma}} (\sin t)^{\frac{1}{\gamma}} |\cos t|^{\frac{\gamma-1}{\gamma}}$$

and therefore, if

$$\int_{s_n}^{\tau_n} [\hat{q}(t) u_n^2(t) - (u_n'(t))^2] dt \geq 0$$

and

$$\int_{\tau_n}^{t_n} [\hat{q}(t) u_n^2(t) - (u_n'(t))^2] dt \geq 0$$

then it will follow from Theorem 3.2 that equation (4.2) is oscillatory. We have since $u_n(t) = 2\sin t \cos t$, $t \in [s_n, \tau_n]$,

$$\begin{aligned} &\int_{s_n}^{\tau_n} [\hat{q}(t) u_n^2(t) - (u_n'(t))^2] dt \\ &= 4h(\gamma)\beta^{\frac{1}{\gamma}} \int_{s_n}^{\tau_n} (\sin t)^{2+\frac{1}{\gamma}} (\cos t)^{3-\frac{1}{\gamma}} dt - 4 \int_{s_n}^{\tau_n} \cos^2(2t) dt \\ &= 4h(\gamma)\beta^{\frac{1}{\gamma}} \int_0^{\frac{\pi}{2}} (\sin t)^{2+\frac{1}{\gamma}} (\cos t)^{3-\frac{1}{\gamma}} dt - 4 \int_0^{\frac{\pi}{2}} \cos^2(2t) dt \\ &= 4h(\gamma)\beta^{\frac{1}{\gamma}} \int_0^{\frac{\pi}{2}} (\sin t)^{2+\frac{1}{\gamma}} (\cos t)^{3-\frac{1}{\gamma}} dt - \pi \\ &= 4h(\gamma)\beta^{\frac{1}{\gamma}} \frac{\Gamma(2 - \frac{1}{2\gamma})\, \Gamma(\frac{3}{2} + \frac{1}{2\gamma})}{2\, \Gamma(\frac{7}{2})} - \pi \end{aligned}$$

where in the last equation we used the formula involving the gamma function (see Whittaker and Watson [**29**, page 256])

$$\int_0^{\frac{\pi}{2}} \cos^{2m-1} x \ \sin^{2n-1} x \ dx = \frac{\Gamma(m)\, \Gamma(n)}{2\, \Gamma(m+n)}$$

for any complex numbers m, n whose real parts are positive. The same result holds also for $\int_{\tau_n}^{t_n} [\hat{q}(t) u_n^2(t) - (u_n'(t))^2] dt$. Therefore, if

$$4h(\gamma)\beta^{\frac{1}{\gamma}} \frac{\Gamma(2 - \frac{1}{2\gamma})\, \Gamma(\frac{3}{2} + \frac{1}{2\gamma})}{2\, \Gamma(\frac{7}{2})} \geq \pi \tag{4.3}$$

then (4.2) is oscillatory.

We would like to remark that the above result improves the example given by Nasr [**21**] by replacing the constant 1 by $h(\gamma)$ and by correcting an error in a calculation involving the gamma function. In particular, if $\gamma = 2$, then $h(\gamma) = 2$ so that the result above improves Nasr's result for the lower bound on β which yields oscillation of all solutions of (4.2). Moreover, still for the case $\gamma = 2$, the inequality (4.3) holds if

$$\beta \geq \frac{25\,\pi^3}{36\left(\Gamma(\frac{3}{4})\right)^4} \approx 9.522,$$

where we have used (see [**20**]) $\Gamma(m+1) = m\Gamma(m)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Example 4.4. We consider the following linear forced differential equation

$$(\sqrt{t}\,x')' + \left(\alpha + \frac{\beta}{\sqrt{t}}\right) x = \sin\sqrt{t}, \tag{4.4}$$

where $\alpha \geq 0$, $\beta \geq 0$ (Wong in [**30**] considered the case $\alpha = 1$, $\beta = 0$). We are interested in conditions under which all solutions of (4.4) are oscillatory. Since the forcing term has zeros at $(n\pi)^2$ we let $u(t) = \sin\sqrt{t}$ and for any $T \geq 0$ choose n such that $(n\pi)^2 \geq T$. We let $s_1 = (n\pi)^2$, $t_1 = (n+1)^2\pi^2$. Then

$$\begin{aligned} Q_1[u] &= \int_{s_1}^{t_1} [q(t)u^2(t) - p(t)(u'(t))^2]dt \\ &= \int_{s_1}^{t_1} \left[\left(\alpha + \frac{\beta}{\sqrt{t}}\right) u^2(t) - \sqrt{t}\,(u'(t))^2\right] dt. \end{aligned}$$

Making the change of variables $\tau = \sqrt{t}$ we get

$$\begin{aligned} Q_1[u] &= \int_{n\pi}^{(n+1)\pi} \left[2(\alpha\tau + \beta)\sin^2\tau - \frac{1}{2}\cos^2\tau\right] d\tau \\ &= \frac{1}{2}\alpha(2n+1)\pi^2 + \pi\left(\beta - \frac{1}{4}\right), \end{aligned}$$

after some routine calculations. Hence we conclude that for any $\alpha > 0$ for all $\beta \geq 0$ (this gives the Wong example $\alpha = 1$, $\beta = 0$ as a special case) all solutions of (4.4) are oscillatory; and we also get that if $\beta > \frac{1}{4}$, then all solutions of (4.4) are oscillatory.

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