

OSCILLATION AND ASYMPTOTIC BEHAVIOR OF A THIRD-ORDER NONLINEAR DYNAMIC EQUATION

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Dedicated to Professors Jack W. Macki and James S. Muldowney

ABSTRACT. In this paper we establish some new oscillation criteria for the third-order nonlinear dynamic equation

$$(c(t) ((a(t)x^\Delta(t))^\Delta)^\gamma)^\Delta + f(t, x(t)) = 0, \quad t \in [a, \infty)_{\mathbb{T}}$$

on time scales, where $\gamma \geq 1$ is a quotient of odd integers. Our results not only unify the oscillation theory for third-order nonlinear differential and difference equations but also are new for the q -difference equations and can be applied on different time scales. The results improve some of the main results in the literature in the case when $\gamma = 1$.

1. INTRODUCTION

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [21], is an area of mathematics that has recently received a lot of attention. It was partly created in order to unify the study of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice - once for differential equations and once again for difference equations.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary closed subset of the reals. In this way, results not only related to the set of real numbers or to the set of integers but those pertaining to more general time scales are obtained. The three most popular examples of calculus on time scales are differential calculus, difference calculus [23], and quantum calculus [22].

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Dynamic equations on a time scale have an enormous potential for modeling a variety of applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. Since then several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal, Bohner, O'Regan, and Peterson [1] and the references cited therein. For more details, we refer the reader to the books by Bohner and Peterson [7], [8], which summarize and organize much of time scale calculus.

In recent years, there has been a great deal of research activity concerning the oscillation and nonoscillation of solutions of ordinary dynamic equations on time scales; we refer the reader to the papers [2-12], [15-20], and [24-26]. To the best of our knowledge, the only paper which deals with the third order dynamic equation is due to Erbe, Peterson and Saker [19]. In this paper, the authors considered the third-order nonlinear (Emden-Fowler type) dynamic equation

$$(1.1) \quad (c(t) ((a(t)x^\Delta(t))^\Delta)^\Delta + q(t)f(x(t))) = 0, \quad t \geq t_0,$$

where $c(t)$, $a(t)$ and $q(t)$ are positive real-valued rd -continuous functions that satisfy

$$(1.2) \quad \int_{t_0}^{\infty} q(t)\Delta t = \infty,$$

and

$$(1.3) \quad \int_{t_0}^{\infty} \frac{1}{c(t)}\Delta t = \infty, \quad \int_{t_0}^{\infty} \frac{1}{a(t)}\Delta t = \infty,$$

$f \in C(\mathbb{R}, \mathbb{R})$ and satisfies $uf(u) > 0$ and $f(u)/u \geq K > 0$, for $u \neq 0$. They established some sufficient conditions insure that every solution of (1.2) either oscillates or has a finite limit as $t \rightarrow \infty$.

Following this trend, in the present paper, we consider the third-order nonlinear dynamic equation of the form

$$(1.4) \quad [c(t) [(a(t)x^\Delta(t))^\Delta]^\gamma]^\Delta + f(t, x(t)) = 0, \quad t \geq t_0$$

on a time scale \mathbb{T} . Throughout this paper we will assume that $a(t) > 0$ and $c(t) > 0$ are rd -continuous on \mathbb{T} , $\gamma \geq 1$ is the quotient of odd integers, and there exists a positive rd -continuous function $q(t)$ such that

$$(1.5) \quad |f(t, u)| \geq q(t) |u^\gamma| \quad \text{and} \quad uf(t, u) > 0, \quad u \neq 0.$$

We also assume

$$(1.6) \quad \int_{t_0}^{\infty} \frac{1}{c^{\frac{1}{\gamma}}(t)} \Delta t = \infty, \quad \int_{t_0}^{\infty} \frac{1}{a(t)} \Delta t = \infty.$$

Occasionally, we will make the additional assumption

$$(1.7) \quad \int_{t_0}^{\infty} \frac{1}{a(s)} \int_s^{\infty} \left[\frac{1}{c(v)} \int_v^{\infty} q(u) \Delta u \right]^{\frac{1}{\gamma}} \Delta v \Delta s = \infty,$$

where here it is understood that

$$\int_{t_0}^{\infty} q(t) \Delta t < \infty, \quad \int_{t_0}^{\infty} \left[\frac{1}{c(v)} \int_v^{\infty} q(u) \Delta u \right]^{\frac{1}{\gamma}} \Delta v < \infty.$$

Our attention is restricted to those solutions of (1.4) which exist on some half line $[t_x, \infty)$ and satisfy

$$\sup\{|x(t)| : t > t_0\} > 0 \quad \text{for any } t_0 \geq t_x.$$

Since we are interested in the asymptotic behavior of solutions, we will suppose that the time scale \mathbb{T} under consideration is not bounded above. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. A solution x of (1.4) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1.4) is said to be oscillatory if all its solutions are oscillatory. Of course one could also consider the expression $f(t, x^{\sigma}(t))$ rather than $f(t, x(t))$ in (1.4) and proceed in much the same way to study this equation.

In this paper, we will establish some sufficient conditions which insure that every solution of (1.4) either oscillates or has a finite limit at ∞ . To prove the main results, we will use a Riccati type substitution and we will use the inequality (which can easily be proved by the mean value theorem)

$$(1.8) \quad A^{\gamma} - B^{\gamma} \geq \gamma B^{\gamma-1}(A - B),$$

where A and B are nonnegative constants and $\gamma \geq 1$. Furthermore, equality holds in (1.8) iff $A = B$. We also use Keller's chain rule [7, Theorem 1.90] which states that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable, then $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$ is delta differentiable and

$$(1.9) \quad (f \circ g)^{\Delta}(t) = \int_0^1 f'(x(t) + h\mu(t)x^{\Delta}(t)) dh x^{\Delta}(t).$$

When $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$, $\mu(t) = 0$, $x^{\Delta}(t) = x'(t)$ and (1.4) becomes the third order nonlinear (Emden–Fowler type) differential equation

$$(1.10) \quad (c(t) \left((a(t)x'(t))' \right)^{\gamma})' + f(t, x(t)) = 0, \quad t \in [t_0, \infty).$$

When $\mathbb{T} = \mathbb{N}$, $\sigma(t) = t + 1$, $\mu(t) = 1$, $x^\Delta(t) = \Delta x(t) = x(t + 1) - x(t)$. In this case, (1.4) becomes the third-order difference equation

$$(1.11) \quad \Delta(c(n) (\Delta(a(n)\Delta x(n)))^\gamma) + f(n, x(n)) = 0, \quad n \in [n_0, \infty)_{\mathbb{N}}.$$

If $\mathbb{T} = h\mathbb{N}_0$, $h > 0$, $\sigma(t) = t + h$, $\mu(t) = h$, $x^\Delta(t) = \Delta_h x(t) := \frac{x(t+h)-x(t)}{h}$. In this case, (1.4) becomes the third-order difference equation with step size h

$$(1.12) \quad \Delta_h(c(t) (\Delta_h(a(t)\Delta_h x(t)))^\gamma) + f(t, x(t)) = 0, \quad t \in [0, \infty)_{h\mathbb{N}_0},$$

If $\mathbb{T} = p^{\mathbb{N}_0} = \{t : t = p^k, k \in \mathbb{N}_0, p > 1\}$, $\sigma(t) = pt$, $\mu(t) = (p-1)t$, $x^\Delta(t) = D_p x(t) := \frac{x(pt)-x(t)}{(p-1)t}$ (D_q is the so-called quantum derivative which has important applications in quantum mechanics [22]). In this case (1.4) becomes the third order p -difference equation

$$(1.13) \quad D_p(c(t) (D_p(a(t)D_p x(t)))^\gamma) + f(t, x(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Also, the results can be applied to many other time scales. For example, one can consider $\mathbb{T} = \mathbb{N}_0^2 = \{t = n^2 : n \in \mathbb{N}_0\}$, where $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$, $x^\Delta(t) = \frac{x((\sqrt{t}+1)^2)-x(t)}{1+2\sqrt{t}}$, and $\mathbb{T} = \mathbb{T}_n = \{H_n : n \in \mathbb{N}_0\}$ where $\{H_n\}$ is the so-called sequence of harmonic numbers defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N}_0.$$

Here $\sigma(H_n) = H_{n+1}$, $\mu(H_n) = \frac{1}{n+1}$, $x^\Delta(H_n) = (n+1)\Delta_n x(H_n)$.

The paper is organized as follows: In Section 2, we present some basic definitions concerning the calculus on time scales and some useful lemmas which we will use in the proofs of our main results. In Section 3, we use the Riccati transformation technique to establish some sufficient conditions that ensure that all solutions of (1.4) oscillate or have a finite limit at ∞ . In the case where $\gamma = 1$, the main results improve the results in [19], since the condition (1.2) is not required. Some examples are considered to illustrate the main results. Our results not only unify the oscillation of third-order differential and difference equations but also can be applied to different types of time scales, i.e, the results can be applied to equations (1.12)–(1.13) and to any time scale with $\sup \mathbb{T} = \infty$.

2. MAIN RESULTS

In this section, we establish some sufficient conditions that guarantee every solution $x(t)$ of (1.4) either oscillates or has a finite limit. The following two lemmas will be used in the proofs of the main results. The proofs of these two lemmas are similar to the proofs in [19], and since no major new principles are involved, we omit the details.

Lemma 1. Assume that (1.5) and (1.6) hold, and suppose that $x(t)$ is an eventually positive solution of (1.4). Then there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that one of the following cases holds:

$$(1) \quad x(t) > 0, \quad x^\Delta(t) > 0, \quad (a(t)x^\Delta(t))^\Delta > 0$$

or

$$(2) \quad x(t) > 0, \quad x^\Delta(t) < 0, \quad (a(t)x^\Delta(t))^\Delta > 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$.

Lemma 2. Assume that (1.5) and (1.6) hold. Let $x(t)$ be a solution of (1.4) such that Case (1) of Lemma 1 holds for $t \in [t_1, \infty)_{\mathbb{T}}$. Then

$$x^\Delta(t) \geq \frac{\delta(t, t_1) c^{\frac{1}{\gamma}}(t)}{a(t)} (a(t)x^\Delta(t))^\Delta, \quad t \in [t_1, \infty)_{\mathbb{T}}$$

where $\delta(t, t_1) := \int_{t_1}^t \frac{\Delta s}{c^{\frac{1}{\gamma}}(s)}$.

For any number $d \in \mathbb{R}$, we define the positive part, d_+ , of d by

$$d_+ := \max\{0, d\}.$$

Theorem 1. Assume that (1.5) and (1.6) hold and given any $t_3 \in [t_0, \infty)_{\mathbb{T}}$ there is a $t_1 \in [t_3, \infty)_{\mathbb{T}}$ and a positive differentiable function $\alpha(t)$ on $[t_1, \infty)_{\mathbb{T}}$ such that

$$(2.1) \quad \limsup_{t \rightarrow \infty} \int_{t_2}^t \left(\alpha(s)q(s) - \frac{a^\gamma(s) (\alpha^\Delta(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^\gamma(s, t_1) \alpha^\gamma(s)} \right) \Delta s = \infty.$$

for $t_2 > t_1$. Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or $\lim_{t \rightarrow \infty} x(t)$ exists (finite).

Proof. Assume $x(t)$ is a nonoscillatory solution of (1.4). Without loss of generality, we may pick $t_1 \in [t_0, \infty)$ sufficiently large so that Lemma 1 holds and (2.1) holds for $t_2 > t_1$. We will consider only this case, because the proof when $x(t) < 0$ is similar. From Lemma 1, there are two possible cases. First, we consider Case (1). Then

$$x(t) > 0, \quad x^\Delta(t) > 0, \quad (a(t)x^\Delta(t))^\Delta > 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. Define the Riccati type function w by

$$(2.2) \quad w(t) := \alpha(t) \frac{c(t) \left[(a(t)x^\Delta(t))^\Delta \right]^\gamma}{x^\gamma(t)}, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

Note that $w(t) > 0$. Using the product rule, we obtain

$$(2.3) \quad w^\Delta = c^\sigma \left(\left((ax^\Delta)^\Delta \right)^\sigma \right)^\gamma \left[\frac{\alpha}{x^\gamma} \right]^\Delta + \frac{\alpha}{x^\gamma} \left[c \left[(ax^\Delta)^\Delta \right]^\gamma \right]^\Delta.$$

In view of (1.4) and (1.5), we have

$$\begin{aligned}
 w^\Delta &\leq -\alpha q + c^\sigma \left(\left((ax^\Delta)^\Delta \right)^\sigma \right)^\gamma \left(\frac{\alpha^\Delta x^\gamma - \alpha(x^\gamma)^\Delta}{x^\gamma (x^\sigma)^\gamma} \right) \\
 (2.4) \quad &= -\alpha q + \frac{\alpha^\Delta}{\alpha^\sigma} w^\sigma - c^\sigma \left(\left((ax^\Delta)^\Delta \right)^\sigma \right)^\gamma \frac{\alpha(x^\gamma)^\Delta}{x^\gamma (x^\sigma)^\gamma}.
 \end{aligned}$$

From Lemma 1, we see that $x^\sigma(t) \geq x(t)$, and then from Keller's chain rule (1.9), we obtain

$$\begin{aligned}
 (x^\gamma(t))^\Delta &= \gamma \int_0^1 [x(t) + h\mu(t)x^\Delta(t)]^{\gamma-1} dh x^\Delta(t) \\
 &= \gamma \int_0^1 [hx(t) + (1-h)x(t)]^{\gamma-1} dh x^\Delta(t) \\
 &\geq \gamma \int_0^1 [hx(t) + (1-h)x(t)]^{\gamma-1} dh x^\Delta(t) \\
 (2.5) \quad &= \gamma(x(t))^{\gamma-1} x^\Delta(t).
 \end{aligned}$$

From (2.4) and (2.5), we obtain

$$w^\Delta \leq -\alpha q + \frac{\alpha^\Delta}{\alpha^\sigma} w^\sigma - \gamma c^\sigma \left(\left((ax^\Delta)^\Delta \right)^\sigma \right)^\gamma \frac{\alpha x^\Delta}{(x^\sigma)^{\gamma+1}}.$$

Then, from Lemma 2, we have

$$(2.6) \quad w^\Delta \leq -\alpha q + \frac{\alpha^\Delta}{\alpha^\sigma} w^\sigma - \gamma \alpha c^\sigma \left[\left((ax^\Delta)^\Delta \right)^\sigma \right]^\gamma \frac{\delta c^{\frac{1}{\gamma}} (ax^\Delta)^\Delta}{a (x^\sigma)^{\gamma+1}}.$$

Since $(c((ax^\Delta)^\Delta)^\gamma)^\Delta \leq 0$, we have

$$(ax^\Delta)^\Delta \geq \left(\frac{c^\sigma}{c} \right)^{\frac{1}{\gamma}} \left((ax^\Delta)^\Delta \right)^\sigma.$$

Using this in (2.6), we obtain

$$(2.7) \quad w^\Delta(t) \leq -\alpha(t)q(t) + \frac{(\alpha^\Delta(t))_+}{\alpha^\sigma(t)} w^\sigma(t) - \frac{\gamma \alpha(t) \delta(t)}{(\alpha^\sigma)^\lambda(t) a(t)} (w^\sigma)^\lambda(t),$$

where $\lambda = (\gamma + 1)/\gamma$. From (1.8), we have that

$$(2.8) \quad \lambda B^{\lambda-1} A - A^\lambda \leq (\lambda - 1) B^\lambda$$

Setting

$$A^\lambda = \frac{\gamma \alpha \delta}{(\alpha^\sigma)^\lambda a} (w^\sigma)^\lambda, \quad B^{\lambda-1} = \frac{1}{\lambda} \frac{(\alpha^\Delta)_+ w^\sigma}{\alpha^\sigma A},$$

we get

$$\begin{aligned}
& \frac{(\alpha^\Delta)_+}{\alpha^\sigma} w^\sigma - \frac{\gamma \alpha \delta}{(\alpha^\sigma)^\lambda a} (w^\sigma)^\lambda \\
&= \lambda B^{\lambda-1} A - A^\lambda \\
&\leq (\lambda - 1) B^\lambda, \quad \text{by (2.8)} \\
&= (\lambda - 1) \left(\frac{(\alpha^\Delta)_+ w^\sigma}{\lambda \alpha^\sigma A} \right)^{\frac{\lambda}{\lambda-1}} \\
&= (\lambda - 1) \left(\frac{a^{\frac{1}{\lambda}} (\alpha^\Delta)_+}{\lambda \gamma^{\frac{1}{\lambda}} (\alpha^{\frac{1}{\lambda}})_+ \delta^{\frac{1}{\lambda}}} \right)^{\frac{\lambda}{\lambda-1}} \\
(2.9) \quad &= \frac{a^\gamma ((\alpha^\Delta)_+)^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \delta^\gamma \alpha^\gamma}.
\end{aligned}$$

Thus, from (2.7) and (2.9) we obtain for $t \in [t_2, \infty)_{\mathbb{T}}$

$$(2.10) \quad w^\Delta(t) < - \left[\alpha(t) q(t) - \frac{a^\gamma(t) ((\alpha^\Delta(t))^{\gamma+1})}{(\gamma + 1)^{\gamma+1} \delta^\gamma(t, t_1) \alpha^\gamma(t)} \right].$$

Integrating (2.10) from t_2 to t , we get

$$(2.11) \quad -w(t_2) < w(t) - w(t_2) < - \int_{t_2}^t \left[\alpha(s) q(s) - \frac{a^\gamma(s) (\alpha^\Delta(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \delta^\gamma(s, t_1) \alpha^\gamma(s)} \right] \Delta s,$$

which yields

$$\int_{t_2}^t \left[\alpha(s) q(s) - \frac{a^\gamma(s) (\alpha^\Delta(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \delta^\gamma(s, t_1) \alpha^\gamma(s)} \right] \Delta s < w(t_2),$$

for all large t . This contradicts (2.1) so we conclude there are no nonoscillatory solutions which satisfy Case (1) in Lemma 1.

Next, we assume that Case (2) in Lemma 1 holds, that is, we assume that

$$x(t) > 0, \quad x^\Delta(t) < 0, \quad (a(t)x^\Delta(t))^\Delta > 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. In this case, it follows that

$$\lim_{t \rightarrow \infty} x(t) \text{ exists (finite).}$$

□

Theorem 2. *In addition to the assumptions of Theorem 1, assume (1.7) holds. Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or satisfies $\lim_{t \rightarrow \infty} x(t) = 0$.*

Proof. From the proof of Theorem 1 it suffices to show that if there is a solution satisfying Case (2) in Lemma 1, that is, if

$$x(t) > 0, \quad x^\Delta(t) < 0, \quad (a(t)x^\Delta(t))^\Delta > 0$$

for $t \in [t_1, \infty)_{\mathbb{T}}$, then $\lim_{t \rightarrow \infty} x(t) = 0$. In this case we know that $\lim_{t \rightarrow \infty} x(t) = L$, where $0 \leq L < \infty$. If $L > 0$ we can show that this leads to a contradiction as follows. Integrating both sides of (1.4) from t to ∞ , and using (1.5) and the fact that $(a(t)x^\Delta(t))^\Delta > 0$, we obtain

$$-c(t) \left[(a(t)x^\Delta(t))^\Delta \right]^\gamma + \int_t^\infty q(u)x^\gamma(u)\Delta u \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Hence,

$$\left[\frac{1}{c(t)} \int_t^\infty q(u)x^\gamma(u)\Delta u \right]^\frac{1}{\gamma} \leq (a(t)x^\Delta(t))^\Delta, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating again from s to t , ($t \geq s \geq t_1$), we have

$$\int_s^t \left[\frac{1}{c(v)} \int_v^\infty q(u)x^\gamma(u)\Delta u \right]^\frac{1}{\gamma} \Delta v \leq a(t)x^\Delta(t) - a(s)x^\Delta(s).$$

Since $x^\Delta(t) < 0$, we have

$$\frac{1}{a(s)} \int_s^t \left[\frac{1}{c(v)} \int_v^\infty q(u)x^\gamma(u)\Delta u \right]^\frac{1}{\gamma} \Delta v \leq -x^\Delta(s).$$

Letting $t \rightarrow \infty$ and then integrating from t_1 to t , we obtain

$$\int_{t_1}^t \frac{1}{a(s)} \int_s^\infty \left[\frac{1}{c(v)} \int_v^\infty q(u)x^\gamma(u)\Delta u \right]^\frac{1}{\gamma} \Delta v \Delta s \leq -x(t) + x(t_1) \leq x(t_1).$$

Hence, by using the fact that $x(t)$ is decreasing to L , we have

$$L \int_{t_1}^t \frac{1}{a(s)} \int_s^\infty \left[\frac{1}{c(v)} \int_v^\infty q(u)\Delta u \right]^\frac{1}{\gamma} \Delta v \Delta s \leq x(t_1).$$

This contradicts (1.7) and the proof is complete. \square

Example 1. Let $\mathbb{T} = p^{\mathbb{N}_0} := \{1, p, p^2, p^3, \dots\}$, $p > 1$, then for the so-called p -difference equation

$$(2.12) \quad ((x^{\Delta\Delta})^\gamma)^\Delta + \frac{1}{t^{2\gamma+1}} x^\gamma = 0,$$

every solution either oscillates or approaches zero as $t \rightarrow \infty$. This follows from Theorem 2 since we can verify the following

$$\int_v^\infty \frac{1}{u^{2\gamma+1}} \Delta u = \frac{(p-1)p^{2\gamma}}{p^{2\gamma}-1} \frac{1}{v^{2\gamma}}, \quad \int_s^\infty \left[\int_v^\infty \frac{1}{u^{2\gamma+1}} \Delta u \right]^\frac{1}{\gamma} \Delta v = \frac{p^3(p-1)^\frac{1}{\gamma}}{(p^{2\gamma}-1)^\frac{1}{\gamma}} \frac{1}{s}$$

and

$$\int_{t_0}^{\infty} \int_s^{\infty} \left[\int_v^{\infty} \frac{1}{u^{2\gamma+1}} \Delta u \right]^{\gamma} \Delta v \Delta s = \infty,$$

so that (1.7) holds for this case. It can be shown that if $1 < p < 2$, then there is a solution of the form $x(t) = \frac{1}{t^{\beta}}$ for some $\beta > 0$ depending on p . Hence, for $1 < p < 2$ this is a solution of (2.12) that approaches zero as $t \rightarrow \infty$.

From Theorem 1, we may also obtain many results concerning the asymptotic behavior of solutions of (1.4) by different choices of $\alpha(t)$. For instance, by choosing $\alpha(t) = 1$, $\alpha(t) = t$, $t \geq t_0$ respectively, we get the next two corollaries of Theorem 1.

Corollary 1. Assume that (1.5) and (1.6), and

$$\int_{t_0}^{\infty} q(t) \Delta t = \infty$$

hold. Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or has a finite limit at ∞ .

Corollary 2. Assume that (1.5) and (1.6) hold. In addition, assume that for any $t_3 \in [t_0, \infty)_{\mathbb{T}}$ there is a $t_1 \in [t_3, \infty)_{\mathbb{T}}$, such that

$$\limsup_{t \rightarrow \infty} \int_{t_2}^t \left[sq(s) - \frac{a^{\gamma}(s)}{(\gamma+1)^{\gamma+1} s^{\gamma} \delta^{\gamma}(s, t_1)} \right] \Delta s = \infty$$

holds for $t_2 > t_1$. Then every solution of (1.4) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or has a finite limit at ∞ . If, in addition, (1.7) holds, then every solution is either oscillatory or approaches zero as $t \rightarrow \infty$.

Next, we present some new oscillation results for (1.4) by using an integral averaging technique of Kamenev-type.

Theorem 3. Assume that (1.5) and (1.6) hold. In addition, assume that for any $t_3 \in [t_0, \infty)_{\mathbb{T}}$ there is a $t_1 \in [t_3, \infty)_{\mathbb{T}}$ and a positive differentiable function $\alpha(t)$ on $[t_1, \infty)_{\mathbb{T}}$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_2}^t (t-s)^m \left[\alpha(s)q(s) - \frac{a^{\gamma}(s) (\alpha^{\Delta}(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^{\gamma}(s, t_1) \alpha^{\gamma}(s)} \right] \Delta s = \infty, \quad (2.13)$$

holds for $t_2 > t_1$, where m is an odd positive integer. Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or satisfies $\lim_{t \rightarrow \infty} x(t)$ exists (finite). If, in addition, (1.7) holds, then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or approaches zero as $t \rightarrow \infty$.

Proof. Assume equation (1.4) is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$; then, without loss of generality, we can pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large so that the conclusions of Lemma 1 hold and such that (2.13) holds for $t_2 > t_1$. First, if Case (1) in Lemma 1 holds, then again by defining $w(t)$ by (2.2) as in Theorem 1, we have $w(t) > 0$ and (2.10) holds. Then, from (2.10), we have

$$R(t) < -w^{\Delta}(t),$$

for $t \in [t_2, \infty)_{\mathbb{T}}$, where

$$R(t) := \alpha(t)q(t) - \frac{a^{\gamma}(t) (\alpha^{\Delta}(t))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^{\gamma}(t, t_1) \alpha^{\gamma}(t)}.$$

Therefore,

$$\int_{t_2}^t (t-s)^m R(s) \Delta s \leq - \int_{t_2}^t (t-s)^m w^{\Delta}(s) \Delta s.$$

Integrating by parts and using the power rule for differentiation [7, Theorem 1.24] we get

$$\begin{aligned} & \int_{t_2}^t (t-s)^m R(s) \Delta s \\ & < (-1)^{m+1} \{[(s-t)^m w(s)]_{t_2}^t \\ & + (-1)^{m+1} \int_{t_2}^t \sum_{j=0}^{m-1} (t-\sigma(s))^j (t-s)^{m-j+1} w^{\sigma}(s) \Delta s\} \\ & = (t-t_2)^m w(t_2) + (-1)^m \int_{t_2}^t \sum_{j=0}^{m-1} (\sigma(t)-s)^j (t-s)^{m-j-1} w(\sigma(s)) \Delta s. \end{aligned}$$

Now since m is an odd integer, we have

$$\int_{t_2}^t (t-s)^m R(s) \Delta s < (t-t_2)^m w(t_2).$$

Therefore,

$$\frac{1}{t^m} \int_{t_2}^t (t-s)^m R(s) \Delta s < \left(\frac{t-t_2}{t}\right)^m w(t_2).$$

Hence,

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_2}^t (t-s)^m R(s) \Delta s < \infty,$$

which is a contradiction to (2.13). Therefore, if $x(t)$ is a nonoscillatory solution of (1.4), then Case (2) of Lemma 1 holds and so again $\lim_{t \rightarrow \infty} x(t)$ exists (finite). Finally if (1.7) holds then Theorem 2 applies. \square

Theorem 4. Assume that (1.5) and (1.6) hold, and given any $t_3 \in [t_0, \infty)_{\mathbb{T}}$ there is a $t_1 \in [t_3, \infty)_{\mathbb{T}}$ and a positive differentiable function $\alpha(t)$ on $[t_1, \infty)_{\mathbb{T}}$ such that

$$(2.14) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_2}^t \left[(t-s)^m \alpha(s) q(s) - D(s, t_1) \frac{B^{\gamma+1}(t, s)}{(t-s)^{m\gamma}} \right] \Delta s = \infty,$$

for $t_2 > t_1$, where $m > 1$, $D(s, t_1) := \frac{a^\gamma(\alpha^\sigma)^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^\gamma(s, t_1) \alpha^\gamma(s)}$, and

$$B(t, s) := (t-s)^m \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} - m(t-\sigma(s))^{m-1}, \quad t \geq s \geq t_0.$$

Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or satisfies $\lim_{t \rightarrow \infty} x(t)$ exists (finite). If, in addition, (1.7) holds, then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or approaches zero as $t \rightarrow \infty$.

Proof. Assume $x(t)$ is a nonoscillatory solution of (1.4). Without loss of generality, we may choose $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large so the conclusions of Lemma 1 hold and for any $t_2 > t_1$ (2.14) holds. If Case (1) in Lemma 1 holds, we proceed as in the proof of Theorem 1 to show that (2.7) holds for $t \geq t_2$. Multiplying (2.7) by $(t-s)^m$ and integrating from t_2 to t , we have

$$(2.15) \quad \begin{aligned} & \int_{t_2}^t (t-s)^m \alpha(s) q(s) \Delta s \\ & \leq - \int_{t_2}^t (t-s)^m w^\Delta(s) \Delta s + \int_{t_2}^t (t-s)^m \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} w^\sigma(s) \Delta s \\ & \quad - \int_{t_2}^t (t-s)^m \frac{\gamma \alpha(s) \delta(s, t_1)}{(\alpha^\sigma(s))^\lambda a(s)} (w^\sigma(s))^\lambda \Delta s. \end{aligned}$$

Integration by parts gives

$$(2.16) \quad - \int_{t_2}^t (t-s)^m w^\Delta(s) \Delta s = (t-t_2)^m w(t_2) + \int_{t_2}^t ((t-s)^m)^{\Delta s} w^\sigma(s) \Delta s.$$

Next, we show that if $t \geq \sigma(s)$ and $m \geq 1$, then

$$(2.17) \quad ((t-s)^m)^{\Delta s} \leq -m(t-\sigma(s))^{m-1}.$$

If $\mu(s) = 0$, it is easy to see that (2.17) is an equality. If $\mu(s) \neq 0$, then we have

$$\begin{aligned} ((t-s)^m)^{\Delta s} &= \frac{1}{\mu(s)} [(t-\sigma(s))^m - (t-s)^m] \\ &= -\frac{1}{\sigma(s)-s} [(t-s)^m - (t-\sigma(s))^m]. \end{aligned}$$

Using inequality (1.8), we have for $t \geq \sigma(s)$

$$[(t-s)^m - (t-\sigma(s))^m] \geq m(t-\sigma(s))^{m-1}(\sigma(s)-s),$$

and from this we see that (2.17) holds. From (2.15)–(2.17), we obtain

$$\begin{aligned}
 & \int_{t_2}^t (t-s)^m \alpha(s) q(s) \Delta s \\
 & \leq (t-t_2)^m w(t_2) + \int_{t_2}^t \left[(t-s)^m \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} - m(t-\sigma(s))^{m-1} \right] w^\sigma(s) \Delta s \\
 (2.18) \quad & - \int_{t_2}^t (t-s)^m \frac{\gamma \alpha(s) \delta(s, t_1)}{(\alpha^\sigma(s))^\lambda a(s)} (w^\sigma(s))^\lambda \Delta s.
 \end{aligned}$$

Now, we let $h(u) := Bu - Au^\lambda$, $u \in \mathbb{R}$, where

$$B := (t-s)^m \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} - m(t-\sigma(s))^{m-1}, \quad A := \frac{\gamma \alpha(s) \delta(s, t_1) (t-s)^m}{(\alpha^\sigma(s))^\lambda a(s)}.$$

Using the fact that h takes on its maximum value on \mathbb{R} at $u^* = \left(\frac{B\gamma}{A(\gamma+1)} \right)^\gamma$, we obtain, after some simple calculations,

$$\max_{u \in \mathbb{R}} h = h(u^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}.$$

Then, from (2.18),

$$\begin{aligned}
 & \int_{t_2}^t (t-s)^m \alpha(s) q(s) \Delta s \\
 & \leq (t-t_2)^m w(t_2) + \int_{t_2}^t \frac{a^\gamma(s) (\alpha^\sigma(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^\gamma(s, t_1) \alpha^\gamma(s)} \frac{B^{\gamma+1}(t, s)}{(t-s)^{m\gamma}} \Delta s.
 \end{aligned}$$

This easily leads to a contradiction of (2.14). The remainder of the proof is similar to the end of the proof of Theorem 3. \square

The following theorem gives a Philos-type oscillation criteria for equation (1.4).

First, let us introduce now the class of functions \mathfrak{R} which will be extensively used in the sequel. Let $\mathbb{D}_0 \equiv \{(t, s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t, s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ is said to belong to the class \mathfrak{R} if

(i) $H(t, t) = 0$, $t \geq t_0$, $H(t, s) > 0$, on \mathbb{D}_0 ,

(ii) H has a continuous Δ -partial derivative $H^{\Delta_s}(t, s)$ on \mathbb{D}_0 with respect to the second variable. (H is rd -continuous function if H is an rd -continuous function in t and s).

Theorem 5. Assume that (1.5) and (1.6) hold, and assume given any $t_3 \in [t_0, \infty)_{\mathbb{T}}$ there is $t_1 \in [t_3, \infty)_{\mathbb{T}}$ and a positively differentiable function $\alpha(t)$

and a function $H \in \mathfrak{R}$ such that for $t_2 > t_1$

$$(2.19) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t \left[H(t, s) \alpha(s) q(s) - E(s, t_1) \frac{C^{\gamma+1}(t, s)}{H^\gamma(t, s)} \right] = \infty,$$

where $E(s, t_1) := \frac{(\alpha^\sigma)^{\gamma+1}(s) a^\gamma(s)}{(\gamma+1)^{\gamma+1} \alpha^\gamma(s) \delta^\gamma(s, t_1)}$ and

$$C(t, s) := H(t, s) \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma} + H^{\Delta s}(t, s).$$

Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or satisfies $\lim_{t \rightarrow \infty} x(t)$ exists (finite). If, in addition, (1.7) holds, then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or approaches zero as $t \rightarrow \infty$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.4). Without loss of generality, we may assume that there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large so that the conclusions of Lemma 1 hold and (2.19) holds for $t_2 > t_1$. If Case (1) of Lemma 1 holds then proceeding as in the proof of Theorem 1, we see that (2.7) holds for $t \geq t_2$. From (2.7), it follows that

$$(2.20) \quad \begin{aligned} & \int_{t_2}^t H(t, s) \alpha(s) q(s) \Delta s \\ & \leq - \int_{t_2}^t H(t, s) w^\Delta(s) \Delta s + \int_{t_2}^t H(t, s) \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} w^\sigma(s) \Delta s \\ & \quad - \int_{t_2}^t H(t, s) \frac{\gamma \alpha(s) \delta(s, t_1)}{(\alpha^\sigma(s))^\lambda a(s)} (w^\sigma(s))^\lambda \Delta s, \quad \left(\lambda = \frac{\gamma+1}{\gamma} \right). \end{aligned}$$

Integrating by parts and using $H(t, t) = 0$, we have

$$\int_{t_2}^t H(t, s) w^\Delta(s) \Delta s = -H(t, t_2) w(t_2) - \int_{t_2}^t H^{\Delta s}(t, s) w^\sigma(s) \Delta s.$$

It then follows from (2.20) that

$$\begin{aligned} & \int_{t_2}^t H(t, s) \alpha(s) q(s) \Delta s \\ & \leq H(t, t_2) w(t_2) + \int_{t_2}^t H^{\Delta s}(t, s) w^\sigma(s) \Delta s + \int_{t_2}^t H(t, s) \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} w^\sigma(s) \Delta s \\ & \quad - \int_{t_2}^t H(t, s) \frac{\gamma \alpha(s) \delta(s, t_1)}{(\alpha^\sigma(s))^\lambda a(s)} (w^\sigma(s))^\lambda \Delta s. \end{aligned}$$

Hence,

$$\begin{aligned}
 & \int_{t_2}^t H(t, s) \alpha(s) q(s) \Delta s \\
 & \leq H(t, t_2) w(t_2) + \int_{t_2}^t \left[H(t, s) \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} + H^{\Delta_s}(t, s) \right] w^\sigma(s) \Delta s \\
 (2.21) \quad & - \int_{t_2}^t H(t, s) \frac{\gamma \alpha(s) \delta(s, t_1)}{(\alpha^\sigma(s))^\lambda a(s)} (w^\sigma(s))^\lambda \Delta s.
 \end{aligned}$$

Therefore, as in Theorem 4, by letting

$$A := H(t, s) \frac{\gamma \alpha(s) \delta(s, t_1)}{(\alpha^\sigma(s))^\lambda a(s)}, \quad B := H(t, s) \frac{(\alpha^\Delta(s))_+}{\alpha^\sigma(s)} + H^{\Delta_s}(t, s),$$

we have

$$\begin{aligned}
 & \int_{t_2}^t H(t, s) \alpha(s) q(s) \Delta s \\
 & \leq H(t, t_2) w(t_2) + \int_{t_2}^t \frac{(\alpha^\sigma)^{\gamma+1}(s) a^\gamma(s) C^{\gamma+1}(t, s)}{(\gamma+1)^{\gamma+1} \alpha^\gamma(s) \delta^\gamma(s, t_1) H^\gamma(t, s)} \Delta s.
 \end{aligned}$$

Then, for all $t \geq t_2$ we have

$$\int_{t_2}^t \left[H(t, s) \alpha(s) q(s) - E(s, t_1) \frac{C^{\gamma+1}(t, s)}{H^\gamma(t, s)} \right] \Delta s < H(t, t_2) w(t_2),$$

and this implies that

$$\frac{1}{H(t, t_2)} \int_{t_2}^t \left[H(t, s) \alpha(s) q(s) - E(s, t_1) \frac{C^{\gamma+1}(t, s)}{H^\gamma(t, s)} \right] \Delta s < w(t_2),$$

for all large t , which contradicts (2.19). \square

As an immediate consequence of Theorem 5, we have following.

Corollary 3. *Let the assumption (2.19) in Theorem 4 be replaced by*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t H(t, s) \alpha(s) q(s) \Delta s = \infty,$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t \left(\frac{(\alpha^\sigma(s))^{\gamma+1} a^\gamma(s) C^{\gamma+1}(t, s)}{(\gamma+1)^{\gamma+1} \alpha^\gamma(s) \delta^\gamma(s, t_1) H^\gamma(t, s)} \right) \Delta s < \infty.$$

Then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or satisfies $\lim_{t \rightarrow \infty} x(t)$ exists (finite). If, in addition, (1.7) holds, then every solution of (1.4) is either oscillatory on $[t_0, \infty)_{\mathbb{T}}$ or approaches zero as $t \rightarrow \infty$.

Remark 1. *With an appropriate choice of the functions H and α one can derive a number of oscillation criteria for equation (1.4) on different types of time scales. Consider, for example the function $H(t, s) = (t-s)^\lambda$, $(t, s) \in \mathbf{D}$ with $\lambda > 1$. We see that H belongs to the class \mathfrak{R} and then we obtain a Kamenev-type oscillation criterion. Also, one can use the so-called falling function (see [23]) $H(t, s) := (t-s)^{\underline{k}}$ where $t^{\underline{k}} := t(t-1)\dots(t-k+1)$, $t^0 := 1$. In this case*

$$H^{\Delta_s}(t, s) = ((t-s)^{\underline{k}})^{\Delta_s} = -k(t-s-2k-3)^{\underline{k-1}} \geq -(k)(t-s)^{\underline{k-1}}.$$

Additional examples may also be given. We leave this to the interested reader.

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