

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A THIRD-ORDER NONLINEAR DYNAMIC EQUATION ON TIME SCALES

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ABSTRACT. In this paper, we will establish some sufficient conditions which guarantee that every solution of the third order nonlinear dynamic equation

$$(c(t)(a(t)x^\Delta(t))^\Delta)^\Delta + q(t)f(x(t)) = 0, \quad t \geq t_0,$$

oscillates or converges to zero.

1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [16] in order to unify continuous and discrete analysis. Not only can this theory of the so-called “dynamic equations” **unify** the theories of differential equations and difference equations, but also it is able to **extend** these classical cases to cases “in between”, e.g., to so-called q-difference equations. A time scale \mathbb{T} is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [5]). A book on the subject of time scales by Bohner and Peterson [5] summarizes and organizes much of the time scale calculus. For the notions used below we refer to [5] and to the next section, where we recall some of the main tools used in the subsequent sections of this paper.

In the second order case, oscillation theories for differential and difference equations are well established, see [1, 2], even though the discrepancies in some of the results in these two theories are not well understood. In the last year there has been much research activity concerning the oscillation and

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nonoscillation of solutions of some dynamic equations on time scales, and we refer the reader to the papers [3, 4, 6-15, 18, 19]. Following this trend, in this paper, we shall consider the third-order nonlinear dynamic equation

$$(1.1) \quad (c(t)(a(t)x^\Delta(t))^\Delta)^\Delta + q(t)f(x(t)) = 0, \quad t \geq t_0,$$

where the functions $c(t)$, $a(t)$, $q(t)$ are positive, real-valued, rd-continuous functions defined on the time scale interval $[a, b]$ (throughout $a, b \in \mathbb{T}$ with $a < b$) and the following conditions hold.

We assume throughout that $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous with $uf(u) > 0$, $u \neq 0$ and satisfies the following condition: For each $k > 0$ there exists $M = M_k > 0$ such that

$$(1.2) \quad f(u)/u \geq M, \quad |u| \geq k.$$

We note that the function $f(u) = |u|^\gamma \operatorname{sgn} u$ with $\gamma \geq 1$ satisfies (1.2). Notice also that the condition (1.2) essentially says that the quotient $f(u)/u$ is bounded away from 0 if $|u|$ is.

Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. By a solution of (1.1) we mean a nontrivial real-valued function x satisfying equation (1.1) for $t \geq a$. A solution x of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$.

In this paper we obtain some oscillation criteria for (1.1). The paper is organized as follows: In the next section, we present some basic definitions concerning the calculus on time scales and state and prove some useful lemmas. In Section 3, we will use the Riccati transformation technique to give some sufficient conditions in terms of the coefficients and the graininess function which guarantee that every solution of (1.1) is oscillatory or converges to zero. To the best of our knowledge nothing is known regarding the qualitative behavior of (1.1) on time scales up to now.

2. SOME PRELIMINARIES ON TIME SCALES AND SOME LEMMAS

In this section, we present some basic definitions concerning the calculus on time scales which are contained in [5], and then we state and prove some lemmas which we will need in the proofs of our main results. A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . On any time

scale \mathbb{T} we define the forward jump operator σ and the graininess function μ by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \mu(t) := \sigma(t) - t.$$

A point $t \in \mathbb{T}$ with $\sigma(t) = t$ is called right-dense, while t is referred to as being right-scattered if $\sigma(t) > t$. The backward jump operator and left-dense and left-scattered points are defined in a similar way.

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be *rd-continuous* if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points. The (delta) *derivative* of $f : \mathbb{T} \rightarrow \mathbb{T}$ at a right-dense point t is defined by

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists, and if t is right-scattered and f is continuous at t we define the (delta) derivative at t by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

The derivative and the shift operator σ are related by the useful formula

$$(2.1) \quad f^\sigma = f + \mu f^\Delta, \quad \text{where} \quad f^\sigma := f(\sigma(t)).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable function f and g :

$$(2.2) \quad (fg)^\Delta = f^\Delta g + f^\sigma g^\Delta, \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

By using the product rule form (2.2), the derivative of $f(t) = (t - \alpha)^m$ for $m \in \mathbb{N}$, and $\alpha \in \mathbb{R}$ can be calculated (see [5, Theorem 1.24]) as

$$(2.3) \quad f^\Delta(t) = \sum_{\nu=0}^{m-1} (\sigma(t) - \alpha)^\nu (t - \alpha)^{m-\nu-1}.$$

For $a, b \in \mathbb{T}$, and a differentiable function f , the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t) \Delta t = f(b) - f(a).$$

The integration by parts formula follows from (2.2) reads

$$(2.4) \quad \int_a^b f(t)g^\Delta(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t)) \Delta t,$$

and improper integrals are defined in the usual way by

$$\int_a^\infty f(t)\Delta t = \lim_{b \rightarrow \infty} \int_a^b f(t)\Delta t.$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have $\sigma(t) = t$, $\mu(t) = 0$,

$$f^\Delta(t) = f'(t), \text{ and } \int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

and in case $\mathbb{T} = \mathbb{Z}$ we have $\sigma(t) = t + 1$, $\mu(t) = 1$,

$$f^\Delta(t) = \Delta f(t) = f(t+1) - f(t), \text{ and } \int_a^b f(t)\Delta t = \sum_{i=a}^{b-1} f(i).$$

Now, we state and prove some useful lemmas, which we will use in the proofs of our main results. We begin with the following lemma.

Lemma 1. *Suppose that x is an eventually positive solution of (1.1) and*

$$(2.5) \quad \int_{t_0}^\infty \frac{1}{c(t)}\Delta t = \int_{t_0}^\infty \frac{1}{a(t)}\Delta t = \infty.$$

Then there is a $t_1 \in [t_0, \infty)$ such that either

$$(i.) \quad x(t) > 0, \quad x^\Delta(t) > 0, \quad (a(t)x^\Delta(t))^\Delta > 0, \quad t \in [t_1, \infty),$$

or

$$(ii.) \quad x(t) > 0, \quad x^\Delta(t) < 0, \quad (a(t)x^\Delta(t))^\Delta > 0, \quad t \in [t_1, \infty).$$

Proof. Let x be an eventually positive solution of (1.1). Then there exists $t_1 \in [t_0, \infty)$ such that $x(t) > 0$ for $t \in [t_1, \infty)$. From (1.1) we have

$$(c(t)(a(t)x^\Delta(t))^\Delta)^\Delta = -q(t)f(x(t)) < 0$$

for $t \in [t_1, \infty)$. Hence $c(t)(a(t)x^\Delta(t))^\Delta$ is strictly decreasing on $[t_1, \infty)$. We claim that $c(t)(a(t)x^\Delta(t))^\Delta > 0$ on $[t_1, \infty)$. Assume not, then there is a $t_2 \in [t_1, \infty)$ such that

$$c(t)(a(t)x^\Delta(t))^\Delta < 0, \quad t \in [t_2, \infty).$$

Then we can choose a negative constant C and $t_3 \in [t_2, \infty)$ such that

$$c(t)(a(t)x^\Delta(t))^\Delta \leq C < 0$$

for $t \in [t_3, \infty)$. Dividing by $c(t)$ and integrating from t_3 to t , we obtain

$$a(t)x^\Delta(t) \leq a(t_3)x^\Delta(t_3) + C \int_{t_3}^t \frac{\Delta s}{c(s)}.$$

Letting $t \rightarrow \infty$, then $a(t)x^\Delta(t) \rightarrow -\infty$ by (2.5). Thus, there is a $t_4 \in [t_3, \infty)$ such that for $t \in [t_4, \infty)$,

$$a(t)x^\Delta(t) \leq a(t_4)x^\Delta(t_4) < 0.$$

Dividing by $a(t)$ and integrating from t_4 to t we obtain

$$x(t) - x(t_4) \leq a(t_4)x^\Delta(t_4) \int_{t_4}^t \frac{\Delta s}{a(s)},$$

which implies that $x(t) \rightarrow -\infty$ as $t \rightarrow \infty$ by (2.5), a contradiction with the fact that $x(t) > 0$. Hence we have

$$(a(t)x^\Delta(t))^\Delta > 0, \quad t \in [t_1, \infty).$$

This implies that $a(t)x^\Delta(t)$ is strictly increasing on $[t_1, \infty)$. It follows from this that either $a(t)x^\Delta(t) < 0$ on $[t_1, \infty)$ or $a(t)x^\Delta(t)$ is eventually positive and the proof is complete. \square

Lemma 2. *Assume that (1.2) and*

$$(2.6) \quad \int_{t_0}^{\infty} q(t)\Delta t = \infty$$

hold, and x is a solution of (1.1) that satisfies Case (ii) in Lemma 1. Then

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Proof. Let x be a solution of (1.1) satisfying Case (ii) in Lemma 1, that is,

$$x(t) > 0, \quad x^\Delta(t) < 0, \quad (a(t)x^\Delta(t))^\Delta > 0, \quad t \in [t_1, \infty).$$

Then

$$\lim_{t \rightarrow \infty} x(t) = b \geq 0.$$

Assume $b > 0$ and we now show that this leads to a contradiction. From (1.1) and (1.2) with $k = b$, there exists $M = M_b > 0$ such that

$$\begin{aligned} (c(t)(a(t)x^\Delta(t))^\Delta)^\Delta &= -q(t)f(x(t)) \\ &\leq -Mq(t)x(t) \leq -Mq(t)b, \end{aligned}$$

for $t \in [t_1, \infty)$. Let

$$u(t) := c(t)(a(t)x^\Delta(t))^\Delta, \quad t \in [t_1, \infty),$$

then we have

$$u^\Delta(t) \leq -Mq(t)b, \quad t \in [t_1, \infty).$$

Integrating the last inequality from t_1 to t , we have

$$(2.7) \quad u(t) \leq u(t_1) - bM \int_{t_1}^t q(s)\Delta s.$$

Using (2.6) it is possible to choose a $t_2 \in [t_1, \infty)$, sufficiently large, such that for all $t \in [t_2, \infty)$

$$u(t) < 0,$$

which is a contradiction, and this completes the proof. \square

Lemma 3. *Assume that (1.2) holds and x is a solution of (1.1) satisfying Case (i) of Lemma 1. Then there exists $t_1 \in [t_0, \infty)$ such that*

$$(2.8) \quad x^\Delta(t) \geq \frac{\delta(t, t_1)c(t)}{a(t)} (a(t)x^\Delta(t))^\Delta \quad \text{for } t \geq t_1,$$

where $\delta(t, t_1) = \int_{t_1}^t \frac{\Delta s}{c(s)}$.

Proof. From Case (i) of Lemma 1 we have x is a solution of (1.1) satisfying

$$x(t) > 0, \quad x^\Delta(t) > 0, \quad (a(t)x^\Delta(t))^\Delta > 0,$$

for $t \geq t_1$. Using x is a solution of (1.1) we get

$$(c(t)(a(t)x^\Delta(t))^\Delta)^\Delta < 0$$

and hence $c(t)(a(t)x^\Delta(t))^\Delta$ is decreasing on $[t_1, \infty)$. Hence

$$(2.9) \quad \begin{aligned} a(t)x^\Delta(t) &= a(t_1)x^\Delta(t_1) + \int_{t_1}^t \frac{c(s)(ax^\Delta)^\Delta(s)}{c(s)} \Delta s \\ &\geq c(t)\delta(t, t_1)(ax^\Delta)^\Delta(t), \quad t \geq t_1, \end{aligned}$$

and this leads to (2.8) and the proof is complete. \square

3. MAIN RESULTS

In this section, we establish some sufficient conditions which guarantee that every solution x of (1.1) oscillates on $[t_0, \infty)$ or converges as $t \rightarrow \infty$.

Theorem 1. *Assume that (1.2) and (2.5) hold. Furthermore, assume that there exists a positive function r such that r^Δ is rd-continuous on $[t_0, \infty)$ and for all $M > 0$ and all sufficiently large t_1 ,*

$$(3.1) \quad \limsup_{t \rightarrow \infty} \int_{t_1}^t \left(Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right) \Delta s = \infty.$$

Then every solution x of (1.1) is oscillatory or $\lim_{t \rightarrow \infty} x(t)$ exists (finite).

Proof. Let x be a nonoscillatory solution of (1.1). We only consider the case when $x(t)$ is eventually positive, since the case when $x(t)$ is eventually negative is similar. By Lemma 1 either Case (i) or Case (ii) in Lemma 1 holds. Assume $x(t)$ satisfies Case (i) in Lemma 1. Define the ‘‘Riccati’’ type function w by

$$(3.2) \quad w(t) = r(t) \frac{c(t) (a(t)x^\Delta(t))^\Delta}{x(t)}, \quad t \geq t_1.$$

By the product rule

$$(3.3) \quad w^\Delta(t) = c(\sigma(t))(ax^\Delta)^\Delta(\sigma(t)) \left[\frac{r(t)}{x(t)} \right]^\Delta + \frac{r(t)}{x(t)} \left(c(t) (a(t)x^\Delta(t))^\Delta \right)^\Delta.$$

Using (1.1) and (1.2) we have with $k = x(t_1) > 0$ and $M = M_k > 0$

$$\begin{aligned} w^\Delta(t) &\leq -Mr(t)q(t) + c(\sigma(t))(ax^\Delta)^\Delta(\sigma(t)) \left(\frac{r^\Delta(t)x(t) - r(t)x^\Delta(t)}{x(t)x(\sigma(t))} \right) \\ &= -Mr(t)q(t) + \frac{r^\Delta(t)}{r(\sigma(t))}w(\sigma(t)) \\ &\quad - c(\sigma(t))(ax^\Delta)^\Delta(\sigma(t)) \frac{r(t)x^\Delta(t)}{x(t)x(\sigma(t))}. \end{aligned}$$

Using (2.8), we obtain

$$\begin{aligned} w^\Delta(t) &\leq -Mr(t)q(t) + \frac{r^\Delta(t)}{r(\sigma(t))}w(\sigma(t)) \\ &\quad - c(\sigma(t))(ax^\Delta)^\Delta(\sigma(t))r(t) \frac{\delta(t, t_1)}{a(t)} \frac{c(t)(a(t)x^\Delta(t))^\Delta}{x(t)x(\sigma(t))}. \end{aligned}$$

Now, since $x^\Delta(t) > 0$, we have that $x(\sigma(t)) \geq x(t)$, also since

$$(c(t)(a(t)x^\Delta(t))^\Delta)^\Delta \leq 0$$

we have

$$c(t)(a(t)x^\Delta(t))^\Delta \geq c(\sigma(t)) ((ax^\Delta)^\Delta)(\sigma(t)).$$

Using these two inequalities we get

$$\begin{aligned} w^\Delta(t) &\leq -Mr(t)q(t) + \frac{r^\Delta(t)}{r(\sigma(t))}w(\sigma(t)) \\ &\quad - \frac{r(t)}{r^2(\sigma(t))} \frac{\delta(t, t_1)}{a(t)} \frac{r^2(\sigma(t))c^2(\sigma(t)) (ax^\Delta)^\Delta(\sigma(t))}{x^2(\sigma(t))}. \end{aligned}$$

Using the definition of $w(t)$ we obtain

$$w^\Delta(t) \leq -Mr(t)q(t) + \frac{r^\Delta(t)}{r(\sigma(t))}w(\sigma(t)) - \frac{r(t)}{r^2(\sigma(t))} \frac{\delta(t, t_1)}{a(t)} w^2(\sigma(t)).$$

Hence,

$$(3.3) \quad w^\Delta(t) \leq -Mr(t)q(t) + \frac{r^\Delta(t)}{r(\sigma(t))}w(\sigma(t)) - Q(t) \frac{w^2(\sigma(t))}{r^2(\sigma(t))},$$

where

$$(3.4) \quad Q(t) = r(t) \frac{\delta(t, t_1)}{a(t)}.$$

From (3.3) we have

$$(3.5) \quad \begin{aligned} w^\Delta(t) &\leq -Mr(t)q(t) + \frac{(r^\Delta(t))^2}{4Q(t)} - \left[\frac{\sqrt{Q(t)}}{r(\sigma(t))} w(\sigma(t)) - \frac{r^\Delta(t)}{2\sqrt{Q(t)}} \right]^2 \\ &\leq - \left[Mr(t)q(t) - \frac{(r^\Delta(t))^2}{4Q(t)} \right]. \end{aligned}$$

Then, by (3.4)

$$(3.6) \quad w^\Delta(t) \leq - \left[Mr(t)q(t) - \frac{(r^\Delta(t))^2 a(t)}{4r(t)\delta(t, t_1)} \right].$$

Integrating (3.6) from t_1 to t , we obtain

$$(3.7) \quad -w(t_1) \leq w(t) - w(t_1) \leq - \int_{t_1}^t \left[Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s,$$

which yields

$$\int_{t_1}^t \left[Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s \leq w(t_1)$$

for all large t . This is contrary to (3.1) and so Case (i) is not possible. If Case (ii) in Lemma 1 holds, then clearly $\lim_{t \rightarrow \infty} x(t)$ exists (finite). \square

Remark 1. We note that if in Theorem 1 we replace the assumption (1.2) with the assumption that there exists an $M_0 > 0$ such that

$$f(u)/u \geq M_0 > 0, \quad \text{for all } u \neq 0,$$

then the conclusions of Theorem 1 hold.

Using Theorem 1, we get the following results.

Corollary 1. Assume that (1.2) and (2.5) hold. Furthermore, assume that for all $M > 0$ and all sufficiently large t_1 ,

$$(3.8) \quad \limsup_{t \rightarrow \infty} \int_{t_1}^t \left[Msq(s) - \frac{a(s)}{4s\delta(s, t_1)} \right] \Delta s = \infty,$$

then every solution x of (1.1) is either oscillatory or $\lim_{t \rightarrow \infty} x(t)$ exists (finite).

Proof. This follows from Theorem 1 by taking $r(t) = t$. \square

Corollary 2 (Leighton–Wintner Theorem). Assume that (1.2), (2.5), and (2.6) hold. Then every solution x of (1.1) is oscillatory or $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Taking $r(t) = 1$ in Theorem 1, we get by the proof of Theorem 1 that every solution of (1.1) oscillates on $[t_0, \infty)$ or satisfies Case (ii) in Lemma 1. Then, by Lemma 2, we get that $\lim_{t \rightarrow \infty} x(t) = 0$. \square

We now give a simple example where Corollary 1 applies, but Corollary 2 does not.

Example 1. Let $\mathbb{T} = p^{\mathbb{N}_0}$, where $p > 1$ is a constant. Take $a(t) = c(t) = 1$ and $q(t) = \frac{1}{t\sigma(t)}$. Since $(\frac{1}{t})^\Delta = -\frac{1}{t\sigma(t)}$, we get that

$$\int_1^\infty q(t)\Delta t = 1.$$

But

$$\int_t^\infty tq(t)\Delta t = \int_1^\infty \frac{1}{\sigma(t)}\Delta t = \sum_{k=0}^\infty \frac{1}{p^{k+1}}(p-1)p^k = \frac{p-1}{p} \sum_{k=0}^\infty 1 = \infty.$$

Using this it is easy to see that the hypotheses of Corollary 1 hold, but the hypotheses of Corollary 2 do not hold. Hence, by Corollary 1, $x^{\Delta\Delta} + \frac{1}{t\sigma(t)}f(x) = 0$, where f satisfies (1.2), is oscillatory on $\mathbb{T} = p^{\mathbb{N}_0}$.

Next, we present some new oscillation results for (1.1), by using an integral averaging condition of Kamenev type.

Theorem 2. Assume that (1.2), (2.5), and (2.6) hold. Further assume there is a positive function r such that r^Δ is rd-continuous on $[t_0, \infty)$ and that for all $M > 0$ and sufficiently large t_1 ,

$$(3.9) \quad \limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_1}^t (t-s)^m \left[Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s = \infty,$$

where $m \geq 1$. Then every solution of (1.1) is either oscillatory or $\lim_{t \rightarrow \infty} x(t)$ exists (finite).

Proof. Proceeding as in Theorem 1, we assume that (1.1) has a nonoscillatory solution, say $x(t) > 0$ for all $t \geq t_1$ where t_1 is chosen so large that Lemma 1 and Lemma 3 hold. By Lemma 1 there are two possible cases. First, if the Case (i) holds, then by defining again $w(t)$ by (3.2) as in Theorem 1 we have $w(t) > 0$ and (3.5) holds. Then from (3.5) we have

$$\left[Mr(t)q(t) - \frac{(r^\Delta(t))^2 a(t)}{4r(t)\delta(t, t_1)} \right] \leq -w^\Delta(t).$$

Therefore,

$$(3.10) \quad \int_{t_1}^t (t-s)^m \left[Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s \leq - \int_{t_1}^t (t-s)^m w^\Delta(s) \Delta s.$$

An integration by parts of the right hand side and leads to

$$\begin{aligned}
 \int_{t_1}^t (t-s)^m w^\Delta(s) \Delta s &= (t-s)^m w(s) \Big|_{s=t_1}^{s=t} - \int_{t_1}^t h(t,s) w(\sigma(s)) \Delta s \\
 (3.11) \qquad \qquad \qquad &= -(t-t_1)^m w(t_1) - \int_{t_1}^t h(t,s) w(\sigma(s)) \Delta s
 \end{aligned}$$

where $h(t,s) := ((t-s)^m)^{\Delta s}$. Note that since

$$h(t,s) = \begin{cases} -m(t-s)^{m-1}, & \mu(s) = 0 \\ \frac{(t-\sigma(s))^m - (t-s)^m}{\mu(s)}, & \mu(s) > 0, \end{cases}$$

and $m \geq 1$, $h(t,s) \leq 0$ for $t \geq \sigma(s)$. It follows from (3.11) that

$$\int_{t_1}^t (t-s)^m w^\Delta(s) \Delta s \geq -(t-t_1)^m w(t_1).$$

Then from (3.10) we have

$$\int_{t_1}^t (t-s)^m \left[Mr(s)q(s) - \frac{(r^\Delta(s, t_1))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s \leq (t-t_1)^m w(t_1).$$

Then

$$\frac{1}{t^m} \int_{t_1}^t (t-s)^m \left[Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s \leq \left(\frac{t-t_1}{t} \right)^m w(t_1).$$

Hence,

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_1}^t (t-s)^m \left[Mr(s)q(s) - \frac{(r^\Delta(s))^2 a(s)}{4r(s)\delta(s, t_1)} \right] \Delta s \leq w(t_1),$$

which is a contradiction of (3.9). If Case (ii) holds, then as before, $\lim_{t \rightarrow \infty} x(t)$ exists (finite) and the proof is complete. \square

Note that when $r(t) = 1$, then (3.9) reduces to

$$(3.12) \qquad \qquad \qquad \lim_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m q(s) \Delta s = \infty,$$

which can be considered as the extension of Kamenev type oscillation criteria for second order differential equations (see [11]).

When $\mathbb{T} = \mathbb{R}^+ := [0, \infty)$, then (3.12) becomes

$$\lim_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m q(s) ds = \infty,$$

when $\mathbb{T} = \mathbb{N}_0$, then (3.12) becomes

$$\lim_{n \rightarrow \infty} \frac{1}{n^m} \sum_{k=0}^{n-1} (n-k)^m q(k) = \infty,$$

and when $\mathbb{T} = p^{\mathbb{N}_0}$, where $p > 1$ is a constant, then (3.12) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{1}{p^{nm}} \sum_{k=0}^{n-1} p^k (p^n - p^k)^m q(p^k) = \infty.$$

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