Oscillation Criteria for Nonlinear Damped Dynamic Equations on Time Scales

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Abstract. We present new oscillation criteria for the second order nonlinear damped delay dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(x^{\Delta\sigma}(t))^{\gamma}+q(t)f\left(x(\tau(t))\right)=0.$$

Our results generalize and improve some known results for oscillation of second order nonlinear delay dynamic equation. Our results are illustrated with examples.

1. Introduction

In this paper, we are concerned with oscillation behavior of the second order nonlinear damped delay dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+p(t)(x^{\Delta\sigma}(t))^{\gamma}+q(t)f\left(x(\tau(t))\right)=0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where γ is a quotient of odd positive integers, r, p and q are positive $rd-$continuous functions on $\mathbb{T}$, and the so-called delay function $\tau:\mathbb{T}\to\mathbb{T}$ satisfies $\tau(t)\le t$ for $t\in\mathbb{T}$ and $\lim_{t\to\infty}\tau(t)=\infty$. The function $f\in C(\mathbb{R},\mathbb{R})$ is assume to satisfy $uf(u)>0$ and $\left|\frac{f(u)}{u^{\gamma}}\right|\ge K$, for $u\neq 0$ and for some $K>0$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. By a solution of (1.1) we mean a nontrivial real–valued function $x\in C^1_{rd}[T_x,\infty)$, $T_x\ge t_0$ which has the property that $r(t)\left(x^{\Delta}(t)\right)^{\gamma}\in C^1_{rd}[T_x,\infty)$ and satisfies equation (1.1) on $[T_x,\infty)$, where C_{rd} is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

Recently there has been a large number of papers devoted to second order nonlinear dynamic equations on time scales. For example. Agarwal *et al* [**1**] considered the second order delay dynamic equation

$$x^{\Delta\Delta}(t)+q(t)x(\tau(t))=0, \tag{1.2}$$

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and established some sufficient conditions for oscillation of (1.2). Zhang *et al* [**19**] study the oscillation of the second order nonlinear delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)f(x(t-\tau)) = 0, \tag{1.3}$$

and the second order nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + q(t)f(x(\sigma(t))) = 0, \tag{1.4}$$

and established the equivalence of the oscillation of (1.3) and (1.4), from which they obtained some oscillation criteria and comparison theorems for (1.3). Sahiner [**15**] considered the second order nonlinear delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)f(x(\tau(t))) = 0, \tag{1.5}$$

and obtained some sufficient conditions for oscillation of (1.5) by means of the Riccati transformation technique. Erbe *et al* [**10**] extended Sahiner's result to the second order nonlinear delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)f(x(\tau(t))) = 0,$$

Erbe *et al* [**8**] considered the pair of second order dynamic equations

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(t) = 0, \tag{1.6}$$

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(\sigma(t)) = 0,$$

and established some necessary and sufficient conditions for nonoscillation of Hille-Kneser type. Saker [**16**] examined oscillation for half-linear dynamic equations (1.6), where $\gamma > 1$ is an odd positive integer and Agarwal *et al* [**2**] studied oscillation for the same equation (1.6), where $\gamma > 1$ is the quotient of odd positive integers. These results can not be applied when $0 < \gamma \leq 1$. Very recently Erbe *et al* [**6**] and [**7**] considered the half-linear delay dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(\tau(t)) = 0,$$

for the cases $\gamma \geq 1$ and $0 < \gamma \leq 1$ respectively. In addition, it was assumed that

$$r^{\Delta}(t) \geq 0, \tag{1.7}$$

and

$$\int_{t_0}^{\infty} \tau^{\gamma}(t)q(t)\Delta t = \infty. \tag{1.8}$$

Both of the two cases

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty, \tag{1.9}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} < \infty, \tag{1.10}$$

were considered. Therefore it is of great interest to study the equation (1.1) without necessarily assuming the conditions (1.7)-(1.10). We will still assume $\gamma > 0$ is the quotient of odd positive integers and, hence our results will improve and extend many known results on half-linear oscillation.

Note that in the special case when $\mathbb{T}=\mathbb{R}$, then

$$\sigma(t)=t,\ \mu(t)=0,\ g^{\Delta}(t)=g'(t),\ \int_a^b g(t)\Delta t=\int_a^b g(t)dt,$$

and (1.1) becomes the second order nonlinear damped delay differential equation

$$(r(t)(x'(t))^{\gamma})'+p(t)(x'(t))^{\gamma}+q(t)f\left(x(\tau(t))\right)=0. \tag{1.11}$$

The oscillation of equation (1.11) when $r\left(t\right)=1,\ p(t)=0$ and $f(x(\tau(t)))=x^{\gamma}(\tau(t))$, was studied by Agarwal et al [**3**] and proved that if

$$\limsup_{t\to\infty} t^{\gamma}\int_t^{\infty} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} ds>1,$$

then every solution of (1.11) oscillates. When $\mathbb{T}=\mathbb{Z}$, then

$$\sigma(t)=t+1,\ \mu(t)=1,\ g^{\Delta}(t)=\Delta g(t),\ \int_a^b g(t)\Delta t=\sum_{t=a}^{b-1} g(t),$$

and (1.1) becomes the second order nonlinear damped delay difference equation

$$\Delta(r(t)(\Delta x(t))^{\gamma})+p(t)(\Delta x(t))^{\gamma}+q(t)f\left(x(\tau(t))\right)=0. \tag{1.12}$$

The oscillation of equation (1.12) when $r\left(t\right)=1,\ p(t)=0$ and $f(x(\tau(t)))=x(t)$, was studied by Thandapani *et al* [**17**] where $\gamma>0$, $p(t)$ is a positive sequence, and it was shown that every solution of (1.12) is oscillatory, if

$$\sum_{n=n_0}^{\infty} q(n)=\infty.$$

We will see that our results not only unify some of the known oscillation results for differential and difference equations but can be applied to other cases to determine the oscillatory behavior. Note that, if $\mathbb{T}=h\mathbb{Z}$, $h>0$, then

$$\sigma(t)=t+h,\ \mu(t)=h,\ y^{\Delta}(t)=\Delta_h y(t)=\frac{y(t+h)-y(t)}{h},$$

$$\int_a^b g(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}} g(a+kh)h,$$

and (1.1) becomes the second order nonlinear damped delay difference equation

$$\Delta_h(r(t)(\Delta_h x(t))^{\gamma})+p(t)(\Delta_h x(t))^{\gamma}+q(t)f\left(x(\tau(t))\right)=0. \tag{1.13}$$

If

$$\mathbb{T}=q^{\mathbb{N}_0}=\{t:t=q^k,\ k\in\mathbb{N}_0,\ q>1\},$$

then

$$\sigma(t)=qt,\ \mu(t)=(q-1)t,\ x^{\Delta}(t)=\Delta_q x(t)=(x(q\,t)-x(t))/(q-1)\,t,$$

$$\int_{t_0}^{\infty} g(t)\Delta t=\sum_{k=n_0}^{\infty} g(q^k)\mu(q^k),$$

where $t_0=q^{n_0}$, and (1.1) becomes the second order $q-$nonlinear damped delay difference equation

$$\Delta_q(r(t)(\Delta_q x(t))^{\gamma})+p(t)(\Delta_q x(t))^{\gamma}+q(t)f\left(x(\tau(t))\right)=0. \tag{1.14}$$

If

$$\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\},$$

then

$$\sigma(t) = (\sqrt{t}+1)^2,\ \mu(t) = 1+2\sqrt{t},\ \Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1+2\sqrt{t}},$$

and (1.1) becomes the second order nonlinear damped delay difference equation

$$\Delta_N(r(t)(\Delta_N x(t))^\gamma) + p(t)(\Delta_N x(t))^\gamma + q(t) f\left(x(\tau(t))\right) = 0. \tag{1.15}$$

If $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$ where H_n denotes the set of numbers defined by

$$H_0 = 0,\ H_n = \sum_{k=1}^{n} \frac{1}{k},\ n \in \mathbb{N},$$

then

$$\sigma(H_n) = H_{n+1},\ \mu(H_n) = \frac{1}{n+1},\ y^\Delta(t) = \Delta_{H_n} y(H_n) = (n+1)\Delta y(H_n),$$

and (1.1) becomes the second order nonlinear damped delay difference equation

$$\Delta_{H_n}(r(H_n)(\Delta_{H_n} x(H_n))^\gamma) + p(H_n)(\Delta_{H_n} x(H_n))^\gamma + q(H_n) f\left(x(\tau(H_n))\right) = 0. \tag{1.16}$$

If

$$\mathbb{T} = \mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\},$$

then

$$\sigma(t) = \sqrt{t^2+1},\ \mu(t) = \sqrt{t^2+1} - t,\ x^\Delta(t) = \Delta_2 x(t) = \frac{x(\sqrt{t^2+1}) - x(t)}{\sqrt{t^2+1} - t},$$

and (1.1) becomes the second order nonlinear delay damped difference equation

$$\Delta_2(r(t)(\Delta_2 x(t))^\gamma) + p(t)(\Delta_2 x(t))^\gamma + q(t) f\left(x(\tau(t))\right) = 0. \tag{1.17}$$

We recall that for a discrete time scale, we have

$$\int_a^b g(t)\Delta t = \sum_{t \in [a,b)_{\mathbb{T}}} g(t)\mu(t).$$

We will utilize in the following a Riccati transformation technique to establish oscillation criteria for (1.1), where γ is the quotient of odd positive integers. These results improve and generalize the results that have been established in [**2**], [**6**], [**7**], and [**16**] and our results are essentially new for equations (1.11)–(1.17). Also, interesting examples that illustrate the importance of our results are included in section 3 below.

2. Main Results

Throughout this paper, we let

$$d_+(t) := \max\{0, d(t)\} \quad \text{and} \quad d_-(t) := \max\{0, -d(t)\},$$

$$\theta(a, b; u) := \frac{\int_u^a \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_u^b \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}, \qquad \alpha(t, u) := \theta(\tau(t), \sigma(t); u),$$

$$P(t) := \frac{1}{r^\sigma(t)}(\delta^\Delta(t) r^\sigma(t) - \delta(t)p(t)), \quad a(t) := e_{\frac{p(t)}{r^\sigma(t)}}(t, t_0),$$

and

$$R(t) := a(t)r(t), \quad Q(t) := a(t)q(t), \quad \beta(t) := \int_{\tau(t)}^{\infty} \frac{\Delta s}{R^{\frac{1}{\gamma}}(s)}.$$

The first two Lemmas give some sufficient conditions in order that a positive solution of (1.1) is eventually increasing.

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{R^{\frac{1}{\gamma}}(t)} = \infty, \tag{2.1}$$

holds and (1.1) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$x^{\Delta}(t) > 0, \quad \textit{and} \quad (r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0, \quad \textit{on } [T, \infty)_{\mathbb{T}}.$$

Proof. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. From (1.1), we have

$$(r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + p(t)(x^{\Delta\sigma}(t))^{\gamma} = -q(t) f\left(x(\tau(t))\right) < 0. \tag{2.2}$$

For $t \in [t_1, \infty)_{\mathbb{T}}$. we can write the left hand side of (2.2) in the form

$$(r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} a(t) + \frac{p(t)}{r^{\sigma}(t)} (r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\sigma} a(t) < 0,$$

which implies

$$\left(R(t) \left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0.$$

Now, we claim that $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. If not, then there exists $t_2 \geq t_1$ such that $x^{\Delta}(t_2) < 0$. Using the fact that $R(t) \left(x^{\Delta}(t)\right)^{\gamma}$ is decreasing, we obtain, for $t \in [t_2, \infty)_{\mathbb{T}}$

$$R(t)(x^{\Delta}(t))^{\gamma} < c := R(t_2)(x^{\Delta}(t_2))^{\gamma} < 0.$$

Integrating from t_2 to t, we find that

$$x(t) < x(t_2) + c^{\frac{1}{\gamma}} \int_{t_2}^{t} \frac{\Delta s}{R^{\frac{1}{\gamma}}(s)},$$

for $t \geq t_2$. Condition (2.1) implies that $x(t)$ is eventually negative, which is a contradiction. Therefore, $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ and hence, from (1.1), we get $(r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0$ on $[t_1, \infty)_{\mathbb{T}}$. The proof is complete. □

Lemma 2.2. *Assume that*

$$\int_{t_0}^{\infty} \left[\frac{1}{R(t)} \int_{t_0}^{t} Q(s)\beta^{\gamma}(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta t = \infty, \tag{2.3}$$

holds and (1.1) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$x^{\Delta}(t) > 0, \quad \textit{and} \quad (r(t) \left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0, \quad \textit{on } [T, \infty)_{\mathbb{T}}.$$

PROOF. As in the proof of Lemma 2.1, assume there is a $t_2 \geq t_0$ such that $x^{\Delta}(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Pick $t_3 \geq t_2$ so that $\tau(t) \geq t_2$, for $t \geq t_3$. Using the fact that $R(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is decreasing, we obtain

$$\begin{aligned} -x(\tau(t)) &< x(\infty) - x(\tau(t)) = \int_{\tau(t)}^{\infty} \frac{\left(R(s)\left(x^{\Delta}(s)\right)^{\gamma}\right)^{\frac{1}{\gamma}}}{R^{\frac{1}{\gamma}}(s)} \Delta s \\ &\leq \left(R(\tau(t))\left(x^{\Delta}(\tau(t))\right)^{\gamma}\right)^{\frac{1}{\gamma}} \int_{\tau(t)}^{\infty} \frac{\Delta s}{R^{\frac{1}{\gamma}}(s)} \\ &\leq \left(R(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma}\right)^{\frac{1}{\gamma}} \int_{\tau(t)}^{\infty} \frac{\Delta s}{R^{\frac{1}{\gamma}}(s)} = L\beta(t), \end{aligned}$$

where $L := \left(R(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma}\right)^{\frac{1}{\gamma}} < 0$. From (1.1), we get, for $t \geq t_3$

$$\begin{aligned} \left(R(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} &= -Q(t) f(x(\tau(t))) \leq -KQ(t) x^{\gamma}(\tau(t)) \\ &\leq KL^{\gamma} Q(t) \beta^{\gamma}(t), \end{aligned}$$

Hence, for $t \geq t_3$, we have

$$\begin{aligned} R(t)(x^{\Delta}(t))^{\gamma} &\leq R(t_3)(x^{\Delta}(t_3))^{\gamma} + KL^{\gamma} \int_{t_3}^{t} Q(u)\beta^{\gamma}(u)\Delta u \\ &\leq KL^{\gamma} \int_{t_3}^{t} Q(u)\beta^{\gamma}(u)\Delta u. \end{aligned}$$

It follows from this last inequality that

$$x(t) - x(t_3) \leq K^{\frac{1}{\gamma}} L \int_{t_3}^{t} \left[\frac{1}{R(s)} \int_{t_3}^{s} Q(u)\beta^{\gamma}(u)\Delta u\right]^{\frac{1}{\gamma}} \Delta s.$$

Hence by (2.3), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is a positive solution of (1.1). Thus $x^{\Delta}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$ and hence, from (1.1), we get $\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0$ on $[t_1, \infty)_{\mathbb{T}}$. □

LEMMA 2.3. *Assume that*

$$\int_{t_0}^{\infty} \left[\frac{1}{R(t)} \int_{t_0}^{t} Q(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta t = \infty, \tag{2.4}$$

holds and (1.1) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then either there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$x^{\Delta}(t) > 0, \quad \text{and} \quad \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \quad \text{on } [T, \infty)_{\mathbb{T}},$$

or $\lim_{t\to\infty} x(t) = 0$.

PROOF. As in the proof of Lemma 2.1, assume there is a $t_2 \geq t_0$ such that $x^{\Delta}(t) < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Thus, we get $\lim_{t\to\infty} x(t) = l \geq 0$. If we assume $l > 0$, then $x(\tau(t)) \geq l$, for $t \geq t_3$. From (1.1), we get

$$\begin{aligned} \left(R(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} &= -Q(t) x^{\gamma}(\tau(t)) \\ &\leq -l^{\gamma} Q(t), \end{aligned}$$

Hence, for $t \geq t_3$, we have

$$\begin{aligned} R(t)(x^{\Delta}(t))^{\gamma} &\leq R(t_3)(x^{\Delta}(t_3))^{\gamma} - l^{\gamma}\int_{t_3}^{t} Q(u)\Delta u \\ &\leq -l^{\gamma}\int_{t_3}^{t} Q(u)\Delta u. \end{aligned}$$

It follows from this last inequality that

$$x(t) - x(t_3) \leq -l\int_{t_3}^{t}\left[\frac{1}{R(s)}\int_{t_3}^{s} Q(u)\Delta u\right]^{\frac{1}{\gamma}}\Delta s.$$

Hence by (2.4), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is a positive solution of (1.1). Thus, $l = 0$. This completes the proof. □

LEMMA 2.4. *Assume that there exists $T \geq t_0$, sufficiently large, such that*

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad \text{and} \quad (r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0, \quad \text{on } [T,\infty)_{\mathbb{T}}.$$

Then

$$x(\tau(t)) > \alpha\,(t,T)\,x^{\sigma}\,(t)\,, \quad \text{for } t \geq T_1 \geq T.$$

PROOF. Since $r(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[T,\infty)$. We can choose $T_1 \geq T$ so that $\tau(t) \geq T$, for $t \geq T_1$. Then, we have that

$$\begin{aligned} x^{\sigma}\,(t) - x(\tau(t)) &= \int_{\tau(t)}^{\sigma(t)} \frac{(r(s)\left(x^{\Delta}(s)\right)^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s \\ &\leq (r(\tau(t))\left(x^{\Delta}(\tau(t))\right)^{\gamma})^{\frac{1}{\gamma}}\int_{\tau(t)}^{\sigma(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}, \end{aligned}$$

and so

$$\frac{x^{\sigma}\,(t)}{x(\tau(t))} \leq 1 + \frac{(r(\tau(t))\left(x^{\Delta}(\tau(t))\right)^{\gamma})^{\frac{1}{\gamma}}}{x(\tau(t))}\int_{\tau(t)}^{\sigma(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}. \tag{2.5}$$

Also, we see that

$$\begin{aligned} x(\tau(t)) &> x(\tau(t)) - x\,(T) = \int_{T}^{\tau(t)} \frac{(r(s)\left(x^{\Delta}(s)\right)^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s \\ &\geq (r(\tau(t))\left(x^{\Delta}(\tau(t))\right)^{\gamma})^{\frac{1}{\gamma}}\int_{T}^{\tau(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}, \end{aligned}$$

and hence

$$\frac{(r(\tau(t))\left(x^{\Delta}(\tau(t))\right)^{\gamma})^{\frac{1}{\gamma}}}{x(\tau(t))} < \left(\int_{T}^{\tau(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\right)^{-1}. \tag{2.6}$$

Therefore, (2.5) and (2.6) imply

$$\frac{x^{\sigma}\,(t)}{x(\tau(t))} < \int_{T}^{\sigma(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\left(\int_{T}^{\tau(t)}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\right)^{-1},$$

and hence we get the desired inequality

$$x(\tau(t)) > \alpha\,(t,T)\,x^{\sigma}\,(t)\,, \quad \text{for } t \geq T_1 \geq T.$$

This completes the proof. □

¿From Lemmas 2.1-2.2, we get the following oscillation criteria for equation (1.1).

THEOREM 2.1. *Assume one of the conditions* (2.1) *or* (2.3) *holds. Furthermore, suppose that there exists a positive* $\Delta-$*differentiable function* $\delta(t)$ *such that, for all sufficiently large* T,

$$\limsup_{t\to\infty}\int_T^t\left[K\alpha^{\gamma}\left(s,T\right)\delta(s)q(s)-\frac{r(s)(P_+(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s=\infty. \tag{2.7}$$

Then every solution of equation (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

PROOF. Assume (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemmas 2.1-2.2 on $[T,\infty)_{\mathbb{T}}$ with $x(\tau(t))>0$ on $[T,\infty)_{\mathbb{T}}$. In particular, we have

$$x(\tau(t))>0,\ x^{\Delta}(t)>0,\ (r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta}<0,\quad \text{for } t\ge T.$$

Consider the generalized Riccati substitution

$$w(t)=\delta(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}. \tag{2.8}$$

By the product rule and then the quotient rule

$$\begin{aligned} w^{\Delta}(t) &= \delta^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\sigma}+\delta(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\Delta}\\ &= \delta^{\Delta}(t)\left(\frac{r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)}\right)^{\sigma}+\delta(t)\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}}{x^{\gamma\sigma}(t)}-\delta(t)\frac{r(t)(x^{\Delta}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma\sigma}(t)}. \end{aligned}$$

¿From (1.1) and the definition of $w(t)$ and $P(t)$, we have

$$w^{\Delta}(t)\le\frac{P(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-K\delta(t)q(t)\left(\frac{x(\tau(t))}{x^{\sigma}(t)}\right)^{\gamma}-\delta(t)\frac{r(t)(x^{\Delta}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma\sigma}(t)}.$$

Using the fact that $r(t)(x^{\Delta}(t))^{\gamma}$ is strictly decreasing and the definition of $w(t)$, we obtain

$$w^{\Delta}(t)\le\frac{P(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-K\delta(t)q(t)\left(\frac{x(\tau(t))}{x^{\sigma}(t)}\right)^{\gamma}-\frac{\delta(t)w^{\sigma}(t)}{\delta^{\sigma}(t)}\frac{(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)}.$$

¿From Lemma 2.4, we get

$$w^{\Delta}(t)\le\frac{P(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-K\alpha^{\gamma}(t,T)\delta(t)q(t)-\frac{\delta(t)w^{\sigma}(t)}{\delta^{\sigma}(t)}\frac{(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)}. \tag{2.9}$$

By the Pötzsche chain rule ([**4**, Theorem 1.90]), we obtain

$$\begin{aligned} (x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1\left[x(t)+h\mu(t)x^{\Delta}(t)\right]^{\gamma-1}dh\ x^{\Delta}(t)\\ &= \gamma\int_0^1\left[(1-h)\,x(t)+hx^{\sigma}(t)\right]^{\gamma-1}dh\ x^{\Delta}(t)\\ &\ge \begin{cases} \gamma\left(x(t)\right)^{\gamma-1}x^{\Delta}(t), & \gamma>1\\ \gamma\left(x^{\sigma}(t)\right)^{\gamma-1}x^{\Delta}(t), & 0<\gamma\le 1 \end{cases}. \end{aligned}$$

If $0<\gamma\le 1$, we have that

$$w^{\Delta}(t)\le \frac{P(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-K\alpha^{\gamma}\left(t,T\right)\delta(t)q(t)-\frac{\gamma\delta(t)w^{\sigma}(t)}{\delta^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{x^{\sigma}(t)}{x(t)}\right)^{\gamma}, \tag{2.10}$$

whereas if $\gamma>1$, we have that

$$w^{\Delta}(t)\le \frac{P(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-K\alpha^{\gamma}\left(t,T\right)\delta(t)q(t)-\frac{\gamma\delta(t)w^{\sigma}(t)}{\delta^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\frac{x^{\sigma}(t)}{x(t)}. \tag{2.11}$$

Using the fact that $x(t)$ is strictly increasing and $r(t)(x^{\Delta}(t))^{\gamma}$ is strictly decreasing, we get that

$$x^{\sigma}(t)\ge x(t),\quad x^{\Delta}(t)\ge\left(\frac{r^{\sigma}(t)}{r(t)}\right)^{\frac{1}{\gamma}}(x^{\Delta}(t))^{\sigma}. \tag{2.12}$$

¿From (2.10), (2.11) and (2.12), we obtain

$$w^{\Delta}(t)\le \frac{P_{+}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-K\alpha^{\gamma}\left(t,T\right)\delta(t)q(t)-\frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{\frac{1}{\gamma}}(t)}. \tag{2.13}$$

where $\lambda:=\frac{\gamma+1}{\gamma}$. Define $A>0$ and $B>0$ by

$$A^{\lambda}:=\frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{\frac{1}{\gamma}}(t)},\quad B^{\lambda-1}:=\frac{(r^{\frac{1}{\gamma}}(t))^{\frac{1}{\lambda}}P_{+}(t)}{\lambda\gamma^{\frac{1}{\lambda}}(\delta(t))^{\frac{1}{\lambda}}}.$$

Then, using the inequality

$$\lambda AB^{\lambda-1}-A^{\lambda}\le(\lambda-1)B^{\lambda}, \tag{2.14}$$

we get that

$$\begin{aligned}\frac{P_{+}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\frac{\gamma\delta(t)(w^{\sigma}(t))^{\lambda}}{(\delta^{\sigma}(t))^{\lambda}r^{\frac{1}{\gamma}}(t)} &= \lambda AB^{\lambda-1}-A^{\lambda}\\ &\le (\lambda-1)B^{\lambda}\\ &= \frac{r(t)(P_{+}(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}.\end{aligned}$$

¿From this last inequality and (2.13) we get

$$w^{\Delta}(t)\le\frac{r(t)(P_{+}(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}-K\delta(t)\alpha^{\gamma}\left(t,T\right)q(t)$$

Integrating both sides from T to t we get

$$\int_{T}^{t}\left[K\alpha^{\gamma}\left(s,T\right)\delta(s)q(s)-\frac{r(s)(P_{+}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s\le w(T)-w(t)\le w(T),$$

which leads to a contradiction to (2.7). □

By choosing $\delta(t)=1$ and $\delta(t)=t$, $t\ge t_0$ in Theorem 2.1 we have the following oscillation results.

Corollary 2.1. *Assume one of the conditions* (2.1) *or* (2.3) *holds and, for all sufficiently large* T,

$$\int_{T}^{\infty}\alpha^{\gamma}\left(t,T\right)q(t)\Delta t=\infty. \tag{2.15}$$

Then every solution of equation (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

COROLLARY 2.2. *Assume one of the conditions* (2.1) *or* (2.3) *holds and, for all sufficiently large* T,

$$\limsup_{t\to\infty}\int_T^t\left[Ks\alpha^{\gamma}\left(s,T\right)q(s)-\frac{r(s)(P_+(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s=\infty, \tag{2.16}$$

where $P(t)=1-\frac{tp(t)}{r^{\sigma}(t)}$. *Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$

We are now ready to state and prove Philos-type oscillation criteria for the equation (1.1).

THEOREM 2.2. *Assume one of the conditions* (2.1) *or* (2.3) *holds. Furthermore, suppose that there exist functions* $H,h\in C_{rd}\left(\mathbb{D},\mathbb{R}\right)$, *where* $\mathbb{D}\equiv\{(t,s):t\geq s\geq t_0\}$ *such that*

$$H\left(t,t\right)=0,\quad t\geq t_0,\ H\left(t,s\right)>0,\quad t>s\geq t_0, \tag{2.17}$$

and H *has a nonpositive continuous* Δ*-partial derivative* $H^{\Delta_s}\left(t,s\right)$ *with respect to the second variable and satisfies*

$$H^{\Delta_s}\left(t,s\right)+H\left(t,s\right)\frac{P(s)}{\delta^{\sigma}\left(s\right)}=-\frac{h\left(t,s\right)}{\delta^{\sigma}\left(s\right)}\left(H\left(t,s\right)\right)^{\frac{\gamma}{\gamma+1}}, \tag{2.18}$$

and, for all sufficiently large T,

$$\limsup_{t\to\infty}\frac{1}{H\left(t,T\right)}\int_T^t\left[K\alpha^{\gamma}\left(s,T\right)\delta\left(s\right)q\left(s\right)H\left(t,s\right)-\frac{\left(h_-\left(t,s\right)\right)^{\gamma+1}r\left(s\right)}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s=\infty, \tag{2.19}$$

where $\delta(t)$ *is a positive* $\Delta-$*differentiable function. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

PROOF. Assume (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemmas 2.1-2.2 on $[T,\infty)_{\mathbb{T}}$ with $x(\tau(t))>0$ on $[T,\infty)_{\mathbb{T}}$. In particular, we have

$$x(\tau(t))>0,\ x^{\Delta}(t)>0,\ (r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta}<0,\quad \text{for } t\geq T.$$

We define $w\left(t\right)$ also, as in Theorem 2.1. From (2.13) with $P_+(t)$ replaced by $P(t)$, we have

$$K\alpha^{\gamma}(t,T)\delta(t)q(t)\leq -w^{\Delta}(t)+\frac{P(t)}{\delta^{\sigma}(t)}w^{\sigma}(t)-\frac{\gamma\delta(t)}{r^{\frac{1}{\gamma}}(t)(\delta^{\sigma}(t))^{\lambda}}(w^{\sigma}(t))^{\lambda}. \tag{2.20}$$

Multiplying both sides of (2.20), with t replaced by s, by $H\left(t,s\right)$, integrating with respect to s from T to t, $t\geq T$,

$$\int_T^t H\left(t,s\right)K\alpha^{\gamma}(s,T)\delta(s)q(s)\Delta s\leq -\int_T^t H\left(t,s\right)w^{\Delta}\left(s\right)\Delta s$$

$$+\int_T^t H\left(t,s\right)\frac{P(s)}{\delta^{\sigma}\left(s\right)}w^{\sigma}\left(s\right)\Delta s-\int_T^t H\left(t,s\right)\frac{\gamma\delta(s)}{r^{\frac{1}{\gamma}}(s)(\delta^{\sigma}(s))^{\lambda}}(w^{\sigma}(s))^{\lambda}\Delta s.$$

Integrating by parts and using (2.17) and (2.18), we obtain

$$\int_T^t H(t,s)\,K\alpha^\gamma(s,T)\delta(s)q(s)\Delta s$$

$$\leq \quad H(t,T)\,w(T) + \int_T^t H^{\Delta_s}(t,s)\,w^\sigma(s)\,\Delta s$$

$$+\int_T^t H(t,s)\,\frac{P(s)}{\delta^\sigma(s)} w^\sigma(s)\,\Delta s - \int_T^t H(t,s)\,\frac{\gamma\delta(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda \Delta s$$

$$\leq H(t,T)\,w(T)$$

$$+\int_T^t \left[-\frac{h(t,s)}{\delta^\sigma(s)}\left(H(t,s)\right)^{\frac{1}{\lambda}} w^\sigma(s) - H(t,s)\,\frac{\gamma\delta(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda\right]\Delta s$$

$$\leq H(t,T)\,w(T)$$

$$+\int_T^t \left[\frac{h_-(t,s)}{\delta^\sigma(s)}\left(H(t,s)\right)^{\frac{1}{\lambda}} w^\sigma(s) - H(t,s)\,\frac{\gamma\delta(s)}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}(w^\sigma(s))^\lambda\right]\Delta s. \tag{2.21}$$

Again, define $A > 0$ and $B > 0$ by

$$A^\lambda := \frac{\gamma H(t,s)\,\delta(s)(w^\sigma(s))^\lambda}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda}, \quad B^{\lambda-1} := \frac{h_-(t,s)\,r^{\frac{1}{\gamma+1}}(s)}{\lambda\left(\gamma\delta(s)\right)^{\frac{1}{\lambda}}},$$

and using the inequality (2.14), we obtain

$$\frac{h_-(t,s)}{\delta^\sigma(s)}\left(H(t,s)\right)^{\frac{1}{\lambda}} w^\sigma(s) - \frac{\gamma H(t,s)\,\delta(s)(w^\sigma(s))^\lambda}{r^{\frac{1}{\gamma}}(s)(\delta^\sigma(s))^\lambda} = \lambda A B^{\lambda-1} - A^\lambda$$

$$\leq (\lambda-1)B^\lambda = \frac{h_-^{\gamma+1}(t,s)\,r(s)}{(\gamma+1)^{\gamma+1}\delta^\gamma(t)}.$$

¿From this last inequality and (2.21), we have

$$\int_T^t \left[K\alpha^\gamma(s,T)\,\delta(s)\,q(s)\,H(t,s) - \frac{\left(h_-(t,s)\right)^{\gamma+1} r(s)}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right]\Delta s \leq H(t,T)\,w(T),$$

and this implies that

$$\frac{1}{H(t,T)}\int_T^t \left[K\alpha^\gamma(s,T)\,\delta(s)\,q(s)\,H(t,s) - \frac{\left(h_-(t,s)\right)^{\gamma+1} r(s)}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right]\Delta s \leq w(T),$$

which contradicts assumption (2.19). This completes the proof. □

Also, by Lemma 2.3, we obtain another oscillation criterion for the equation (1.1) as in Theorems 2.1 and 2.2 and Corollaries 2.1 and 2.2 as follows.

Corollary 2.3. *Assume that* (2.4) *and* (2.7) *hold. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Corollary 2.4. *Assume that* (2.4) *and* (2.15) *hold. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Corollary 2.5. *Assume that* (2.4) *and* (2.16) *hold. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Corollary 2.6. *Assume that* (2.4), (2.17), (2.18) *and* (2.19) *hold. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Remark 2.1. Note that conditions (1.7) and (1.8) are not assumed to hold and condition (1.9) is also not necessary in order that (1.1) be oscillatory, in contrast to the results of [**6**] and [**7**].

We introduce the following notation, for all sufficiently large T,

$$p_* := \liminf_{t\to\infty} \frac{t^\gamma}{r(t)} \int_{\sigma(t)}^{\infty} Q_*(s)\Delta s, \quad q_* := \liminf_{t\to\infty} \frac{1}{t}\int_T^t \frac{s^{\gamma+1}}{r(s)} Q_*(s)\Delta s,$$

$$r_* := \liminf_{t\to\infty} \frac{t^\gamma w^\sigma(t)}{r\,(t)}, \quad R_* := \limsup_{t\to\infty} \frac{t^\gamma w^\sigma(t)}{r(t)},$$

where $Q_*(t) = K\alpha^\gamma(t,T)q\,(t)$, and assume that $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}$. Note that $0 \le l \le 1$. In order for the definition of p_* to make sense we assume that

$$\int_{t_0}^{\infty} Q_*(s)\Delta s < \infty. \tag{2.22}$$

Theorem 2.3. *Assume* (2.1) *holds and* $r(t)$ *is a (delta) differentiable function with* $r^\Delta(t) \ge 0$ *and* (2.22) *holds. Furthermore, assume* $l > 0$ *and*

$$p_* > \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{2.23}$$

or

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.24}$$

Then every solution of equation (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemmas 2.1-2.2 on $[T,\infty)_{\mathbb{T}}$ with $x(\tau(t)) > 0$ on $[T,\infty)_{\mathbb{T}}$. In particular, we have

$$x(\tau(t)) > 0, \; x^\Delta(t) > 0, \; (r(t)\left(x^\Delta(t)\right)^\gamma)^\Delta < 0, \quad \text{for } t \ge T.$$

We define $w\,(t)$ also, as in Theorem 2.1 by putting $\delta(t) = 1$. Note in this case $P_+(t) = 0$. From (2.13), we have

$$w^\Delta(t) \le -Q_*(t) - \frac{\gamma}{r^{\frac{1}{\gamma}}(t)}(w^\sigma(t))^{\frac{\gamma+1}{\gamma}} \le 0, \quad \text{for} \quad t \in [T,\infty)_{\mathbb{T}}. \tag{2.25}$$

First, we assume (2.23) holds. It follows from (2.8) and $r(t)\left(x^\Delta(t)\right)^\gamma$ is strictly decreasing that

$$w(t) = r(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\gamma < \left(\int_{t_0}^t \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\right)^{-\gamma}, \quad \text{for} \quad t \in [T,\infty)_{\mathbb{T}}.$$

Since (2.1) implies $\int_{t_0}^t \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \infty$, we have that $\lim_{t\to\infty} w(t) = 0$. Integrating (2.25) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^\sigma(t) \ge \int_{\sigma(t)}^{\infty} Q_*(s)\Delta s + \gamma\int_{\sigma(t)}^{\infty} \frac{(w^\sigma(s))^{\frac{1}{\gamma}} w^\sigma(s)}{r^{\frac{1}{\gamma}}(s)}\Delta s. \tag{2.26}$$

It follows from (2.26) that

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \geq \frac{t^\gamma}{r(t)}\int_{\sigma(t)}^\infty Q_*(s)\Delta s + \gamma\frac{t^\gamma}{r(t)}\int_{\sigma(t)}^\infty \frac{(w^\sigma(s))^{\frac{1}{\gamma}} w^\sigma(s)}{r^{\frac{1}{\gamma}}(s)}\Delta s. \tag{2.27}$$

Let $\epsilon > 0$, then by the definition of p_* and r_* we can pick $t_1 \in [T,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$\frac{t^\gamma}{r(t)}\int_{\sigma(t)}^\infty Q_*(s)\Delta s \geq p_* - \epsilon, \quad \text{and} \quad \frac{t^\gamma w^\sigma(t)}{r(t)} \geq r_* - \epsilon, \tag{2.28}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. From (2.27) and (2.28) and using the fact $r^\Delta(t) \geq 0$, we get that

$$\begin{aligned}\frac{t^\gamma w^\sigma(t)}{r(t)} &\geq (p_*-\epsilon) + \gamma\frac{t^\gamma}{r(t)}\int_{\sigma(t)}^\infty \frac{s\,(w^\sigma(s))^{\frac{1}{\gamma}}\, s^\gamma w^\sigma(s)}{s^{\gamma+1} r^{\frac{1}{\gamma}}(s)}\Delta s \\ &\geq (p_*-\epsilon) + (r_*-\epsilon)^{1+\frac{1}{\gamma}}\,\frac{t^\gamma}{r(t)}\int_{\sigma(t)}^\infty \frac{\gamma r(s)}{s^{\gamma+1}}\Delta s \\ &\geq (p_*-\epsilon) + (r_*-\epsilon)^{1+\frac{1}{\gamma}}\, t^\gamma \int_{\sigma(t)}^\infty \frac{\gamma}{s^{\gamma+1}}\Delta s. \end{aligned} \tag{2.29}$$

Using the Pötzsche chain rule ([**4**, Theorem 1.90]), we get

$$\begin{aligned}\left(\frac{-1}{s^\gamma}\right)^\Delta &= \gamma\int_0^1 \frac{1}{[s+h\mu(s)]^{\gamma+1}}dh \\ &\leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right) dh \\ &= \frac{\gamma}{s^{\gamma+1}}. \end{aligned} \tag{2.30}$$

Then from (2.29) and (2.30), we have

$$\frac{t^\gamma w^\sigma(t)}{r(t)} \geq (p_*-\epsilon) + (r_*-\epsilon)^{1+\frac{1}{\gamma}}\left(\frac{t}{\sigma(t)}\right)^\gamma.$$

Taking the lim inf of both sides as $t \to \infty$ we get that

$$r_* \geq p_* - \epsilon + (r_*-\epsilon)^{1+\frac{1}{\gamma}}\, l^\gamma.$$

Since $\epsilon > 0$ is arbitrary, we get

$$p_* \leq r_* - r_*^{1+\frac{1}{\gamma}}\, l^\gamma. \tag{2.31}$$

Using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}$$

with $B = 1$ and $A = l^\gamma$ we get that

$$p_* \leq \frac{\gamma^\gamma}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.23). Next, we assume (2.24) holds. Multiplying both sides of (2.25) by $\frac{t^{\gamma+1}}{r(t)}$, and integrating from T to t ($t \geq T$) we get

$$\int_T^t \frac{s^{\gamma+1}}{r(s)} w^\Delta(s)\Delta s \leq -\int_T^t \frac{s^{\gamma+1}}{r(s)} Q_*(s)\Delta s - \gamma\int_T^t \left(\frac{s^\gamma w^\sigma(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} + \int_T^t \left(\frac{s^{\gamma+1}}{r(s)}\right)^{\Delta} w^{\sigma}(s)\Delta s - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q_*(s)\Delta s \\ &\quad -\gamma \int_T^t \left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.\end{aligned}$$

By the quotient rule and applying the Pötzsche chain rule,

$$\begin{aligned}\left(\frac{s^{\gamma+1}}{r(s)}\right)^{\Delta} &= \frac{(s^{\gamma+1})^{\Delta}}{r^{\sigma}(s)} - \frac{s^{\gamma+1}r^{\Delta}(s)}{r(s)r^{\sigma}(s)} \\ &\leq \frac{(\gamma+1)\sigma^{\gamma}(s)}{r^{\sigma}(s)} \\ &\leq \frac{(\gamma+1)\sigma^{\gamma}(s)}{r(s)}. \qquad (2.32)\end{aligned}$$

Hence

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q_*(s)\Delta s + \int_T^t (\gamma+1)\left(\frac{\sigma^{\gamma}(s)w^{\sigma}(s)}{r(s)}\right)\Delta s \\ &\quad - \gamma \int_T^t \left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}} \Delta s.\end{aligned}$$

Let $0 < \epsilon \leq l$ be given, then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that

$$\frac{s}{\sigma(s)} \geq l - \epsilon, \quad s \geq T.$$

It follows that

$$\sigma(s) \leq Ls, \qquad s \geq T \quad \text{where} \quad L := \frac{1}{l-\epsilon} > 0.$$

We then get that

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q_*(s)\Delta s \\ &\quad + \int_T^t \{(\gamma+1)L^{\gamma}\frac{s^{\gamma}w^{\sigma}(s)}{r(s)} - \gamma\left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\frac{\gamma+1}{\gamma}}\}\Delta s.\end{aligned}$$

Let

$$u(s) := \frac{s^{\gamma}w^{\sigma}(s)}{r(s)},$$

then

$$u^{\lambda}(s) = \left(\frac{s^{\gamma}w^{\sigma}(s)}{r(s)}\right)^{\lambda}.$$

where $\lambda = \frac{\gamma+1}{\gamma}$. It follows that

$$\begin{aligned}\frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)}Q_*(s)\Delta s \\ &\quad + \int_T^t \{(\gamma+1)L^{\gamma}u(s) - \gamma u^{\lambda}(s)\}\Delta s.\end{aligned}$$

Again, using the inequality

$$Bu - Au^{\lambda} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}},$$

where A, B are constants, we get

$$\begin{aligned} \frac{t^{\gamma+1}w(t)}{r(t)} &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} Q_*(s)\Delta s \\ &+ \int_T^t \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1)L^{\gamma}]^{\gamma+1}}{\gamma^{\gamma}} \Delta s \\ &\leq \frac{T^{\gamma+1}w(T)}{r(T)} - \int_T^t \frac{s^{\gamma+1}}{r(s)} Q_*(s)\Delta s + L^{\gamma(\gamma+1)}(t-T). \end{aligned}$$

It follows from this that

$$\frac{t^{\gamma}w(t)}{r(t)} \leq \frac{\frac{T^{\gamma+1}w(T)}{r(T)}}{t} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}}{r(s)} Q_*(s)\Delta s + L^{\gamma(\gamma+1)}(1-\frac{T}{t}).$$

Since $w^{\Delta}(t) \leq 0$, we get

$$\frac{t^{\gamma}w^{\sigma}(t)}{r(t)} \leq \frac{\frac{T^{\gamma+1}w(T)}{r(T)}}{t} - \frac{1}{t}\int_T^t \frac{s^{\gamma+1}}{r(s)} Q_*(s)\Delta s + L^{\gamma(\gamma+1)}(1-\frac{T}{t}).$$

Taking the lim sup of both sides as $t \to \infty$ we obtain

$$R_* \leq -q_* + L^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\epsilon)^{\gamma(\gamma+1)}}.$$

Since $\epsilon > 0$ is arbitrary, we get that

$$R_* \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Using this and the inequality (2.31) we get

$$p_* \leq r_* - l^{\gamma} r_*^{1+\frac{1}{\gamma}} \leq r_* \leq R_* \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Therefore

$$p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}}$$

which contradicts (2.24). □

Remark 2.2. Note that condition (1.8) is not assumed to hold, in contrast to the results of [**6**] and [**7**].

3. Examples

In this section, we give some examples to illustrate our main results.

Example 3.1. Consider the nonlinear delay dynamic equation

$$\left(\frac{t^{\gamma-1}}{a(t)}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{(\sigma(t))^{\gamma-1}}{ta^{\sigma}(t)}\left(x^{\Delta\sigma}(t)\right)^{\gamma} + \frac{1}{\alpha^{\gamma}(t,t_0)\,t^2} x^{\gamma}(\tau(t)) = 0, \tag{3.1}$$

where γ is the quotient of odd positive integers and $a(t) = e_{\frac{1}{t}}(t, t_0)$. Here $p(t) = \frac{(\sigma(t))^{\gamma-1}}{ta^\sigma(t)}$, $q(t) = \frac{1}{\alpha^\gamma(t,t_0)t^2}$ and $r(t) = \frac{t^{\gamma-1}}{a(t)}$, then, it is clear that $P(t) = 0$ and the condition (2.1) holds since

$$\int_{t_0}^{\infty} \frac{\Delta t}{R^{\frac{1}{\gamma}}(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}}} = \infty,$$

by Example 5.60 in [**5**]. Also

$$\begin{aligned} &\limsup_{t\to\infty} \int_T^t \left[Ks\alpha^\gamma(s,T)\, q(s) - \frac{r(s)(P_+(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^\gamma} \right] \Delta s \\ = \;& K \limsup_{t\to\infty} \int_{t_0}^t \frac{\Delta s}{s} = \infty, \end{aligned}$$

since $\int_{t_0}^\infty \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty$ implies $\lim_{t\to\infty} \frac{\alpha(s,T)}{\alpha(t,t_0)} = 1$. Then by Corollary 2.2, every solution of (3.1) is oscillatory.

Example 3.2. Consider the nonlinear delay dynamic equation

$$\left(\frac{(t\sigma(t))^\gamma}{a(t)} \left(x^\Delta(t) \right)^\gamma \right)^\Delta + \frac{(\sigma(t))^{\gamma-1}}{t} \left(x^{\Delta\sigma}(t) \right)^\gamma + \frac{t^\gamma}{\alpha^\gamma(t,t_0)} x^\gamma(\tau(t)) = 0, \tag{3.2}$$

where $0 < \gamma \le 1$ is the quotient of odd positive integers, $a(t) = e_{\frac{p(t)}{r^\sigma(t)}}(t, t_0)$ and we assume

$$\int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}}\sigma(t)} = \infty, \quad \text{for } \; 0 < \gamma \le 1, \tag{3.3}$$

for those time scales $[t_0, \infty)_{\mathbb{T}}$, $t_0 > 0$. This holds for many time scales, for example when $\mathbb{T} = q^{\mathbb{N}_0} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$. It is clear $r(t)$ satisfies

$$\int_{t_0}^{\infty} \frac{\Delta t}{R^{\frac{1}{\gamma}}(t)} \le \int_{t_0}^{\infty} \frac{1}{t\sigma(t)} \Delta t = \int_{t_0}^{\infty} \left(\frac{-1}{t} \right)^\Delta \Delta t < \infty.$$

To see that (2.3) holds note that

$$\begin{aligned} \int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^t Q(s)\beta^\gamma(s)\Delta s \right]^{\frac{1}{\gamma}} \Delta t \;&\ge\; \int_{t_0}^{\infty} \left[\frac{1}{(t\sigma(t))^\gamma} \int_{t_0}^t \left(\frac{s\beta(s)}{\alpha(s,t_0)} \right)^\gamma \Delta s \right]^{\frac{1}{\gamma}} \Delta t \\ &\ge\; \int_{t_0}^{\infty} \left[\frac{t-t_0}{(t\sigma(t))^\gamma} \right]^{\frac{1}{\gamma}} \Delta t, \end{aligned}$$

since

$$\beta(t) = \int_{\tau(t)}^{\infty} \frac{\Delta s}{R^{\frac{1}{\gamma}}(s)} = \int_{\tau(t)}^{\infty} \frac{1}{s\sigma(s)} = \int_{\tau(t)}^{\infty} \left(\frac{-1}{s} \right)^\Delta \Delta s = \frac{1}{\tau(t)} \ge \frac{1}{t} \ge \frac{\alpha(t,t_0)}{t}.$$

We can find $0 < k < 1$ such that $t - t_0 > kt$, for $t \ge t_k > t_0$. Therefore, we get

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^t Q(s)\beta^\gamma(s)\Delta s \right]^{\frac{1}{\gamma}} \Delta t > k^{\frac{1}{\gamma}} \int_{t_K}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}}\sigma(t)} \overset{(3.3)}{=} \infty.$$

To apply Corollary 2.1, it remains to prove that condition (2.15) holds, then

$$\int_T^{\infty} \alpha^\gamma(t,T)\, q(t)\Delta t = \int_{t_0}^{\infty} t^\gamma \Delta t = \infty,$$

where we use, as in Example 3.1, $\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty$ implies $\lim_{t\to\infty} \frac{\alpha(s,T)}{\alpha(t,t_0)} = 1$. We conclude that if $[t_0,\infty)_{\mathbb{T}}$, $t_0 > 0$ is a time scale where $\int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}}\sigma(t)} = \infty$, then, by Corollary 2.1, every solution of (3.2) is oscillatory.

EXAMPLE 3.3. Consider the nonlinear dynamic equation

$$\left(\frac{t^{\gamma-1}}{a(t)}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{(\gamma-1)\left(\sigma(t)\right)^{\gamma-1}}{ta^{\sigma}(t)}\left(x^{\Delta\sigma}(t)\right)^{\gamma} + \frac{\eta}{\alpha^{\gamma}(t,t_0)\,t^2}x^{\gamma}(\tau(t)) = 0, \tag{3.4}$$

where $\gamma \geq 1$ is the quotient of odd positive integers, η is a positive constant and $a(t) = e_{\frac{p(t)}{r^{\sigma}(t)}}(t,t_0)$. Here $r(t) = \frac{t^{\gamma-1}}{a(t)}$, $p(t) = \frac{(\gamma-1)(\sigma(t))^{\gamma-1}}{ta^{\sigma}(t)}$ and $q(t) = \frac{\eta}{\alpha^{\gamma}(t,t_0)t^2}$. Note that

$$\int_{t_0}^{\infty} \frac{\Delta t}{R^{\frac{1}{\gamma}}(t)} = \int_{t_0}^{\infty} \frac{\Delta t}{t^{1-\frac{1}{\gamma}}} = \infty,$$

$$\begin{aligned} r^{\Delta}(t) &= \frac{1}{a(t)a^{\sigma}(t)}\left((t^{\gamma-1})^{\Delta}a(t) - (\gamma-1)t^{\gamma-2}a(t)\right) \\ &\geq \frac{1}{a^{\sigma}(t)}\left((\gamma-1)t^{\gamma-2} - (\gamma-1)t^{\gamma-2}\right) = 0, \end{aligned}$$

and, as in Example 3.1,

$$\begin{aligned} p_* &= \liminf_{t\to\infty} \frac{t^{\gamma}}{r(t)}\int_{\sigma(t)}^{\infty} K\alpha^{\gamma}(s,T)q(s)\,\Delta s \\ &= \eta K \liminf_{t\to\infty} ta(t)\int_{\sigma(t)}^{\infty} \frac{\alpha^{\gamma}(s,T)}{\alpha^{\gamma}(s,t_0)}\frac{1}{s^2}\Delta s \\ &\geq \eta K \liminf_{t\to\infty} t\int_{\sigma(t)}^{\infty} \frac{\Delta s}{s^2} \\ &\geq \eta K \liminf_{t\to\infty} t\int_{\sigma(t)}^{\infty} \frac{\Delta s}{s\sigma(s)} = \eta l K > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \end{aligned}$$

if $\eta > \frac{\gamma^{\gamma}}{l^{\gamma^2-1}K(\gamma+1)^{\gamma+1}}$. Then, by Theorem 2.3, we get that (3.4) is oscillatory if $\eta > \frac{\gamma^{\gamma}}{l^{\gamma^2-1}K(\gamma+1)^{\gamma+1}}$.

Additional examples may be readily given. We leave this to interested reader.

4. Applications

In this section, we apply the oscillation criteria to different types of time scales, for example if $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and (1.1) becomes the nonlinear damped delay differential equation

$$(r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + q(t)f\left(x(\tau(t))\right) = 0. \tag{4.1}$$

then we have from Theorems 2.1-2.3 and Corollaries 2.1-2.6 the following oscillation criteria for equation (4.1).

THEOREM 4.1. *Assume one of the conditions*

$$\int_{t_0}^{\infty} \frac{dt}{R^{\frac{1}{\gamma}}(t)} = \infty, \tag{4.2}$$

or

$$\int_{t_0}^{\infty}\left[\frac{1}{R(t)}\int_{t_0}^{t}Q(s)\beta^{\gamma}(s)ds\right]^{\frac{1}{\gamma}}dt=\infty, \tag{4.3}$$

holds. Furthermore, suppose that there exists a positive differentiable function $\delta(t)$ *such that, for all sufficiently large* T,

$$\limsup_{t\to\infty}\int_{T}^{t}\left[K\alpha^{\gamma}\left(s,T\right)\delta(s)q(s)-\frac{r(s)(P_{+}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]ds=\infty. \tag{4.4}$$

Then every solution of equation (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Corollary 4.1. *Assume one of the conditions* (4.2) *or* (4.3) *holds and, for all sufficiently large* T,

$$\int_{T}^{\infty}\alpha^{\gamma}\left(t,T\right)q(t)dt=\infty. \tag{4.5}$$

Then every solution of equation (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Corollary 4.2. *Assume one of the conditions* (4.2) *or* (4.3) *holds and, for all sufficiently large* T,

$$\limsup_{t\to\infty}\int_{T}^{t}\left[Ks\alpha^{\gamma}\left(s,T\right)q(s)-\frac{r(s)(P_{+}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]ds=\infty. \tag{4.6}$$

Then every solution of equation (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Theorem 4.2. *Assume one of the conditions* (4.2) *or* (4.3) *holds. Furthermore, suppose that there exist functions* $H,h\in C\left(\mathbb{D},\mathbb{R}\right)$, *where* $\mathbb{D}\equiv\{(t,s):t\geq s\geq t_0\}$ *such that*

$$H\left(t,t\right)=0,\quad t\geq t_0,\ H\left(t,s\right)>0,\quad t>s\geq t_0, \tag{4.7}$$

$$\frac{\partial H\left(t,s\right)}{\partial s}+H\left(t,s\right)\frac{P(s)}{\delta^{\sigma}\left(s\right)}=-\frac{h\left(t,s\right)}{\delta^{\sigma}\left(s\right)}\left(H\left(t,s\right)\right)^{\frac{\gamma}{\gamma+1}}, \tag{4.8}$$

and, for all sufficiently large T,

$$\limsup_{t\to\infty}\frac{1}{H\left(t,T\right)}\int_{T}^{t}\left[K\alpha^{\gamma}\left(s,T\right)\delta\left(s\right)q\left(s\right)H\left(t,s\right)-\frac{\left(h_{-}\left(t,s\right)\right)^{\gamma+1}r\left(s\right)}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]ds=\infty, \tag{4.9}$$

Then every solution of equation (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Corollary 4.3. *Assume that* (4.4) *holds and*

$$\int_{t_0}^{\infty}\left[\frac{1}{R(t)}\int_{t_0}^{t}Q(s)ds\right]^{\frac{1}{\gamma}}dt=\infty, \tag{4.10}$$

Then every solution of equation (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Corollary 4.4. *Assume that* (4.5) *and* (4.10) *hold. Then every solution of equation* (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Corollary 4.5. *Assume that* (4.6) *and* (4.10) *hold. Then every solution of equation* (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

Corollary 4.6. *Assume that* (4.7), (4.8), (4.9) *and* (4.10) *hold. Then every solution of equation* (4.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

THEOREM 4.3. *Assume* (4.2) *holds and* $r(t)$ *is differentiable function with* $r'(t) \geq 0$ *and*

$$\int_{t_0}^{\infty} \alpha^{\gamma}(t) q(t)\, dt < \infty$$

hold. Furthermore, assume that

$$p_* > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}},$$

or

$$p_* + q_* > 1,$$

where, for all sufficiently large T,

$$p_* := \liminf_{t\to\infty} \frac{Kt^{\gamma}}{r(t)} \int_t^{\infty} \alpha^{\gamma}(s) q(s)\, ds, \quad q_* := \liminf_{t\to\infty} \frac{K}{t} \int_T^t \frac{s^{\gamma+1}}{r(s)} \alpha^{\gamma}(s) q(s)\, ds.$$

Then every solution of equation (4.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the nonlinear damped delay difference equation

$$\Delta(r(t)(\Delta x(t))^{\gamma}) + p(t)(\Delta x(t+1))^{\gamma} + q(t) f(x(\tau(t))) = 0. \tag{4.11}$$

then we have from Theorems 2.1-2.3 and Corollaries 2.1-2.6 the following oscillation criteria for equation (4.11).

THEOREM 4.4. *Assume one of the conditions*

$$\sum_{t=t_0}^{\infty} \frac{1}{R^{\frac{1}{\gamma}}(t)} = \infty, \tag{4.12}$$

or

$$\sum_{t=t_0}^{\infty} \left[\frac{1}{R(t)} \sum_{s=t_0}^{t-1} Q(s)\beta^{\gamma}(s)\right]^{\frac{1}{\gamma}} = \infty, \tag{4.13}$$

holds. Furthermore, suppose that there exists a sequence $\delta(t)$ *such that, for all sufficiently large* N,

$$\limsup_{t\to\infty} \sum_{s=N}^{t-1} \left[K\alpha^{\gamma}(s,N)\,\delta(s) q(s) - \frac{r(s)(P_+(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right] = \infty. \tag{4.14}$$

Then every solution of equation (4.11) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

COROLLARY 4.7. *Assume one of the conditions* (4.12) *or* (4.13) *holds and, for all sufficiently large* N,

$$\sum_{t=N}^{\infty} \alpha^{\gamma}(t,N)\, q(t) = \infty. \tag{4.15}$$

Then every solution of equation (4.11) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

COROLLARY 4.8. *Assume one of the conditions* (4.2) *or* (4.3) *holds and, for all sufficiently large* N,

$$\limsup_{t\to\infty} \sum_{s=N}^{t-1} \left[Ks\alpha^{\gamma}(s,N)\, q(s) - \frac{r(s)(P_+(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right] = \infty. \tag{4.16}$$

Then every solution of equation (4.11) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

THEOREM 4.5. *Assume one of the conditions* (4.12) *or* (4.13) *holds. Furthermore, suppose that there exist two sequences* H, h *on* $\mathbb{D}$, *where* $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$ *such that*

$$H(t,t) = 0, \quad t \geq t_0, \ H(t,s) > 0, \quad t > s \geq t_0, \tag{4.17}$$

$$\Delta_s H(t,s) + H(t,s)\frac{P(s)}{\delta^\sigma(s)} = -\frac{h(t,s)}{\delta^\sigma(s)}(H(t,s))^{\frac{\gamma}{\gamma+1}}, \tag{4.18}$$

and, for all sufficiently large N,
(4.19)

$$\limsup_{t\to\infty}\frac{1}{H(t,N)}\sum_{s=N}^{t-1}\left[K\alpha^\gamma(s,N)\,\delta(s)\,q(s)\,H(t,s) - \frac{(h_-(t,s))^{\gamma+1}r(s)}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right] = \infty,$$

Then every solution of equation (4.11) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

COROLLARY 4.9. *Assume that* (4.14) *holds and*

$$\sum_{t=t_0}^{\infty}\left[\frac{1}{R(t)}\sum_{s=t_0}^{t-1}Q(s)\right]^{\frac{1}{\gamma}} = \infty, \tag{4.20}$$

Then every solution of equation (4.11) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

COROLLARY 4.10. *Assume that* (4.15) *and* (4.20) *hold. Then every solution of equation* (4.11) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

COROLLARY 4.11. *Assume that* (4.16) *and* (4.20) *hold. Then every solution of equation* (4.11) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

COROLLARY 4.12. *Assume that* (4.17), (4.18), (4.19) *and* (4.20) *hold. Then every solution of equation* (4.11) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$ *or tends to zero.*

THEOREM 4.6. *Assume* (4.12) *holds,* $\Delta r(t) \geq 0$ *and*

$$\sum_{t=t_0}^{\infty}\alpha^\gamma(t)q(t) < \infty$$

hold. Furthermore, assume that

$$p_* > \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

or

$$p_* + q_* > 1,$$

where, for all sufficiently large N,

$$p_* := \liminf_{t\to\infty}\frac{Kt^\gamma}{r(t)}\sum_{t=t+1}^{\infty}\alpha^\gamma(s)q(s), \quad q_* := \liminf_{t\to\infty}\frac{K}{t}\sum_{t=N}^{t-1}\frac{s^{\gamma+1}}{r(s)}\alpha^\gamma(s)q(s).$$

Then every solution of equation (4.11) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Similarly, we can state oscillation criteria for many other time scales, e.g., $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, or $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ where H_n is the so-called n-th harmonic number defined by $H_0 = 0$, $H_n = \sum_{k=1}^n \frac{1}{k}$, $n \in \mathbb{N}_0$.

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