Kamenev-type Oscillation Criteria for Second-Order Linear Delay Dynamic Equations

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This paper is dedicated to our good friend John Baxley.

Abstract

In this paper we will establish Kamenev-type criteria for oscillation of the second order delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{T},$$

where $\mathbb{T}$ is a time scale. Our results are not only new for differential and difference equations, but are also new for the generalized difference and q-difference equations and many other dynamic equations on time scales. Our results are new for delay equations and extend some recent results of Medico and Kong. An example is given to illustrate the main results.

Key words: *delay equations, time scales, oscillation*
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1 Introduction

A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Phd. Thesis in 1988 in order to unify continuous and discrete analysis (see

[16]). Not only can this theory of the so-called "dynamic equations on time scales" unify the theories of differential equations and difference equations, but also extends these classical cases to situations "in between", e.g., to the so-called q-difference equations.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and difference equations respectively. Many other interesting time scales exist, and they give rise to a number of applications, among them the study of population dynamic models which are continuous while in season (and may follow a difference scheme with variable step-size), die out in say winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [5]). Since Stefen Hilger introduced the time scale calculus, several authors have expounded on various aspects of the new theory, see the paper by Agarwal et al. [1] and the references cited therein. A book on the subject of time scales, or more generally measure chains, by Bohner and Peterson [5] summarizes and organizes much of time scale calculus. Many of these results that we use in this paper will be summarized in Section 2. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales (we refer the reader to the papers [2-4, 6-15, 24, 35]).

In this paper we will be concerned with the second-order linear delay dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{T}, \tag{1}$$

on a time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where the functions r, q are $rd-$continuous positive functions, and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$. Throughout this paper these assumptions will be supposed to hold. Our attention is restricted to those solutions $x(t)$ of (1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$. A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory. We consider the following two cases:

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty, \tag{2}$$

or

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} < \infty. \tag{3}$$

Došlý and Hilger [8] have considered the second order dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\sigma} = 0, \ t \in \mathbb{T}, \tag{4}$$

and have given necessary and sufficient condition for oscillation of all solutions on unbounded time scales. Unfortunately, the oscillation criteria are not completely satisfactory since additional assumptions need to be imposed on the unknown solutions.

Erbe and Peterson [12] have considered equation (4) and supposed that $r(t)$ is bounded above on $[t_0, \infty)$, $t_0 \in \mathbb{T}$, $h_0 = \inf\{\mu(t) : t \in [t_0, \infty)\} > 0$, and have used the Riccati transformation and proved that if

$$\int_{t_0}^{\infty} p(t)\Delta t = \infty, \tag{5}$$

then every solution is oscillatory in $[t_0, \infty)$. This may be regarded as a sort of Fite–Wintner criterion. It is clear that the results given in [6, 8], can not be applied when $r(t)$ is unbounded, $\mu(t) = 0$ and $p(t) = t^{-\alpha}$ with $\alpha > 1$.

Recently Saker [35] and Bohner and Saker [6] used the Riccati substitution and provided several oscillation criteria for the equation

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)(f \circ x^{\sigma}) = 0,\ t \in \mathbb{T}, \tag{6}$$

when (2) holds, and improved some of the results established in [8, 12].

Also, Erbe, Peterson and Saker [15] used the generalized Riccati transformation techniques and the generalized exponential function and obtained some different oscillation criteria for (6) on time scales, and applied these results to the linear dynamic equations with damping terms to give some sufficient condition for oscillation. Also, for oscillation of second order dynamic equations of Emden-Fowler type we refer to the results in [7].

In the case when $\mathbb{T} = \mathbb{R}$, equation (4) reduces to the second order linear differential equation

$$(r(t)x^{'}(t))^{'} + p(t)x(t) = 0,\ t \in [t_0, \infty]. \tag{7}$$

Numerous oscillation and nonoscillation criteria have been established for equation (7), see for example [37], in which the authors make a survey of many of the results for this equation. It is known [17], when $r(t) = 1$, the condition

$$\lim_{t \to \infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m p(s)\, ds = \infty, \tag{8}$$

plays an important rule in the oscillation of all solutions of equation (7), where $m > 1$ is an integer. However, the condition (8) when $m = 1$ is not sufficient for the oscillation of equation (7).

In recent years, improvements of the discrete analogues of the Kamenev-type criteria have been obtained by several authors for different types of second order difference, neutral difference and partial difference equations. We refer to the results in [18, 19, 20, 22, 26-34, 36, 38-40].

We shall address the following question. Can we obtain oscillation criteria on time scales from which we are able to deduce the results for differential and difference equations and as a special case, cover criteria of the type established by Philos and others? The aim of this paper is to give a positive answer to this question in the time scales setting and also extend the results to delay equations and to the Euler equation. From this we will deduce the sharpness of the results.

The paper is organized as follows: In Section 3, we intend to use the Riccati transformation techniques to obtain some new oscillation criteria of Kamenev-type for equation (1) when (2) holds. Our results unify and extend results due to Philos [25], Medico and Kong [23], and Saker [30]. Moreover, the results in this paper are essentially new in the case when $\mathbb{T} = q^{\mathbb{N}}$, for $q > 1$, i.e., for the q-difference equations and can be extended to other time scales. Finally, in Section 3, we consider equations that satisfy (3) and present some conditions that ensure that all solutions are either oscillatory or converge to zero. An example is considered to illustrate the main results.

2 Some Preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$. We define the forward jump operator on such a time scale by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. The graininess function μ is defined by $\mu(t) := \sigma(t) - t$, for $t \in \mathbb{T}$. We define the time scale interval $[a, \infty)_{\mathbb{T}}$ by

$$[a, \infty)_{\mathbb{T}} := [a, \infty) \cap \mathbb{T}.$$

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^{\Delta}(t)$ of f at $t \in \mathbb{T}$ can be defined by (see [5, Theorem 1.16])

$$f^{\Delta}(t) = \lim_{s \to \infty} \frac{f(t) - f(s)}{t - s},$$

if $\sigma(t) = t$ (in this limit and others in this paper s just takes on values in the time scale $\mathbb{T}$) and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

if f is continuous at t and $\sigma(t) > t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and at all left-dense points left hand limits exist and are finite. If f is differentiable at t, then a useful formula (see [5, Theorem 1.16]) is

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t), \quad \text{where} \quad f^{\sigma}(t) := (f \circ \sigma)(t) = f(\sigma(t)). \tag{9}$$

Assuming f and g are delta differentiable we will make use of the product rule [5, Theorem 1.20]

$$\begin{aligned} (f(t)g(t))^{\Delta} &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) \\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)), \end{aligned} \tag{10}$$

and the quotient rule [5, Theorem 1.20]

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}, \tag{11}$$

provided $g(t)g^\sigma(t) \neq 0$. For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy (delta) integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

The integration by parts formula [5, Theorem 1.77] reads

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(t)g(t)]_a^b - \int_a^b f^\sigma(t)g^\Delta(t)\Delta t, \tag{12}$$

and we define the improper integral

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s,$$

in the standard way. A useful formula is [5, Theorem 1.75]

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t).$$

We now give some examples of what we have discussed so far. First, if $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^\Delta(t) = f'(t), \quad \text{and} \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

If $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t+1, \quad \mu(t) \equiv 1, \quad f^\Delta(t) = \Delta f(t), \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

For $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t+h$, $\mu(t) = h$,

$$f^\Delta(t) = \Delta_h f(t) := \frac{f(t+h) - f(t)}{h}, \text{ and} \int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h.$$

Finally, if $\mathbb{T} = \varrho^{\mathbb{N}_0} = \{t : t = \varrho^k, \, k \in \mathbb{N}_0\}$, where $\varrho > 1$, we have $\sigma(t) = \varrho t$, $\mu(t) = (\varrho - 1)t$

$$x^\Delta(t) = \frac{x(\varrho t) - x(t)}{(\varrho - 1)t} \text{ and } \int_a^\infty f(t)\Delta t = \sum_{k=0}^\infty f(\varrho^k)\mu(\varrho^k).$$

3 Oscillation Criteria

In this section we give some new oscillation criteria of Philos-type for equation (1).

First, we define $\Re$ by $H \in \Re$ provided $H : [a,\infty)_\mathbb{T} \times [a,\infty)_\mathbb{T} \to \mathbb{R}$ satisfies

$$H(t,t) \geq 0, \quad t \geq a, \quad H(t,s) > 0, \quad t > s \geq a,$$

$H^{\Delta_s}(t,s) \le 0$, for $t \ge s \ge a$, and for each fixed t, $H(t,s)$ is right-dense continuous with respect to s. As a simple and important example, note that if $\mathbb{T} = \mathbb{R}$, then $H(t,s) := (t-s)^n$ is in $\Re$.

In what follows it will be assumed that $r^\Delta(t) \ge 0$ and

$$\int_{t_0}^{\infty} \tau(s)p(s)\Delta s = \infty, \tag{13}$$

is satisfied.

Lemma 1 *Let* (2) *and* (13) *be satisfied, and assume that* (1) *has a positive solution x on $[t_0,\infty)_{\mathbb{T}}$. Then there exists a $T \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that*

(i) $x^\Delta(t) > 0$, $x^{\Delta\Delta}(t) < 0$, $x(t) > tx^\Delta(t)$ for $t \in [T,\infty)_{\mathbb{T}}$;

(ii) x is strictly increasing and $x(t)/t$ is strictly decreasing on $[T,\infty)_{\mathbb{T}}$.

Proof: Assume x is a positive solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0,\infty)_{\mathbb{T}}$ so that $t_1 > 0$ and so that $x(\tau(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Then, since x is a solution of (1),

$$(r(t)x^\Delta(t))^\Delta = -p(t)x(\tau(t)) < 0, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Then $r(t)x^\Delta(t)$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that $r(t)x^\Delta(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Assume not, then there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $r(t_2)x^\Delta(t_2) =: c < 0$. Then

$$r(t)x^\Delta(t) \le r(t_2)x^\Delta(t_2) = c, \quad t \in [t_2,\infty)_{\mathbb{T}},$$

and therefore

$$x^\Delta(t) \le \frac{c}{r(t)}, \quad \mathrm{t} \in [t_2,\infty)_{\mathbb{T}}.$$

Integrating, we get

$$x(t) = x(t_2) + \int_{t_2}^{t} x^\Delta(s)\Delta s \le x(t_2) + c\int_{t_2}^{t} \frac{\Delta s}{r(s)} \to -\infty \text{ as } t \to \infty,$$

which implies $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)x^\Delta(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$ and so $x^\Delta(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Since $(r(t)x^\Delta(t))^\Delta < 0$ on $[t_1,\infty)_{\mathbb{T}}$, we have

$$x^{\Delta\Delta}(t) < -\frac{r^\Delta(t)x^\Delta(t)}{r^\sigma(t)} \le 0, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Next let $X(t) := x(t) - tx^\Delta(t)$. Since $X^\Delta(t) = -\sigma(t)x^{\Delta\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, we have that $X(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. We claim there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $X(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Assume not, then $X(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^\Delta = \frac{tx^\Delta(t) - x(t)}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)} > 0, \quad t \in [t_1,\infty)_{\mathbb{T}},$$

which implies that $x(t)/t$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. Pick $t_3 \in [t_1,\infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_1)$, for $t \geq t_3$. Then

$$x(\tau(t))/\tau(t) \geq x(\tau(t_1))/\tau(t_1) =: d > 0,$$

so that $x(\tau(t)) \geq d\tau(t)$ for $t \geq t_3$. Now by integrating both sides of equation (1) from t_3 to t we have

$$r(t)x^{\Delta}(t) - r(t_3)x^{\Delta}(t_3) + \int_{t_3}^{t} p(s)x(\tau(s))\Delta s = 0,$$

which implies that

$$\begin{aligned} r(t_3)x^{\Delta}(t_3) &= r(t)x^{\Delta}(t) + \int_{t_3}^{t} p(s)x(\tau(s))\Delta s \\ &\geq \int_{t_3}^{t} p(s)x(\tau(s))\Delta s \geq d\int_{t_3}^{t} p(s)\tau(s)\Delta s, \end{aligned}$$

so using (13) we get a contradiction. Hence there is a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $X(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{X(t)}{t\sigma(t)} < 0, \quad t \in [t_2,\infty)_{\mathbb{T}}$$

and we have that $\frac{x(t)}{t}$ is strictly decreasing on $[t_2,\infty)_{\mathbb{T}}$. □

Theorem 2 *Assume that* (2) *and* (13) *hold,* $H \in \Re$*, and for* $t > s$ *let*

$$h(t,s) := -\frac{H^{\Delta_s}(t,s)}{\sqrt{H(t,s)}}. \tag{14}$$

If there exists a positive delta differentiable function δ *such that for every* $t_0 \in [a,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\left[\delta(s)p(s)\frac{\tau(s)}{s} - \frac{r(s)\left(\delta^{\sigma}(s)\right)^2}{4\delta(s)}R^2(t,s)\right]\Delta s = \infty, \tag{15}$$

where

$$R(t,s) := h(t,s)/\sqrt{H(t,s)} - \frac{b(s)}{\delta^{\sigma}(s)}, \quad b(t) := \max\{0, \delta^{\Delta}(t)\}.$$

Then every solution of equation (1) *is oscillatory on* $[a,\infty)_{\mathbb{T}}$.

Proof: Suppose to the contrary that (1) is nonoscillatory on $[a,\infty)_{\mathbb{T}}$. Then there is a solution x of (1) and a $t_1 \in [a,\infty)_{\mathbb{T}}$ such that $x(t)$ and $x(\tau(t))$ are positive on $[t_1,\infty)_{\mathbb{T}}$. Make the "Riccati" substitution

$$w(t) := \delta(t)\frac{r(t)x^{\Delta}(t)}{x(t)}, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{16}$$

Then, by Lemma 1, there is a $T \in [t_1, \infty)_{\mathbb{T}}$ such that $w(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. Using the product rule (10) we obtain

$$w^{\Delta}(t) = \left(\frac{\delta(t)}{x(t)}\right)^{\Delta} (r(t)x^{\Delta}(t))^{\sigma} + \frac{\delta(t)}{x(t)} \left(r(t)x^{\Delta}(t)\right)^{\Delta},$$

Using (1), the quotient rule (11), and (16), we obtain

$$\begin{aligned} w^{\Delta}(t) &= -\delta(t)p(t)\frac{x(\tau(t))}{x(t)} + \frac{\delta^{\Delta}(t)}{x^{\sigma}(t)}(rx^{\Delta})^{\sigma}(t) - \frac{(rx^{\Delta})^{\sigma}(t)\delta(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)} \\ &= -\delta(t)p(t)\frac{x(\tau(t))}{x(t)} + \frac{\delta^{\Delta}(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\delta(t)x^{\Delta}(t)}{x(t)\delta^{\sigma}(t)}w^{\sigma}(t). \end{aligned} \tag{17}$$

From Lemma 1, we have $x^{\Delta}(t) > 0$, $\left(r(t)x^{\Delta}(t)\right)^{\Delta} < 0$, and $\frac{x(t)}{t}$ is strictly decreasing for $t \in [T, \infty)_{\mathbb{T}}$. It follows that for $t \in [T, \infty)_{\mathbb{T}}$

$$x(\sigma(t)) \geq x(t), \quad (rx)^{\Delta}(t) \geq (rx^{\Delta})^{\sigma}(t), \quad \frac{x(\tau(t))}{x(t)} \geq \frac{\tau(t)}{t} \tag{18}$$

(for the last inequality we might have to choose T larger so that this inequality is true). Applying the inequalities (18) to (17), we have (using $b(t) \geq 0$)

$$\begin{aligned} w^{\Delta}(t) &\leq -\delta(t)p(t)\frac{\tau(t)}{t} + \frac{b(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\delta(t)r(t)x^{\Delta}(t)}{r(t)x(t)\delta^{\sigma}(t)}w^{\sigma}(t), \\ &\leq -\delta(t)p(t)\frac{\tau(t)}{t} + \frac{b(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\delta(t)((rx)^{\Delta})^{\sigma}(t)}{r(t)x^{\sigma}(t)\delta^{\sigma}(t)}w^{\sigma}(t) \\ &\leq -\delta(t)p(t)\frac{\tau(t)}{t} + \frac{b(t)}{\delta^{\sigma}(t)}w^{\sigma}(t) - \frac{\delta(t)}{r(t)\left(\delta^{\sigma}\right)^{2}(t)}\left(w^{\sigma}\right)^{2}(t) \end{aligned} \tag{19}$$

for $t \in [T, \infty)_{\mathbb{T}}$.

From (19), it follows that

$$\begin{aligned} &\int_{T}^{t} H(t,s)\delta(s)p(s)\frac{\tau(s)}{s}\Delta s \leq -\int_{T}^{t} H(t,s)w^{\Delta}(s)\Delta s \\ &+\int_{T}^{t} H(t,s)\frac{b(s)}{\delta^{\sigma}(s)}w^{\sigma}(s)\Delta s - \int_{T}^{t} H(t,s)\frac{\delta(s)}{r(s)\left(\delta^{\sigma}\right)^{2}(s)}(w^{\sigma})^{2}(s)\Delta s. \end{aligned} \tag{20}$$

Using the integration by parts formula (12), we have

$$\begin{aligned} \int_{T}^{t} H(t,s)w^{\Delta}(s)\Delta s &= H(t,s)w(s)|_{s=T}^{s=t} - \int_{T}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \\ &= H(t,t)w(t) - H(t,T)w(T) - \int_{T}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s \\ &\geq -H(t,T)w(T) - \int_{T}^{t} H^{\Delta_s}(t,s)w^{\sigma}(s)\Delta s. \end{aligned} \tag{21}$$

From (21) and (20) and using (14) we get

$$\int_T^t H(t,s)\delta(s)p(s)\frac{\tau(s)}{s}\Delta s \le H(t,T)w(T) - \int_T^t h(t,s)\sqrt{H(t,s)}w^\sigma(s)\Delta s$$
$$+\int_T^t H(t,s)\frac{b(s)}{\delta^\sigma(s)}w^\sigma(s)\Delta s - \int_T^t H(t,s)\frac{\delta(s)}{r(s)\left(\delta^\sigma\right)^2(s)}(w^\sigma)^2(s)\Delta s. \tag{22}$$

Hence,

$$\begin{aligned}
& \int_T^t H(t,s)\delta(s)p(s)\frac{\tau(s)}{s}\Delta s \\
\le\ & H(t,T)w(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)} - H(t,s)\frac{b(s)}{\delta^\sigma(s)}\right] w^\sigma(s)\Delta s \\
& - \int_T^t H(t,s)\frac{\delta(s)}{r(s)\left(\delta^\sigma\right)^2(s)}(w^\sigma)^2(s)\Delta s.
\end{aligned} \tag{23}$$

Therefore, completing the square,

$$\begin{aligned}
& \int_T^t H(t,s)\delta(s)p(s)\frac{\tau(s)}{s}\Delta s \le H(t,T)w(T) \\
& - \int_T^t \left[\sqrt{\frac{H\delta}{r\left(\delta^\sigma\right)^2}}\, w^\sigma + \frac{\left[h\sqrt{H} - H\frac{b}{\delta^\sigma}\right]}{2\sqrt{\frac{H\delta}{r(\delta^\sigma)^2}}}\right]^2 \Delta s \\
& + \int_T^t H(t,s)\frac{r(s)\left(\delta^\sigma\right)^2(s)}{4\delta(s)}\left[h(t,s)/\sqrt{H(t,s)} - \frac{b(s)}{\delta^\sigma(s)}\right]^2 \Delta s.
\end{aligned} \tag{24}$$

Then, for all $t \ge T$ we have

$$\begin{aligned}
& \int_T^t H(t,s)\left[\delta(s)p(s)\frac{\tau(s)}{s} - \frac{r(s)\left(\delta^\sigma\right)^2(s)}{4\delta(s)}R^2(t,s)\right]\Delta s \\
\le\ & H(t,T)w(T),
\end{aligned} \tag{25}$$

and this implies that

$$\begin{aligned}
& \frac{1}{H(t,T)}\int_T^t H(t,s)\left[\delta(s)p(s)\frac{\tau(s)}{s} - \frac{r(s)\left(\delta^\sigma\right)^2(s)}{4\delta(s)}R^2(t,s)\right]\Delta s \\
\le\ & w(T),
\end{aligned} \tag{26}$$

for all large t, which contradicts (15). Therefore every solution of (1) oscillates on $[t_0,\infty)_{\mathbb{T}}$.
□

As an immediate consequence of Theorem 2 we get the following.

Corollary 3 *Let the assumption* (15) *in Theorem 2 be replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\delta(s)p(s)\frac{\tau(s)}{s}\Delta s = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\frac{r(s)\left(\delta^{\sigma}\right)^2(s)}{4\delta(s)}R^2(t,s)\Delta s < \infty,$$

for all $t_0 \in [a,\infty)_{\mathbb{T}}$ sufficiently large. Then every solution of equation (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

From Theorem 2 we can derive some oscillation criteria for equation (1) on different types of time scales.

If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, $\mu(t)\equiv 0$, $\delta^{\Delta}=\delta'$ and $H^{\Delta_s}(t,s)=\partial H(t,s)/\partial s$. Let $\tau(t)=t-\beta$, where β is a positive constant. Then (15) becomes

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\left[\delta(s)p(s)\frac{s-\beta}{s} - \frac{\delta(s)r(s)A^2(t,s)}{4}\right] ds = \infty, \tag{27}$$

where

$$A(t,s) := h(t,s)/\sqrt{H(t,s)} - \frac{b(s)}{\delta(s)}, \quad b(s) := \max\{0,\delta'(s)\}.$$

Note that when $\delta(t)\equiv 1$ and $r(t)\equiv 1$, the condition (27) reduces to a result due to Philos [25].

If $\mathbb{T}=\mathbb{Z}$, then $\delta^{\Delta}(n)=\Delta\delta(n)=\delta(n+1)-\delta(n)$, $H^{\Delta_s}(m,n)=\Delta_2 H(m,n)=H(m,n+1)-H(m,n)$, and (15) becomes

$$\limsup_{m\to\infty} \frac{1}{H(m,n_0)} \sum_{n=n_0}^{m-1} H(m,n)\left[\delta(n)p(n)\frac{\tau(n)}{n} - \frac{\delta^2(n+1)r(n)}{4\delta(n)}B^2(m,n)\right] = \infty, \tag{28}$$

where

$$B(m,n) := \left(h(m,n)/\sqrt{H(m,n)} - \frac{b(n)}{\delta(n+1)}\right), \quad b(n) := \max\{0,\Delta b(n)\}.$$

If $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\sigma(t)=t+h$, $\mu(t)=h$, $\delta^{\Delta}(t)=\Delta_h\delta(t)=\frac{\delta(t+h)-\delta(t)}{h}$, $H^{\Delta_s}(t,s)=\Delta_2 H(t,s)=\frac{H(t,s+h)-H(t,s)}{h}$ and (15) becomes for t, $s\in h\mathbb{Z}$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} H(t,s)\left[\delta(s)p(s)\frac{\tau(s)}{s} - \frac{\delta^2(s+h)r(s)C^2(t,s)}{4\delta(s)}\right] = \infty, \tag{29}$$

where

$$C(t,s) := \left(h(t,s)/\sqrt{H(t,s)} - \frac{b(s)}{\delta(s+h)}\right), \quad b(s) := \max\{0,\Delta_h\delta(s)\}.$$

When $\mathbb{T}=\mathbb{Z}$, $r(t)\equiv 1$, $\tau(t)=t-k$ and $k\in\mathbb{N}$, we get a result for the second-order delay difference equation

$$x(n+2)-2x(n+1)+x(n)+p(t)x(t-k)=0, \quad t\in[t_0,\infty]. \tag{30}$$

When $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $r(t) \equiv 1$, $\tau(t) = t - k_0 h$ and $k_0 \in \mathbb{N}$ our results are essentially new for the second-order generalized delay-difference equation

$$x(t + 2h) - 2x(t + h) + x(t) + h^2 p(t)x(t - k_0 h) = 0. \tag{31}$$

When $\mathbb{T}{=}q^{\mathbf{N}}$, $r(t) \equiv 1$,$\tau(t) = t - q^{n_0}$ and $n_0 \in \mathbb{N}$ our results are essentially new for the second order q-delay difference equation

$$x(q^2 t) - (q + 1)x(qt) + qx(t) + q(q - 1)^2 t^2 p(t)x(t - q^{n_0}) = 0. \tag{32}$$

When $\mathbb{T} = \mathbb{N}_0^2 = \{t = n^2 : n \in \mathbb{N}_0\}$, then we have $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$ for $t \in \mathbb{T}$, and equation (1) becomes the difference equation

$$\begin{aligned} x((\sqrt{t}+2)^2) \quad &- \quad \frac{4+4\sqrt{t}}{1+2\sqrt{t}} x((\sqrt{t}+1)^2) \\ &+ \quad \frac{3+2\sqrt{t}}{1+2\sqrt{t}} x(t) + (3+2\sqrt{t})(1+2\sqrt{t})p(t)x(\tau(t)) = 0. \end{aligned}$$

Finally, when $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where the H_n's are the so-called harmonic numbers defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}, \; n \in \mathbb{N},$$

then $\sigma(H_n) = H_{n+1}$, $\mu(H_n) = \frac{1}{n+1}$, and equation (1) becomes the difference equation

$$x(H_{n+2}) - \frac{1}{n+2} x(H_{n+1}) + \frac{n+1}{n+2} x(H_n) + \frac{p(H_n)}{(n+1)(n+2)} x(\tau(H_n)) = 0. \tag{33}$$

Our results for equations (32) and (33) are also essentially new and can be applied to many other time scales.

With an appropriate choice of the functions H and δ one can derive from Theorem 2 a number of oscillation criteria for equation (1) for many different types of time scales. For example if $\mathbb{T} = \mathbb{R}$, if $H(t, s) = (t - s)^\lambda$, for $t \geq s \geq a$, where $\lambda \geq 1$ is an integer, $\delta(t) = 1$, and $r(t) = 1$, then (15) reduces to the oscillation criterion of Kamenev-type [17]. Next we give another example of this type for different choices of H and δ.

Example 4 *Consider the delay dynamic equation*

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \tag{34}$$

where the delay function $\delta : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$ and $p(t) > 0$. From Theorem 2, by choosing $H(t, s) = 1$ and $\delta(t) = t$, we see that if

$$\int_a^\infty \left(p(t)\tau(t) - \frac{1}{4t} \right) \Delta t = \infty$$

then equation (34) *is oscillitory. As a special case note that the Euler–Cauchy equation*

$$x^{\Delta\Delta}(t) + \frac{\gamma}{t\sigma(t)} x(\tau(t)) = 0, \tag{35}$$

is oscillatory if

$$\int_a^\infty \left(\frac{\gamma\tau(t)}{\sigma(t)} - \frac{1}{4} \right) \frac{1}{t} \Delta t = \infty.$$

Note that this holds if there is an $\varepsilon > 0$ *such that*

$$\frac{\gamma\tau(t)}{\sigma(t)} > \frac{1}{4} + \varepsilon$$

for all large t*. This is sharp in the case when* $\mathbb{T} = \mathbb{R}$ *and when* $\mathbb{T} = \mathbb{N}$ *(in particular, for the case when* $\tau(t) = t$*). See [21, 30] for additional details for these two cases respectively.*

4 Other Criteria

In this section we consider (1), where r does not satisfy (2), i.e.,

$$\int_a^\infty \frac{1}{r(t)} \Delta t < \infty. \tag{36}$$

We start with the following auxiliary result, whose proof is similar to that which can be found in [35], and so is omitted.

Lemma 5 *[35]: Assume* (36) *holds, and*

$$\int_{t_0}^\infty \frac{1}{r(t)} \int_{t_0}^t p(s) \Delta s \Delta t = \infty. \tag{37}$$

Suppose that x *is a nonoscillatory solution of* (1) *such that there exists* $t_1 \in \mathbb{T}$ *with*

$$x(t)x^\Delta(t) < 0 \quad \textit{for all } t \geq t_1.$$

Then

$$\lim_{t\to\infty} x(t) = 0.$$

Using Lemma 5, we can derive the following criterion.

Theorem 6 *Let the assumptions* (36) *and* (37) *hold, let* $H \in \Re$*, and assume* (14) *holds. If there exists a positive differentiable function* $\delta(t)$ *such that for every* $t_0 \geq a$ *we have that* (15) *holds, then every solution of* (1) *is oscillatory or converges to zero as* $t \to \infty$.

Proof: Assume that x is a nonoscillatory solution of (1). Then x is either eventually positive or eventually negative, i.e., there exists t_0 with $x(t) > 0$ for all $t \geq t_0$ or $x(t) < 0$ for all $t \geq t_0$. Without loss of generality we assume that $x(t)$ is eventually positive. From (1) we have

$$(r(t)x^{\Delta}(t))^{\Delta} = -p(t)x(\tau(t)) < 0,$$

for all large t. Hence rx^{Δ} is an eventually decreasing function and either $x^{\Delta}(t)$ is eventually positive or eventually negative. If $x^{\Delta}(t)$ is eventually positive we can derive a contradiction as in Theorem 2. If $x^{\Delta}(t)$ is eventually negative we see from Lemma 5 that $x(t)$ converges to zero as $t \to \infty$. This completes the proof. $\square$

More examples can be obtained similar to those given following Corollary 3. The details are left to the reader.

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