Some Oscillation Results for Second Order Linear Delay Dynamic Equations

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ABSTRACT. We obtain some oscillation theorems for linear delay dynamic equations on a time scale. We illustrate the results by a number of examples.

1. Preliminary Results

Consider the second order linear delay dynamic equation

$$L[x](t) := (r(t)x^{\Delta}(t))^{\Delta} + \sum_{i=1}^{n} q_i(t)x(\tau_i(t)) = 0. \tag{1.1}$$

We will be interested in obtaining oscillation theorems for (1.1) by comparing the solutions to a related equation without delay of the form

$$(r(t)x^{\Delta})^{\Delta} + \sum_{i=1}^{n} Q_i(t)x^{\sigma} = 0, \tag{1.2}$$

for which many oscillation results are known. We recall that a solution of (1.1) or (1.2) is nonoscillatory if it is eventually of one sign. If a solution changes sign infinitely often it is said to be oscillatory.

Let $\mathbb{T}$ be a time scale (nonempty closed subset of the reals $\mathbb{R}$) which is unbounded above. We assume that the coefficient functions $q_i(t) \geq 0$, $i = 1, 2, \cdots, n$, and $r(t) > 0$ are rd-continuous on the time scale interval $[a, \infty)_{\mathbb{T}} := [a, \infty) \cap \mathbb{T}$, (i.e., $r, q_i \in C_{rd}([a, \infty)_{\mathbb{T}})$. Furthermore, we will assume that $\sum_{i=1}^{n} q_i(t) \not\equiv 0$ (for all large t). We will also assume that the delay functions $\tau_i : [a, \infty)_{\mathbb{T}} \to \mathbb{T}$ are rd-continuous, $\tau_i(t) \leq t$, and $\tau_i(t) \to \infty$ as $t \to \infty$, $i = 1, 2, \cdots, n$. For details concerning calculus

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on time scales and other pertinent definitions, we refer to the books [**3**], [**4**], and [**11**]. Stability and oscillation questions for certain first order delay dynamic equations have been considered in [**1**] for example.

We start with several auxiliary lemmas which are crucial in the proof of the main results. The first lemma is usually referred to as the Riccati technique. Denote

$$S[z] = \frac{z^2}{r(t) + \mu(t)z}.$$

LEMMA 1.1 ([**12**], [**6**]). *The equation*

$$L_{rq}[x] := (r(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0 \tag{1.3}$$

is nonoscillatory if and only if there is a function z satisfying the Riccati dynamic inequality

$$z^{\Delta}(t) + q(t) + S[z](t) \leq 0 \tag{1.4}$$

with $r(t) + \mu(t)z(t) > 0$ for large t.

That is, if $x(t)$ is a solution of (1.3) that is of one sign for all large $t \in [a, \infty)_{\mathbb{T}}$, then $z(t) := \frac{r(t)x^{\Delta}(t)}{x(t)}$ satisfies (1.4) with $r(t) + \mu(t)z(t) > 0$ for large t. Conversely, if $z(t)$ solves (1.4) with $r(t) + \mu(t)z(t) > 0$ for large t, then (1.3) has a solution $x(t)$ which is of one sign for all large t.

We will use this lemma to show that a nonoscillatory solution of (1.1) leads to a solution of the Riccati dynamic inequality (1.4). In order to do this, we introduce the auxiliary functions $H(t, t_1)$ and $\eta_i(t, t_1)$ defined by

$$H(t, t_1) := \int_{t_1}^{t} \frac{1}{r(s)}\, \Delta s, \quad \text{and} \quad \eta_i(t, t_1) := \frac{H(\tau_i(t), t_1)}{H(\sigma(t), t_1)}, \quad l \leq i \leq n.$$

We may then establish the following result.

LEMMA 1.2. *Let $x(t)$ be a solution of* (1.1) *which satisfies*

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad (r(t)x^{\Delta}(t))^{\Delta} \leq 0$$

for all $t \geq \tau_i(t) \geq T \geq a$. Then for each $1 \leq i \leq n$ we have

$$x(\tau_i(t)) > \eta_i(t, T)x^{\sigma}(t), \quad t \geq \tau_i(t) > T.$$

PROOF. For $t \geq \tau_i(t) > T \geq a$ we have

$$\begin{aligned} x(\sigma(t)) - x(\tau_i(t)) &= \int_{\tau_i(t)}^{\sigma(t)} x^\Delta(s)\Delta s \\ &= \int_{\tau_i(t)}^{\sigma(t)} \frac{1}{r(s)} r(s) x^\Delta(s)\Delta s \\ &\leq r(\tau_i(t))x^\Delta(\tau_i(t)) \int_{\tau_i(t)}^{\sigma(t)} \frac{\Delta s}{r(s)} \end{aligned}$$

which yields

$$x^\sigma(t) \leq x(\tau_i(t)) + r(\tau_i(t))x^\Delta(\tau_i(t))H(\sigma(t), \tau_i(t)).$$

Dividing both sides of this inequalilty by $x(\tau_i(t))$ we get

$$\frac{x^\sigma(t)}{x(\tau_i(t))} \leq 1 + \frac{r(\tau_i(t))x^\Delta(\tau_i(t))}{x(\tau_i(t))} H(\sigma(t), \tau_i(t)). \tag{1.5}$$

Also, we have

$$\begin{aligned} x(\tau_i(t)) - x(T) &= \int_T^{\tau_i(t)} x^\Delta(s)\Delta s \\ &\geq r(\tau_i(t))x^\Delta(\tau_i(t)) \int_T^{\tau_i(t)} \frac{\Delta s}{r(s)} \end{aligned}$$

and so

$$\begin{aligned} x(\tau_i(t)) &\geq x(T) + r(\tau_i(t))x^\Delta(\tau_i(t))H(\tau_i(t), T) \\ &> r(\tau_i(t))x^\Delta(\tau_i(t))H(\tau_i(t), T). \end{aligned}$$

Therefore, we have

$$\frac{r(\tau_i(t))x^\Delta(\tau_i(t))}{x(\tau_i(t))} < \frac{1}{H(\tau_i(t), T)}. \tag{1.6}$$

Hence, from (1.5) and (1.6) we have

$$\begin{aligned} \frac{x^\sigma(t)}{x(\tau_i(t))} &< 1 + \frac{H(\sigma(t), \tau_i(t))}{H(\tau_i(t), T)} \\ &= \frac{H(\sigma(t), T)}{H(\tau_i(t), T)} = \frac{1}{\eta_i(t, T)}. \end{aligned}$$

This gives us the desired result

$$x(\tau_i(t)) > x^\sigma(t)\eta_i(t, T).$$

□

LEMMA 1.3. *Assume $q_i(t) \geq 0$, $1 \leq i \leq n$, and $\sum_{i=1}^n q_i(t) \not\equiv 0$ for large t. Let x be a solution of (1.1) with $x(t) > 0$, $t \in [t_0, \infty)_{\mathbb{T}}$ and assume further that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty.$$

Then there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad \text{and} \quad (r(t)x^{\Delta}(t))^{\Delta} \leq 0$$

for $t \in [T, \infty)_{\mathbb{T}}$.

PROOF. We can suppose that $t_1 \geq t_0$ is such that $x(t) > 0$, $x(\tau_i(t)) > 0$, $t \geq t_1$, for all $1 \leq i \leq n$. Then we have

$$(r(t)x^{\Delta}(t))^{\Delta} = -\sum_{i=1}^{n} q_i(t)x(\tau_i(t)) \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

and so $r(t)x^{\Delta}(t)$ is decreasing for $t \in [t_1, \infty)_{\mathbb{T}}$. Therefore, if $x^{\Delta}(t_2) \leq 0$ for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$, then it follows that

$$r(t)x^{\Delta}(t) \leq 0, \quad t \in [t_2, \infty)_{\mathbb{T}}.$$

If $x^{\Delta}(t_3) < 0$ for some $t_3 \geq t_2$, then an integration gives

$$\begin{aligned} \int_{t_3}^{t} x^{\Delta}(s)\Delta s &= x(t) - x(t_3) \\ &\leq r(t_3)x^{\Delta}(t_3) \int_{t_3}^{t} \frac{\Delta s}{r(s)} \\ &\to -\infty, \quad \text{as} \quad t \to \infty, \end{aligned}$$

which gives us a contradiction. Hence, $x^{\Delta}(t) \equiv 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$ and this means $x(t) \equiv$ *constant* for $t \in [t_2, \infty)_{\mathbb{T}}$. But, then

$$(r(t)x^{\Delta}(t))^{\Delta} \equiv 0 \equiv -\sum_{i=1}^{n} q_i(t)x(\tau_i(t)) \not\equiv 0,$$

which is a contradiction. Hence, it follows that

$$x^{\Delta}(t) > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \quad \text{and} \quad (r(t)x^{\Delta}(t))^{\Delta} \leq 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

□

2. Main Results

We may now apply the previous lemmas to obtain our first oscillation result.

THEOREM 2.1. *Assume $r(t) > 0$ with $\int_a^\infty 1/r(t)\,\Delta t = \infty$ and assume that $q_i(t) \geq 0$, $1 \leq i \leq n$, and $\sum_{i=1}^n q_i(t) \not\equiv 0$, for all sufficiently large t. If*

$$(r(t)x^\Delta)^\Delta + Q(t,T)x^\sigma = 0, \tag{2.1}$$

where, for $t \in (T,\infty)_{\mathbb{T}}$,

$$Q(t,T) := \sum_{i=1}^n \eta_i(t,T)q_i(t),$$

is oscillatory on $(T,\infty)_{\mathbb{T}}$ for all sufficiently large T, then all solutions of (1.1) are oscillatory.

PROOF. If not, assume that $x(t)$ is a solution of (1.1) of one sign for $t \geq t_1 \geq a$ and without loss of generality let us suppose that $x(t) > 0$, $t \in [t_1,\infty)_{\mathbb{T}}$. Then by Lemma 1.2 and Lemma 1.3, there exists a $T \in [t_1,\infty)_{\mathbb{T}}$, sufficiently large, such that

$$x(t) > 0, \quad x^\Delta(t) > 0, \quad t \in [T,\infty)_{\mathbb{T}},$$

$$x(\tau_i(t)) \geq \eta_i(t,T)x^\sigma(t), \quad \tau_i(t) \in (T,\infty)_{\mathbb{T}}, \quad \text{for all} \quad 1 \leq i \leq n,$$

and (2.1) is oscillatory on $[a,\infty)_{\mathbb{T}}$. Consequently, we have that $x(t) > 0$ satisfies $x^\Delta(t) > 0$ and $\quad (r(t)x^\Delta)^\Delta + Q(t,T)x^\sigma \leq 0, \quad t \in (T,\infty)_{\mathbb{T}}$.

If we set $z(t) := \frac{r(t)x^\Delta(t)}{x(t)}$, then $z(t) > 0$ and

$$\begin{aligned} z^\Delta(t) &= \frac{x(t)(r(t)x^\Delta(t))^\Delta - r(t)(x^\Delta(t))^2}{x(t)x^\sigma(t)} \\ &\leq -Q(t,T) - \frac{1}{r(t)}z^2(t)\frac{x(t)}{x^\sigma(t)} \\ &= -Q(t,T) - \frac{z^2(t)}{r(t)}\frac{x(t)}{x(t)+\mu(t)x^\Delta(t)} \\ &= -Q(t,T) - \frac{z^2(t)}{r(t)+\mu(t)z(t)}. \end{aligned}$$

Therefore, since $r(t) + \mu(t)z(t) > 0$ and z is a solution of

$$z^\Delta + Q(t,T) + S[z](t) \leq 0$$

for large t, we get by Lemma 1.1 that the linear equation

$$(r(t)x^\Delta)^\Delta + Q(t,T)x^\sigma = 0 \tag{2.2}$$

is nonoscillatory on $(T,\infty)_{\mathbb{T}}$. This contradiction proves the result. □

We may establish a number of corollaries by using Theorem 2.1 and known criteria for linear second order dynamic equations (cf. [2], [5-8], [10] and [12-15]). For example, we have the following result.

Corollary 2.2. *Assume*

$$\int_T^\infty \frac{\Delta s}{r(s)} = \infty = \int_T^\infty Q(s,T)\Delta s.$$

Then all solutions of (1.1) *are oscillatory.*

Proof. Corollary 2.2 follows from the Fite–Wintner–Leighton criterion which says that all solutions of (1.3) are oscillatory (cf. [**2**]) if

$$\int_a^\infty \frac{1}{r(t)}\Delta t = \infty = \int_a^\infty q(t)\Delta t.$$

□

For the case $r(t) \equiv 1$ and a single delay, (1.1) becomes

$$x^{\Delta\Delta} + q(t)x(\tau(t)) = 0. \tag{2.3}$$

In this case, $\eta(t,T) = \frac{\tau(t)-T}{\sigma(t)-T}$, so that

$$Q(t,T) = \frac{\tau(t)-T}{\sigma(t)-T}q(t) \sim \frac{\tau(t)}{\sigma(t)}q(t) \quad \text{as} \quad t \to \infty.$$

Therefore, if

$$\int_a^\infty \frac{\tau(t)}{\sigma(t)}q(t)\Delta t = \infty,$$

then all solutions of (2.3) are oscillatory.

We next consider the dynamic equation

$$x^{\Delta\Delta} + \frac{\gamma}{t\tau(t)}x(\tau(t)) = 0. \tag{2.4}$$

We have the following.

Corollary 2.3. *All solutions of* (2.4) *are oscillatory if* $\gamma > \frac{1}{4}$ *and* $\lim_{t\to\infty} \frac{\mu(t)}{t} = 0$.

Proof. We use the fact that all solutions of

$$x^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)}x^\sigma = 0 \tag{2.5}$$

are oscillatory if $\gamma > \frac{1}{4}$ and $\lim_{t\to\infty} \frac{\mu(t)}{t} = 0$ (cf. [**12**]) along with Theorem 2.1. □

3. Examples

In this section we give examples of our main results.

EXAMPLE 3.1. If $\mathbb{T}$ is any time scale with $\lim_{t\to\infty}\frac{\mu(t)}{t}=0$ (e.g., $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$), then all solutions of (2.5), or more generally

$$x^{\Delta\Delta}+\sum_{i=1}^{n}\frac{\gamma_i}{t\tau_i(t)}x(\tau_i(t))=0$$

are oscillatory provided

$$\bar{\gamma}:=\sum_{i=1}^{n}\gamma_i>\frac{1}{4}.$$

To see this, we observe that in this case

$$\begin{aligned} Q(t,T) &= \sum_{i=1}^{n}\eta_i(t,T)q_i(t) \\ &= \sum_{i=1}^{n}\frac{\gamma_i}{t\tau_i(t)}\left(\frac{\tau_i(t)-T}{\sigma(t)-T}\right) \end{aligned}$$

and

$$\sum_{i=1}^{n}\frac{\gamma_i}{t\tau_i(t)}\left(\frac{\tau_i(t)-T}{\sigma(t)-T}\right)\sim\frac{\bar{\gamma}}{t\sigma(t)}\quad\text{as}\quad t\to\infty.$$

Therefore, since (2.5) with γ replaced by $\bar{\gamma}$ is oscillatory, the result now follows from Theorem 2.1.

EXAMPLE 3.2. If $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, then the q-difference equation

$$x^{\Delta\Delta}+\frac{\gamma}{t\sigma(t)}x^{\sigma}=0$$

is oscillatory iff $\gamma>\frac{1}{(\sqrt{q}+1)^2}$ (cf. [**5**], [**12**]). Therefore, the delay q-difference equation (where $\tau_i:\mathbb{T}=q^{\mathbb{N}_0}\to\mathbb{T}$)

$$x^{\Delta\Delta}+\sum_{i=1}^{n}\frac{\gamma_i}{t\tau_i(t)}x(\tau_i(t))=0$$

is oscillatory provided

$$\bar{\gamma}=\sum_{i=1}^{n}\gamma_i>\frac{1}{(\sqrt{q}+1)^2}.$$

Additional examples may be readily given. We leave this to the interested reader.

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