EXPONENTIAL STABILITY OF DYNAMIC EQUATIONS ON TIME SCALES

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[1]

Abstract. In this work we investigate the exponential stability of the zero solution to systems of dynamic equations on time scales. We define suitable Lyapunov-type functions and then formulate appropriate inequalities on these functions that guarantee that the zero solution decay to zero exponentially. Several examples are given.

1. Introduction

This paper considers the exponential stability of the zero solution the first-order dynamic equation

$$x^\Delta = f(t,x), \quad t \geq 0, \tag{1}$$

subject to the initial condition

$$x(t_0) = x_0, \quad t_0 \geq 0, \quad x_0 \in \mathbb{R}, \tag{2}$$

where $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and t is from a so-called "time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$). Throughout the paper we assume that $f(t,0) = 0$. Equation (1) subject to (2) is known as an initial value problem (IVP) on time scales.

If $\mathbb{T} = \mathbb{R}$ then $x^\Delta = x'$ and (1), (2) become the following IVP for ordinary differential equations

$$x' = f(t,x), \quad t \geq 0, \tag{3}$$

$$x(t_0) = x_0, \quad t_0 \geq 0. \tag{4}$$

[1]AMS Subject Classification: 39A12

Keywords: time scale, Lyapunov function, Exponential stability, system of equations

Recently, Peterson and Tisdel [9] used Lyapunov-type functions to formulate some sufficient conditions that ensure all solutions to (1), (2) are bounded and unique. On the other hand, Raffoul [10] used the same idea to obtain boundedness on all solutions of (3) and (4). while in a more classical setting, Hartman [4, Chapter 3] employed Lyapunov-type functions to prove that solutions to (3), (4) are unique. Motivated by [10] and [4] (see also references therein), we investigate the boundedness and uniqueness of solutions to systems of dynamic equations in the more general time scale setting. We define suitable Lyapunov-type functions on time scales and then formulate appropriate inequalities on these functions that guarantee solutions to (1), (2) are uniformly bounded and / or unique. In fact, our theory generalizes some of the results in [10] and [4] for the special case $\mathbb{T} = \mathbb{R}$.

To understand the notation used above and the idea of time scales some preliminary definitions are needed.

Definition A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. We assume that $0 \in \mathbb{T}$ (for convenience) and $\mathbb{T}$ is unbounded above.

Since a time scale may or may not be connected, the concept of the jump operator is useful.

Defn Define the forward jump operator $\sigma(t)$ at t by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \text{ for all } t \in \mathbb{T},$$

and define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ as $\mu(t) = \sigma(t) - t$. Also let $x^\sigma(t) = x(\sigma(t))$, that is x^σ is the composite function $x \circ \sigma$. The jump operator σ then allows the classification of points in a time scale in the following way: If $\sigma(t) > t$ then call the point t right-scattered; while $\sigma(t) = t$ then call the point t right-dense.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition 1. *Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}^n$. Define $x^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t with*

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U \text{ and each } i = 1, \ldots, n.$$

Call $x^\Delta(t)$ the (delta) derivative of $x(t)$ and say that x is (delta) differentiable.

Definition 2. *If $G^\Delta(t) = g(t)$ then define the Cauchy integral by*

$$\int_a^t g(s)\Delta s = G(t) - G(a).$$

For a more general definition of the delta integral see [2], [3].

The following theorem is due to Hilger [5].

Theorem 1. *Assume that $g : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}$.*

(i) If g is differentiable at g then g is continuous at t.

(ii) If g is continuous at t and t is right-scattered then g is differentiable at t with

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\sigma(t) - t}.$$

(iii) If g is differentiable and t is right-dense then

$$g^\Delta(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is differentiable at t then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.

We assume throughout that $t_0 \geq 0$ and $t_0 \in \mathbb{T}$. By the interval $[t_0, \infty)$ we mean the set $[t_0, \infty) \cap \mathbb{T}$.

Definition 3. *Define S to be the set of all functions $x : \mathbb{T} \to \mathbb{R}^n$ such that*

$$S = \{x : x \in C([t_0, \infty); \mathbb{R}^n)\}.$$

A solution to (1) is a function $x \in S$ which satisfies (1) for each $t \geq t_0$.

The theory of time scales dates back to Hilger [5]. The monographs [2], [3] and [6] also provide an excellent introduction.

2. Lyapunov Functions

The following Chain Rule shall be very useful throughout the remainder of the paper.

Theorem 2. *Let $p : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $q : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $p \circ q$ is delta differentiable and*

$$[p(q(t))]^{\Delta} = \left(\int_0^1 p'(q(t) + h\mu(t)q^{\Delta}(t))dh \right) q^{\Delta}(t),$$

Proof. Keller [7] and Potzsche [8]. See also Bohner and Peterson [2], Theorem 1.90. ☐

Definition 4. *Call $V : \mathbb{R}^n \to \mathbb{R}$ a "type I" function when*

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1) + \ldots + V_n(x_n),$$

where each $V_i : \mathbb{R} \to \mathbb{R}$ is continuously differentiable.

Now assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a type I function and x is a solution to (1). Consider

$$\begin{aligned} [V(x(t))]^{\Delta} &= \left[\sum_{i=1}^{n} V_i(x_i(t)) \right]^{\Delta} = \sum_{i=1}^{n} [V_i(x_i(t))]^{\Delta}, \\ &= \sum_{i=1}^{n} \int_0^1 V_i'(x_i(t) + h\mu(t)x_i^{\Delta}(t))dh \; x_i^{\Delta}(t), \\ &= \sum_{i=1}^{n} \int_0^1 V_i'(x_i(t) + h\mu(t)f_i(t, x(t)))f_i(t, x(t)) \; dh, \\ &= \int_0^1 \nabla V(x(t) + h\mu(t)f(t, x(t))) \cdot f(t, x(t)) \; dh, \end{aligned}$$

where $\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$ is the gradient operator.

This motivates us to define $\dot{V} : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\dot{V}(t, x) = \int_0^1 \nabla V(x + h\mu(t)f(t, x)) \cdot f(t, x)dh. \tag{5}$$

Next we find another formula for $\dot{V}(t, x)$. If $\mu(t) = 0$, then we simply get

$$\dot{V}(t, x) = \nabla V(x) \cdot f(t, x).$$

On the other hand if $\mu(t) \neq 0$, then

$$\begin{aligned}
\dot{V}(t,x) &= \int_0^1 \nabla V(x + h\mu(t)f(t,x)) \cdot f(t,x)\, dh \\
&= \sum_{i=1}^{n} V_i'(x_i + h\mu(t)f_i(t,x))f_i(t,x)\, dh \\
&= \sum_{i=1}^{n} \frac{1}{\mu(t)} \int_0^1 V_i'(x_i + h\mu(t)f_i(t,x))\mu(t)f_i(t,x)\, dh \\
&= \sum_{i=1}^{n} \frac{V_i(x_i + \mu(t)f_i(t,x)) - V_i(x_i)}{\mu(t)}.
\end{aligned}$$

Summarizing, we get that

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^{n} [V_i(x_i + \mu(t)f_i(t,x)) - V_i(x_i)]/\mu(t), & \text{when } \mu(t) \neq 0, \\ \nabla V(x) \cdot f(t,x), & \text{when } \mu(t) = 0. \end{cases} \tag{6}$$

If, in addition to the above, $V : \mathbb{R}^n \to [0, \infty)$ then we call V a
type I Lyapunov function. Sometimes the domain of V will be a subset D of $\mathbb{R}^n$.

Note that $V = V(x)$ and even if the vector field associated with the dynamic equation is autonomous then $\dot{V}$ still depends on t (and x of course) when the graininess function of $\mathbb{T}$ is nonconstant.

Using formulas (5) and (6) we can easily calculate $\dot{V}(t,x)$ for each of the following examples:

Example 1 Let $V(x) = \sum_{i=1}^{n} x_i^{1/2}$, for $x \in D$, where

$$D = \{x \in \mathbb{R}^n : x_i > 0,\ x_i + \mu(t)f_i(t,x) \geq 0,\ i = 1, 2, \cdots, n\}.$$

Then

$$\dot{V}(t,x) = \sum_{i=1}^{n} \frac{f_i(t,x)}{\sqrt{x_i + \mu(t)f_i(t,x)} + \sqrt{x_i}}.$$

Example 2 Let $V(x) = \sum_{i=1}^{n} a_i x_i^2$, for $x \in \mathbb{R}^n$ and $a_i > 0$, $i = 1, 2, \cdots, n$. For $x \in \mathbb{R}^n$, we define the associated weighted vector by

$$w(x) := (a_1x_1, a_2x_2, \cdots, a_nx_n).$$

Then

$$\dot{V}(t,x) = 2w(x) \cdot f(t,x) + \mu(t)w(f(t,x)) \cdot f(t,x).$$

In particular, if $V(x) = \|x\|^2 = \sum_{i=1}^{n} x_i^2$, then

$$\dot{V}(t,x) = 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2. \tag{7}$$

Example 3 Let $V(x) = \sum_{i=1}^{n} a_i x_i^4$ for $x \in \mathbb{R}^n$ and $a_i > 0$, $i = 1, 2, \cdots, n$. Then

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^{n} a_i[(x_i + \mu(t)f_i(t,x))^4 - x_i^4]/\mu(t), & \text{when } \mu(t) \neq 0, \\ \sum_{i=1}^{n} 4a_i x_i^3 f_i(t,x), & \text{when } \mu(t) = 0. \end{cases}$$

Example 4 Let $V(x) = \sum_{i=1}^n a_i x_i^{4/3}$ for $x \in \mathbb{R}^n$ and $a_i > 0$, $i = 1, 2, \cdots, n$. Then

$$\dot{V}(t,x) = \begin{cases} \sum_{i=1}^n a_i[(x_i + \mu(t) f_i(t,x))^{4/3} - x_i^{4/3}]/\mu(t), & \text{when } \mu(t) \neq 0, \\ \sum_{i=1}^n 4 a_i x_i^{1/3} f_i(t,x)/3, & \text{when } \mu(t) = 0. \end{cases}$$

Example 5 For Lyapunov functions which may not be power functions, let

$$V(x) = \sum_{i=1}^n \int_0^{x_i} p_i(u) du,$$

where each $p_i : \mathbb{R} \to \mathbb{R}^+$ is continuous. Then

$$\begin{aligned} \dot{V}(t,x) &= \sum_{i=1}^n \int_0^1 p_i(x_i + h\mu(t) f_i(t,x)) f_i(t,x) dh. \\ &= P(t,x) \cdot f(t,x), \end{aligned}$$

where

$$P(t,x) := \left(\int_0^1 p_1(x_1 + h\mu(t) f_1(t,x)) dh, \ldots, \int_0^1 p_n(x_n + h\mu(t) f_n(t,x)) dh \right).$$

Note that if $\mathbb{T} = \mathbb{R}$, then

$$P(t,x) = P(x) = (p_1(x_1), \ldots, p_n(x_n)).$$

3. Exponential Stability

In this section we present some results on the exponential stability of the zero solution of (1), (2). First a few more preliminaries.

Definition 5. *Assume $g : \mathbb{T} \to \mathbb{R}$. Define and denote $g \in C_{rd}(\mathbb{T}; \mathbb{R})$ as right-dense continuous (rd-continuous) if g is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} g(s)$ exists and is finite, at every left-dense point $t \in \mathbb{T}$.*

Now define the so-called set of regressive functions, $\mathcal{R}$, by

$$\mathcal{R} = \{p : \mathbb{T} \to \mathbb{R};\ p \in C_{rd}(\mathbb{T}; \mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0 \text{ on } \mathbb{T}\}$$

and define the set of positively regressive functions by

$$\mathcal{R}^+ = \{p \in \mathcal{R} :\ 1 + p(t)\mu(t) > 0 \text{ on } \mathbb{T}\}.$$

For $p \in \mathcal{R}$, the generalized exponential function $e_p(\cdot, t_0)$ on a time scale $\mathbb{T}$ is ([2, Theorem 2.35]) the unique solution to the IVP

$$x^\Delta = p(t)x, \quad x(t_0) = x_0.$$

If $p \in \mathcal{R}^+$, then [2, Theorem 2.48] $e_p(t, t_0) > 0$ for $t \in \mathbb{T}$. It follows from Bernoulli's inequality ([2, Theorem 6.2]) that for any time scale, if the constant $\lambda \in \mathcal{R}^+$, then

$$0 < e_{\ominus\lambda}(t, t_0) \leq \frac{1}{1 + \lambda(t - t_0)}, \quad t \geq t_0.$$

It follows that

$$\lim_{t \to \infty} e_{\ominus\lambda}(t, t_0) = 0.$$

In particular, if $\mathbb{T} = \mathbb{R}$, then $e_{\ominus\lambda}(t,t_0) = e^{-\lambda(t-t_0)}$ and if $\mathbb{T} = \mathbb{Z}^+$ then $e_{\ominus\lambda}(t,t_0) = (1+\lambda)^{-(t-t_0)}$. For the growth of generalized exponential functions on time scales see Bodine and Lutz [1]. With all this in mind we make the following definition.

Definition 6. *We say the zero solution of the IVP* (1), (2) $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$ *is exponentially stable if there exists a positive constant* d *and constant* $C \in \mathbb{R}^+$ *and a* $M \in \mathcal{R}^+$ *such that for any solution* $x(t,t_0,x_0)$ *of the IVP* (1), (2)

$$\|x(t,t_0,x_0)\| \leq C\Big(||x_0||, t_0\Big)(e_{\ominus M}(t,t_0))^d, \quad \textit{for all } t \in [t_0,\infty).$$

The zero solution of (1.1) is said to be uniformly exponentially stable if C *is independent of* t_0.

Note that if $\mathbb{T} = \mathbb{R}$, then $(e_{\ominus\lambda}(t,t_0))^d = e^{-\lambda d(t-t_0)}$ and if $\mathbb{T} = \mathbb{Z}^+$, then $(e_{\ominus\lambda}(t,t_0))^d = (1+\lambda)^{-d\lambda(t-t_0)}$.

We are now ready to present some results.

Theorem 3. *Assume* $D \subset \mathbb{R}^n$ *and there exists a type I Lyapunov function* $V : D \to [0,\infty)$ *such that for all* $(t,x) \in [0,\infty) \times D$*:*

$$W(\|x\|) \;\leq\; V(x) \;\leq\; \phi(\|x\|); \tag{8}$$

$$\dot{V}(t,x) \;\leq\; \frac{\psi(\|x\|) + Le_{\ominus\delta}(t,t_0)}{1+\mu(t)M}; \tag{9}$$

$$\psi(\phi^{-1}(V(x))) \;+\; MV(x) \leq \gamma e_{\ominus\delta}(t,t_0) \tag{10}$$

where W, ϕ, ψ *are continuous functions such that* $\phi, W : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to (-\infty, 0]$, ψ *is nonincreasing,* W *is strictly increasing with* $W(\|x\|) \neq 0$ *for* $x \neq 0$ *and* $W(0) = 0$ *and* ϕ^{-1} *exists;* L, M *and* γ *are nonnegative constants with* $\delta > M$*. then all solutions of* (1), (2) *that stay in* D *satisfy*

$$||x|| \;\leq\; W^{-1}\Big\{\Big(V(x_0) + \frac{K}{\delta \ominus M}\Big)e_{\ominus M}(t,t_0)\Big\}; \textit{for all } \; t \geq t_0.$$

Proof. Let x be a solution to (1), (2) that stays in D for all $t \geq t_0 \geq 0$. For $0 < M < \delta$ consider

$$\begin{aligned}
& [V(x(t))e_M(t,t_0)]^{\Delta} \\
= \;& \dot{V}(t,x(t))e^{\sigma}_M(t,t_0) + V(x(t))e^{\Delta}_M(t,t_0), \\
\leq \;& \frac{\psi(\|x(t)\|) + Le_{\ominus\delta}(t,t_0)}{1+\mu(t)M}(1+\mu(t)M)e_M(t,t_0) + MV(x(t))e_M(t,t_0), \quad \text{by (9)}, \\
\leq \;& \Big(\psi(\|x(t)\|) + Le_{\ominus\delta}(t,t_0)\Big)e_M(t,t_0) + MV(x(t))e_M(t,t_0), \\
= \;& \Big(\psi(\|x(t)\|) + Le_{\ominus\delta}(t,t_0) + MV(x(t))\Big)e_M(t,t_0), \\
\leq \;& \Big((\psi(\phi^{-1}(V(x(t))) + MV(x(t)) + Le_{\ominus\delta}(t,t_0)\Big)e_M(t,t_0), \quad \text{by (8)}, \\
\leq \;& (\gamma e_{\ominus\delta}(t,t_0) + Le_{\ominus\delta}(t,t_0))e_M(t,t_0), \quad \text{by (10)} \\
:= \;& Ke_{\ominus\delta}(t,t_0)e_M(t,t_0) \\
= \;& Ke_{M\ominus\delta}(t,t_0).
\end{aligned}$$

Integrating both sides from t_0 to t with $x_0 = x(t_0)$, we obtain

$$\begin{aligned} V(x(t))e_M(t,t_0) &= V(x_0) + \frac{K}{M\ominus\delta}\Big(e_{M\ominus\delta}(t,t_0) - 1\Big), \\ &\leq -\frac{K}{M\ominus\delta} + V(x_0). \end{aligned}$$

It follows that

$$V(x(t)) \leq \Big(V(x_0) + \frac{K}{\delta\ominus M}\Big)e_{\ominus M}(t,t_0) \quad \text{for all } t\in[t_0,\infty). \tag{11}$$

Thus by (8),

$$\|x(t)\| \leq W^{-1}\Big\{\Big(V(x_0) + \frac{K}{\delta\ominus M}\Big)e_{\ominus M}(t,t_0)\Big\}; \text{for all } \ t\geq t_0. \tag{12}$$

This conclues the proof. □

We now provide a special case of Theorem 3 for certain functions ϕ and ψ.

Theorem 4. *Assume $D\subset\mathbb{R}^n$ and there exists a type I Lyapunov function $V : D\to[0,\infty)$ such that for all $(t,x)\in[0,\infty)\times D$:*

$$\lambda_1(t)\|x\|^p \leq V(x) \leq \lambda_2(t)\|x\|^q; \tag{13}$$

$$\dot{V}(t,x) \leq \frac{-\lambda_3(t)\|x\|^r + Le_{\ominus\delta}(t,0)}{1+M\mu(t)}; \tag{14}$$

$$V(x) - V^{r/q}(x) \leq \gamma e_{\ominus\delta}(t,0); \tag{15}$$

where $\lambda_1(t),\lambda_2(t)$ and $\lambda_3(t)$ are positive functions where $\lambda_1(t)$ is nondecreasing; p, q, r are positive constants; L and γ are nonnegative constants, and $\delta > \inf_{t\geq t_0\geq 0}\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}$. Then the zero solution of (1), (2) *is exponentially stable.*

Proof. As in the proof of Theorem 3, let x be a solution to (1), (2) that stays in D for all $t\geq t_0\geq 0$. Let $M = \inf_{t\geq t_0\geq 0}\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}\in\mathcal{R}^+$. Then, $0<M<\delta$ and so $e_M(t,t_0)$ is well defined and positive. Since $\frac{\lambda_3(t)}{[\lambda_2(t)]^{r/q}}\geq M$, we have

$$\begin{aligned} & [V(x(t))e_M(t,t_0)]^{\Delta} \\ = \;& \dot{V}(t,x(t))e_M^{\sigma}(t,t_0) + V(x(t))e_M^{\Delta}(t,t_0), \\ \leq \;& \Big(-\lambda_3(t)\|x\|^r + Le_{\ominus\delta}(t,t_0)\Big)e_M(t,t_0) + MV(x(t))e_M(t,t_0) \\ \leq \;& \Big(\frac{-\lambda_3(t)}{[\lambda_2(t)]^{r/q}}V^{r/q}(x) + MV(x) + Le_{\ominus\delta}(t,t_0)\Big)e_M(t,t_0) \\ \leq \;& \Big(M(V(x(t)) - V^{r/q}(x)) + Le_{\ominus\delta}(t,t_0)\Big)e_M(t,t_0), \quad \text{by (14)}, \\ \leq \;& (\gamma e_{\ominus\delta}(t,t_0) + Le_{\ominus\delta}(t,t_0))e_M(t,t_0), \quad \text{by (15)} \\ := \;& Ke_{\ominus\delta}(t,t_0)e_M(t,t_0) \\ = \;& Ke_{M\ominus\delta}(t,t_0), \end{aligned}$$

where $K = M\gamma + L$. Integrating both sides from t_0 to t with $x_0 = x(t_0)$, and by invoking condition (13) and the fact that $\lambda_(t) \geq \lambda_(t_0)$ we obtain

$$\|x(t)\| \leq \lambda_1(t)^{-1/p}\Big\{\Big(V(x_0) + \frac{K}{\delta \ominus M}\Big)e_{\ominus M}(t,t_0)\Big\}^{1/p} \tag{16}$$

$$\leq \lambda_1(t_0)^{-1/p}\Big\{\Big(V(x_0) + \frac{K}{\delta \ominus M}\Big)e_{\ominus M}(t,t_0)\Big\}^{1/p}; \text{for all } \ t \geq t_0. \tag{17}$$

This conclues the proof. $\square$

Remark 1. *In Theorem 4, if $\lambda_i, i = 1,2,3$ are positive constants, then the zero solution of (1), (2) is uniformly exponentially stable. The proof follows of the Remark from Theorem 4 by taking $\delta > \frac{\lambda_3}{[\lambda_2]^{r/q}}$ and $M = \frac{\lambda_3}{[\lambda_2]^{r/q}}$.*

Theorem 5. *Assume $D \subset \mathbb{R}^n$ and there exists a type I Lyapunov function $V : D \to [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$:*

$$\lambda_1\|x\|^p \leq V(x), \tag{18}$$

$$\dot{V}(t,x) \leq \frac{-\lambda_3 V(x) + Le_{\ominus\delta}(t,0)}{1 + \varepsilon\mu(t)}; \tag{19}$$

where $\lambda_1, \lambda_3, p, \delta > 0, L \geq 0$ are constants and $\varepsilon = min\{\lambda_3, \delta\}$. Then the zero solution of (1), (2) is uniformly exponentially stable.

Proof Let x be a solution to (1), (2) that stays in D for all $t \in [t_0,\infty)$. Since $\varepsilon \in \mathcal{R}^+$, $e_\varepsilon(t,0)$ is well defined and positive. Now consider

$$\begin{aligned}
[V(x(t))e_\varepsilon(t,0)]^\Delta &= \dot{V}(t,x(t))e^\sigma_\varepsilon(t,0) + \varepsilon V(x(t))e_\varepsilon(t,0), \\
&\leq \Big(-\lambda_3 V(x(t)) + Le_{\ominus\delta}(t,0)\Big)e_\varepsilon(t,0) + \varepsilon V(x(t))e_\varepsilon(t,0), \quad \text{by (19)}, \\
&= e_\varepsilon(t,0)[\varepsilon V(x(t)) - \lambda_3 V(x(t)) + Le_{\ominus\delta}(t,0)] \\
&\leq e_\varepsilon(t,0)Le_{\ominus\delta}(t,0) \\
&\leq Le_{\varepsilon\ominus\delta}(t,0)
\end{aligned}$$

Integrating both sides from t_0 to t we obtain

$$\begin{aligned}
V(x(t))e_\varepsilon(t,0) &\leq V(x_0)e_\varepsilon(t_0,0) + \frac{L}{\delta \ominus \varepsilon}e_{\varepsilon\ominus\delta}(t,0) \\
&\leq \Big(V(x_0) + \frac{L}{\delta \ominus \varepsilon}\Big)e_\varepsilon(t_0,0).
\end{aligned}$$

Dividing both sides of the above inequality by $e_\varepsilon(t_0,0)$, we obtain

$$\begin{aligned}
V(x(t)) &\leq \Big(V(x_0) + \frac{L}{\delta \ominus \varepsilon}\Big)e_\varepsilon(t_0,0)e_{\ominus\varepsilon}(t,0) \\
&= \Big(V(x_0) + \frac{L}{\delta \ominus \varepsilon}\Big)e_{\ominus\varepsilon}(t,t_0).
\end{aligned}$$

The proof is completed by invoking condition (18).

4. Examples

We now present some examples to illustrate the theory developed in Section 3.
Example 1 Consider the IVP

$$x^{\Delta} = ax + bx^{1/3} e_{\ominus\delta}(t,0), \quad x(t_0) = x_0, \tag{20}$$

where a, b are constants, $x_0 \in \mathbb{R}$, and $t_0 \in [0,\infty)$. If there is a constant $M = \lambda_3 > 0$ such that

$$(2a + a^2\mu(t) + 1)(1 + \lambda_3\mu(t)) \leq -\lambda_3, \tag{21}$$

and

$$\Big(\frac{2}{3}(\mu(t)b^2)^{3/2} + \frac{|2b + 2ab\mu(t)|^3}{3}\Big)(1 + \lambda_3\mu(t)) \leq L,$$

for some constant $L \geq 0$ and all $t \in [0,\infty)$, with $0 < M < \delta$, then the zero solution of (20) is exponentially stable.
Proof We shall show that under the above assumptions, the conditions of Theorem 4 are satisfied. Choose $D = \mathbb{R}$ and $V(x) = x^2$ so $p = q = 2$, $\lambda_1(t) = \lambda_2(t) = 1$ and (13) holds. Now from (7)

$$\begin{aligned}
\dot{V}(t,x) &= 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2, \\
&= 2x(ax + bx^{1/3}e_{\ominus\delta}(t,0)) + \mu(t)(ax + bx^{1/3}e_{\ominus\delta}(t,0))^2, \\
&\leq (2a + a^2\mu(t))x^2 + |2b + 2ab\mu(t)|x^{4/3}e_{\ominus\delta}(t,0) + b^2\mu(t)x^{2/3}(e_{\ominus\delta}(t,0))^2.
\end{aligned}$$

To further simplify the above inequality, we make use of Young's inequality, which says for any two nonnegative real numbers w and z, we have

$$wz \leq \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } 1/e + 1/f = 1.$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$\begin{aligned}
x^{4/3}|2b + 2ab\mu(t)|e_{\ominus\delta}(t,0) &\leq \Big[\frac{(x^{4/3})^{3/2}}{3/2} + \frac{|2b + 2ab\mu(t)|^3(e_{\ominus\delta}(t,0))^3}{3}\Big] \\
&= \frac{2}{3}x^2 + \frac{|2b + 2ab\mu(t)|^3(e_{\ominus\delta}(t,0))^3}{3},
\end{aligned}$$

and

$$\begin{aligned}
x^{2/3}b^2\mu(t)(e_{\ominus\delta}(t,0))^2 &\leq \frac{(x^{2/3})^3}{3} + \frac{\Big(|b^2\mu(t)|(e_{\ominus\delta}(t,0))^2\Big)^{3/2}}{3/2} \\
&= \frac{x^2}{3} + \frac{2}{3}(\mu(t)b^2)^{3/2}(e_{\ominus\delta}(t,0))^3.
\end{aligned}$$

Thus, putting everything together we arrive at

$$\begin{aligned}
\dot{V}(t,x) &\leq (2a + \mu(t)a^2 + 1)x^2 \\
&+ \Big[\frac{2}{3}(\mu(t)b^2)^{3/2} + \frac{|2b + 2ab\mu(t)|^3}{3}\Big](e_{\ominus\delta}(t,0))^3 \\
&\leq (2a + \mu(t)a^2 + 1)x^2 \\
&+ \Big[\frac{2}{3}(\mu(t)b^2)^{3/2} + \frac{|2b + 2ab\mu(t)|^3}{3}\Big]e_{\ominus\delta}(t,0).
\end{aligned} \tag{22}$$

Dividing and multiplying the right hand side by $(1+\lambda_3\mu(t))$ we see that (15) holds under the above assumptions with $r=2$ and $\gamma=0$. Therefore all the conditions of Theorem 4 are satisfied and we conclude by Remark 1, that the zero solution of (20) is uniformly exponentially stable.

Case 1: If $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$ and (21) reduces to $2a+1\le -\lambda_3$. If $a<-1/2$ then we take $\lambda_3=-(2a+1)>0$ and we can choose $L=2/3|b|^3$, concluding that the zero solution to (20) is uniformly exponentially stable.

Remark 2. *If* $\mathbb{T}=\mathbb{Z}^+$ *then* $\mu(t)=1$ *and condition* (21) *can not be satisfied for positive* λ_3.

To get around it we will have to adjust inequality (22) as follow.

$$\begin{aligned} x^{4/3}|2b+2ab\mu(t)|e_{\ominus\delta}(t,0) &\le |2b+2ab\mu(t)|\Big[\frac{(x^{4/3})^{3/2}}{3/2}+\frac{(e_{\ominus\delta}(t,0))^3}{3}\Big] \\ &= \frac{2}{3}|2b+2ab\mu(t)|x^2+\frac{|2b+2ab\mu(t)|}{3}(e_{\ominus\delta}(t,0))^3, \end{aligned}$$

and

$$\begin{aligned} x^{2/3}b^2\mu(t)(e_{\ominus\delta}(t,0))^2 &\le b^2\mu(t)\Big[\frac{(x^{2/3})^3}{3}+\frac{(e_{\ominus\delta}(t,0))^2)^{3/2}}{3/2}\Big] \\ &= b^2\mu(t)\frac{x^2}{3}+\frac{2}{3}\mu(t)b^2e_{\ominus\delta}(t,0). \end{aligned}$$

Hence, inequality (22) becomes

$$\begin{aligned} \dot{V}(t,x) &\le (2a+\mu(t)a^2+\frac{2}{3}|2b+2ab\mu(t)|+\frac{\mu(t)b^2}{3})x^2 \\ &+ \Big[\frac{|2b+2ab\mu(t)|+\frac{2}{3}\mu(t)b^2}{3}\Big]e_{\ominus\delta}(t,0). \end{aligned} \tag{23}$$

Now, if $\mathbb{T}=\mathbb{Z}^+$ then $\mu(t)=1$ and a condition similar to (21) is satisfied for positive λ_3 if

$$2a+a^2+\frac{2}{3}|2b+2ab|+\frac{b^2}{3}<0.$$

Note that the inequality is satisfied for $a=-\frac{4}{5}$ and $b=\frac{1}{5}$. Then we may define λ_3 by $\lambda_3=-(2a+a^2+\frac{2}{3}|2b+2ab|+\frac{b^2}{3})>0$.

Case 2: If $\mathbb{T}=h\mathbb{N}_0=\{0,h,2h,\ldots\}$ then $\mu(t)=h$ and (21) reduces to

$$(2a+a^2h+1)\le -\lambda_3/(1+\lambda_3 h).$$

Therefore we want to find those $h>0$ such that

$$ha^2+2a+1<0.$$

Now the polynomial

$$p(a):=ha^2+2a+1,$$

will have distinct real roots

$$a_1(h)=(-1-\sqrt{1-h})/h$$
$$a_2(h)=(-1+\sqrt{1-h})/h$$

if $0 < h < 1$. Therefore if $0 < h < 1$ and $a_1(h) < a < a_2(h)$, then

$$A := ha^2 + 2a + 1 < 0.$$

Now, for such an h, let λ_3 be defined by

$$-\lambda_3(1 + \lambda_3 h) = A < 0,$$

that is

$$\lambda_3 := -A/(1 + hA).$$

Therefore if $0 < h < 1$ then for $a_1(h) < a < a_2(h)$ the zero solution is uniformly exponentially stable by Theorem 4.

Remark 3. *It is interesting to note that*

$$\lim_{h\to 0+} a_2(h) = \lim_{h\to 0+} (-1 + \sqrt{1-h})/h = -1/2,$$

and

$$\lim_{h\to 0+} a_1(h) = \lim_{h\to 0+} (-1 - \sqrt{1-h})/h = -\infty,$$

recalling that if $\mathbb{T} = \mathbb{R}$ then for $-\infty < a \le -1/2$ then the zero solution is uniformly exponentially stable.

Example Consider the the following system of IVPs for $t \ge t_0 \ge 0$

$$x_1^{\Delta} = -ax_1 + ax_2, \tag{24}$$

$$x_2^{\Delta} = -ax_1 - ax_2, \tag{25}$$

$$(x_1(t_0), x_2(t_0)) = (c, d), \tag{26}$$

for certain constants $a > 0$; c and d. If there is a constant $\lambda_3 > 0$ such that for all $t \in [0, \infty)$

$$\lambda_3/(1 + \lambda_3\mu(t)) \le 2a(1 - a\mu(t)), \tag{27}$$

then all solutions to (24) - (26) are uniformly bounded.

Proof We will show that, under the above assumptions, the conditions of Theorem 5 are satisfied. Choose $D = \mathbb{R}^2$ and $V(x) = \|x\|^2 = x_1^2 + x_2^2$ so (**??**) holds. From (7) we see that

$$\begin{aligned} \dot{V}(t,x) &= 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2, \\ &= -2a(1 - a\mu(t))\|x\|^2, \\ &\le -\lambda_3\|x\|^2/(1 + \lambda_3\mu(t)), \quad \text{by (27)}, \\ &= -\lambda_3 V(x)/(1 + \lambda_3\mu(t)). \end{aligned}$$

Hence (19) holds under the above assumptions with $L = 0$. Therefore all the conditions of Theorem 5 are satisfied and we conclude that all solutions to (24) - (26) are uniformly bounded.

In fact, if there is a constant K such that

$$0 \le a\mu(t) \le K < 1 \tag{28}$$

for all $t \in [0, \infty)$ then (27) will hold.

Case 1: If $\mathbb{T} = \mathbb{R}$ then $\mu(t) = 0$ and (28) will hold for any $0 \le K < 1$ which, in turn, will make (27) hold and we conclude that all solutions are uniformly bounded.

Case 2: If $\mathbb{T} = \{H_n\}_0^\infty$ defined by

$$H_0 = 0, \quad H_n = \sum_{r=1}^{n} 1/r, \quad n \in \mathbb{N},$$

then $\mu(t) = 1/(n+1)$ and (28) will hold when $a < 1$ which, in turn, will make (27) hold and we conclude that all solutions are uniformly bounded.

Case 3: If $\mathbb{T} = h\mathbb{N}_0$ then $\mu(t) = h$ and (28) will hold when $ah < 1$ which, in turn, will make (27) hold and we conclude that all solutions are uniformly bounded.

Remark 4. *By using standard methods* [2], *the system (24) - (26) has solutions*

$$\begin{aligned} x_1(t) &= c_1 e_{-a+ia}(t, t_0) + c_2 e_{-a-ia}(t, t_0), \\ x_2(t) &= ac_1(-1+i)e_{-a+ia}(t, t_0) - ac_2(1+i)e_{-a-ia}(t, t_0), \end{aligned}$$

and for $\mathbb{T} = h\mathbb{N}_0$ we see by closely investigating these exponentials that when $h < 1/a$ all solutions are uniformly bounded and when $h > 1/a$ all nontrivial solutions are unbounded.

It is interesting to note that even though the eigenvalues of the coefficient matrix

$$A = \begin{pmatrix} -a & a \\ -a & -a \end{pmatrix}, \tag{29}$$

are complex with negative real parts, our system is not stable when $ah > 1$.

5. Uniqueness of Solutions

In this brief section we present a result on the uniqueness of solutions of the IVP (1), (2).

Theorem 6. *Assume that f satisfies*

$$(x_2 - x_1) \cdot (f(t, x_2) - f(t, x_1)) + \mu(t)\|f(t, x_2) - f(t, x_1)\|^2 \leq 0,$$

for $t \in [t_0, \infty)$ and x_1, $x_2 \in \mathbb{R}^n$. Then there is, at most, one solution to the IVP (1), (2).

Proof Let x_1, x_2 be two solutions to (1), (2) and let

$$x(t) = x_2(t) - x_1(t).$$

Choose $V(x) = \|x\|^2$ and note that

$$\begin{aligned} [V(x(t))]^\Delta &= 2x(t) \cdot x^\Delta(t) + \mu(t)\|x^\Delta(t)\|^2, \\ &= (x_2(t) - x_1(t)) \cdot (f(t, x_2(t)) - f(t, x_1(t))) \\ &\quad + \mu(t)\|f(t, x_2(t)) - f(t, x_1(t))\|^2 \leq 0. \end{aligned}$$

Hence $V(x(t))$ is nonincreasing and since $V(x(t_0)) = V(0) = 0$ we conclude that V is identically equal to 0 along $x(t)$. This implies that $x(t) = x_2(t) - x_1(t) = 0$ for $t \in [t_0, \infty)$ and solutions of (1), (2) are unique and this concludes the proof.

Acknowledgement This work was undertaken while the first author was visiting UNSW. The first author would like to thank the NSF for their support under NSF Grant 0072505. The second author gratefully acknowledges the support of the Faculty Research Grants Program, UNSW.

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