

Existence of Solutions to Second-Order BVPs on Time Scales

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Abstract: We consider a boundary value problem (BVP) for systems of second-order dynamic equations on time scales. Using methods involving dynamic inequalities, we formulate conditions under which all solutions to a certain family of systems of dynamic equations satisfy certain a priori bounds. These results are then applied to guarantee the existence of solutions to BVPs for systems of dynamic equations on time scales.

Key words: *Boundary value problem, system of equations, time scale, dynamic equation, existence of solutions*

AMS Subject Classification. 39A10, 39A99.

1. INTRODUCTION

We are concerned here with the existence of solutions to the second-order vector dynamic equation

$$(1.1) \quad y^{\Delta\Delta} = f(t, y^\sigma), \quad t \in [a, b]$$

subject to the boundary conditions

$$(1.2) \quad y(a) = A, \quad y(\sigma^2(b)) = B,$$

where $f : [a, b] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $A, B \in \mathbb{R}^d$. Equation (1.1) subject to (1.2) is called a boundary value problem (BVP) where t comes from a so-called “time scale” \mathbb{T} .

In a recent paper by Henderson, Peterson and Tisdell [5], a priori bound results and existence theorems for solutions to the BVP (1.1), (1.2) were given. The methods used in [5], [6] for guaranteeing a priori bounds and existence of solutions to the BVP (1.1), (1.2) both involved maximum principles. The existence theorems in [5], [6] relied on the nonlinear alternative

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and topological transversality methods, respectively. In this paper we use alternative methods to those used in [5], [6], namely our conditions do not use maximum principles. Instead we formulate conditions on $\|A\|$, $\|B\|$, $[a, \sigma^2(b)]$, and f such that a priori bounds on solutions to a certain family of systems of second-order BVPs will follow. These results are then applied to give existence theorems for solutions to (1.1), (1.2) by using Schaefer's fixed-point theorem (Theorem 1.2). We remark that, although we are considering systems of equations, our results are new even in the scalar case $d = 1$. For more on dynamic equations on time scales we refer to the books by Bohner and Peterson [2], [3]. To see a discussion of recent and future applications of dynamic equations on time scales see the featured front page article in *New Scientist* [11].

To understand the concept of time scales and the above notation, some definitions are useful.

Definition 1.1. Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\},) \text{ for all } t \in \mathbb{T}.$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. For simplicity and clarity denote $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$. Define the graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

Throughout this work the assumption is made that \mathbb{T} has the topology that it inherits from the standard topology on the real numbers \mathbb{R} . Also assume throughout that $a < b$ are points in \mathbb{T} and define the time scale interval $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$ then call the point t right-scattered; while if $\rho(t) < t$ then we say t is left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then call the point t right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then we say t is left-dense. If \mathbb{T} has a left-scattered maximum at m then define $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

We next define the so-called delta derivative. The novice could skip this definition and for the scalar case look at the results stated in Theorem 1.1. In particular in part (2) of Theorem 1.1 we see what the delta derivative is at right-scattered points and in part (3) of Theorem 1.1 we see that at right-dense points the derivative is similar to the definition given in calculus.

Definition 1.2. Fix $t \in \mathbb{T}^\kappa$ and let $y : \mathbb{T} \rightarrow \mathbb{R}^d$. Define $y^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t such that, for all $s \in U$ and each $i = 1, \dots, d$,

$$|[y_i(\sigma(t)) - y_i(s)] - y_i^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|.$$

Call $y^\Delta(t)$ the (delta) derivative of $y(t)$ at t .

Definition 1.3. If $F^\Delta(t) = f(t)$ then define the integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

The following theorem is due to Hilger [7].

Theorem 1.1. Assume that $f : \mathbb{T} \rightarrow \mathbb{R}^d$ and let $t \in \mathbb{T}^k$.

- (1) If f is differentiable at t , then f is continuous at t .
- (2) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

- (3) If f is differentiable and t is right-dense, then

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}.$$

- (4) If f is differentiable at t , then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

Next we define the important concept of right-dense continuity. An important fact concerning right-dense continuity is that every right-dense continuous function has a delta antiderivative [2, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists.

Definition 1.4. We say that $f : \mathbb{T} \rightarrow \mathbb{R}$ is right-dense continuous (and write $f \in C_{rd}(\mathbb{T}; \mathbb{R}^d)$) provided f is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \rightarrow t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

In this paper we will be interested in so-called ‘‘regular’’ time scales which we define as follows:

Definition 1.5. We say a time scale \mathbb{T} is *regular* provided either $\mathbb{T} = \mathbb{R}$ or \mathbb{T} is an isolated time scale (i.e., all points in \mathbb{T} are isolated).

In addition to \mathbb{R} and $h\mathbb{Z} := \{0, \pm h, \pm 2h, \pm 3h, \dots\}$, $h > 0$, there are many other regular time scales, e.g. $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ ([2, Example 1.41]) which is important in the theory of orthogonal polynomials and q -difference equations, and the harmonic numbers $\mathbb{T} = \{t_n = \sum_{k=1}^n \frac{1}{k}, \quad n = 1, 2, 3, \dots\}$ ([2, Example 1.45]). For numerous other examples see Bohner and Peterson [2]. If \mathbb{T} is an isolated time scale and $f : \mathbb{T} \rightarrow \mathbb{R}$, then one can easily show that

$$(1.3) \quad f^{\sigma\Delta}(t) = \frac{\mu(\sigma(t))}{\mu(t)} f^{\Delta\sigma}(t).$$

This formula will be useful in the proof of Lemma 2.1. We next define S to be the set of all functions $y : \mathbb{T} \rightarrow \mathbb{R}^d$ given by

$$S = \{y : y \in C([a, \sigma(b)]; \mathbb{R}^d) \text{ and } y^{\Delta\Delta} \in C_{rd}([a, b]; \mathbb{R}^d)\}.$$

A solution to (1.1) is a function $y \in S$ which satisfies (1.1) for each $t \in [a, b]$. In order to prove the existence of a solution to the BVP (1.1), (1.2) the following fixed point theorem will be used.

Theorem 1.2 (Schaefer's Fixed Point Theorem). *Let X be a Banach space with norm $\|\cdot\|_X$, assume that $T : X \rightarrow X$ is compact, and that*

$$S := \{x \in X : x = \lambda Tx \text{ for some } \lambda \in [0, 1)\}$$

is bounded. Then T has a fixed point y in X . Moreover, if

$$R > \sup\{\|x\|_X : x \in S\},$$

then $\|y\|_X < R$.

Proof. This result appears in Lloyd [10, Theorem 4.4.12]. □

Let y be a solution to (1.1). In what follows below, the maximum principle of Lemma 2.1 will be applied to the ‘‘Lyapunov-type’’ function $r(t) = \|y(t)\|^2$ and then used to show that r is bounded on $[a, \sigma^2(b)]$ (and therefore solutions y are bounded on $[a, \sigma^2(b)]$).

A widely used technique in the theory of BVPs on time scales involves reformulating the BVP (1.1), (1.2) as an equivalent integral equation. In particular the BVP (1.1), (1.2) is equivalent to the integral equation

$$y(t) = \int_a^{\sigma(b)} G(t, s) f(s, y^\sigma(s)) \Delta s + \phi(t), \quad t \in [a, \sigma^2(b)],$$

where

$$G(t, s) = \begin{cases} -\frac{(\sigma(s)-a)(\sigma^2(b)-t)}{\sigma^2(b)-a}, & \sigma(s) \leq t \\ -\frac{(t-a)(\sigma^2(b)-\sigma(s))}{\sigma^2(b)-a}, & t \leq s \end{cases}$$

is the Green's function (see [2], page 171) for the BVP

$$y^{\Delta\Delta} = 0, \quad y(a) = 0, \quad y(\sigma^2(b)) = 0,$$

and

$$\phi(t) = \frac{A\sigma^2(b) - Ba + (B - A)t}{\sigma^2(b) - a}.$$

2. MAIN RESULTS

We will be concerned with the family of BVPs

$$(2.1) \quad y^{\Delta\Delta} = \lambda f(t, y^\sigma), \quad t \in [a, b]$$

$$(2.2) \quad y(a) = \lambda A, \quad y(\sigma^2(b)) = \lambda B,$$

where $\lambda \in [0, 1]$. The following result gives a priori bounds on possible solutions of the family of BVPs (2.1), (2.2). In the following we will let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on \mathbb{R}^d .

Lemma 2.1. *Assume that \mathbb{T} is a regular time scale and f satisfies*

$$(2.3) \quad \|f(t, p)\| \leq 2\alpha \langle p, f(t, p) \rangle + K$$

for $t \in [a, b]$, $p \in \mathbb{R}^d$, where $\alpha \geq 0$, and $K \geq 0$ are constants. If we choose R so that

$$(2.4) \quad R > \alpha\beta^2 + \beta + K \max_{t \in [a, \sigma(b)]} \int_a^{\sigma(b)} |G(t, s)| \Delta s,$$

where G is the Green's function given above and

$$\beta = \max\{\|A\|, \|B\|\},$$

then all solutions y of (2.1), (2.2) satisfy

$$\|y(t)\| < R, \quad t \in [a, \sigma^2(b)],$$

for $\lambda \in [0, 1]$ (note that R is independent of λ).

Proof. Assume y is a solution of (2.1), (2.2) on $[a, \sigma^2(b)]$ for some $\lambda \in [0, 1]$. Then $y(t)$ solves the equivalent integral equation

$$(2.5) \quad y(t) = \phi_\lambda(t) + \int_a^{\sigma(b)} G(t, s) \lambda f(s, y^\sigma(s)) \Delta s,$$

where

$$\phi_\lambda(t) = \lambda \phi(t)$$

and $\phi(t)$ and the Green's function $G(t, s)$ are given above. It follows that

$$\begin{aligned} \|y(t)\| &\leq \int_a^{\sigma(b)} |G(t, s)| \lambda \|f(s, y^\sigma(s))\| \Delta s + \|\phi_\lambda(t)\| \\ &\leq \int_a^{\sigma(b)} |G(t, s)| \lambda \|f(s, y^\sigma(s))\| \Delta s + \beta. \end{aligned}$$

Using (2.3), we obtain

$$\begin{aligned}
\|y(t)\| &\leq \int_a^{\sigma(b)} |G(t, s)| (2\alpha \langle y^\sigma(s), \lambda f(s, y^\sigma(s)) \rangle + \lambda K) \Delta s + \beta \\
(2.6) \quad &\leq \int_a^{\sigma(b)} |G(t, s)| (2\alpha \langle y^\sigma(s), \lambda f(s, y^\sigma(s)) \rangle + K) \Delta s + \beta.
\end{aligned}$$

Let

$$r(t) := \|y(t)\|^2, \quad t \in [a, \sigma^2(b)],$$

then

$$r^\Delta(t) = \langle y^\sigma(t), y^\Delta(t) \rangle + \langle y(t), y^\Delta(t) \rangle,$$

and so, using the product rule, we have

$$\begin{aligned}
&r^{\Delta\Delta}(t) \\
&= \langle (y^\sigma(t))^\Delta, y^{\Delta\sigma}(t) \rangle + \langle y^\sigma(t), y^{\Delta\Delta}(t) \rangle \\
&+ \langle y^\sigma(t), y^{\Delta\Delta}(t) \rangle + \langle y^\Delta(t), y^\Delta(t) \rangle \\
&= c(t) \|y^{\sigma\Delta}(t)\|^2 + 2 \langle y^\sigma(t), y^{\Delta\Delta}(t) \rangle + \|y^\Delta(t)\|^2, \quad \text{using (1.3),}
\end{aligned}$$

where

$$c(t) = \begin{cases} 1, & \mathbb{T} = \mathbb{R} \\ \frac{\mu(\sigma(t))}{\mu(t)}, & \mathbb{T} \text{ is isolated.} \end{cases}$$

Therefore we obtain

$$r^{\Delta\Delta}(t) \geq 2 \langle y^\sigma(t), \lambda f(t, y^\sigma(t)) \rangle.$$

Using this in (2.6), we get

$$\|y(t)\| \leq \alpha \int_a^{\sigma(b)} |G(t, s)| (r^{\Delta\Delta}(t) + K) \Delta s + \beta.$$

Now consider

$$\begin{aligned}
&\int_a^{\sigma(b)} |G(t, s)| r^{\Delta\Delta}(s) \Delta s \\
&= \int_a^t |G(t, s)| r^{\Delta\Delta}(s) \Delta s + \int_t^{\sigma(b)} |G(t, s)| r^{\Delta\Delta}(s) \Delta s \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 := \int_a^t |G(t, s)| r^{\Delta\Delta}(s) \Delta s, \quad I_2 := \int_t^{\sigma(b)} |G(t, s)| r^{\Delta\Delta}(s) \Delta s.$$

Then

$$\begin{aligned}
I_1 &= \int_a^t |G(t, s)| r^{\Delta\Delta}(s) \Delta s \\
&= \frac{\sigma^2(b) - t}{\sigma^2(b) - a} \int_a^t (\sigma(s) - a) r^{\Delta\Delta}(s) \Delta s \\
&= \frac{\sigma^2(b) - t}{\sigma^2(b) - a} \left(\{(s - a) r^\Delta(s)\}_a^t - \int_a^t r^\Delta(s) \Delta s \right) \\
&= \frac{\sigma^2(b) - t}{\sigma^2(b) - a} ((t - a) r^\Delta(t) - r(t) + r(a)) \\
&\leq \frac{\sigma^2(b) - t}{\sigma^2(b) - a} ((t - a) r^\Delta(t) + \|A\|^2) \\
&\leq \frac{\sigma^2(b) - t}{\sigma^2(b) - a} ((t - a) r^\Delta(t) + \beta^2).
\end{aligned}$$

Similarly,

$$I_2 \leq \frac{t - a}{\sigma^2(b) - a} (-(\sigma^2(b) - t) r^\Delta(t) + \beta^2).$$

Hence

$$I_1 + I_2 \leq \beta^2,$$

and so we finally get that

$$\|y(t)\| \leq \alpha\beta^2 + \beta + K \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} |G(t, s)| \Delta s < R$$

for $t \in [a, \sigma^2(b)]$. □

Remark 2.1. If $\mathbb{T} = \mathbb{R}$, it is well known (see e.g. [9], Theorem 5.108) that

$$\int_a^{\sigma(b)} |G(t, s)| \Delta s = \int_a^b |G(t, s)| ds \leq \frac{(b - a)^2}{8}$$

for $t \in [a, \sigma^2(b)] = [a, b]$, and if $\mathbb{T} = \mathbb{Z}$ (see e.g. [8], Exercise 6.20)

$$\int_a^{\sigma(b)} |G(t, s)| \Delta s \leq \frac{(b + 2 - a)^2}{8}$$

for $t \in [a, \sigma^2(b)] = [a, b + 2]$.

Theorem 2.1. *Assume the hypotheses of Lemma 2.1 hold. Then the BVP (1.1), (1.2) has a solution $y(t)$. Furthermore if we choose R so that (2.4) holds, then*

$$\|y(t)\| < R, \quad t \in [a, \sigma^2(b)].$$

Proof. Let X be the Banach space defined by

$$X = \{x : [a, \sigma^2(b)] \rightarrow \mathbb{R}^d \text{ such that } x \text{ is continuous}\},$$

with norm $\|\cdot\|_X$ defined by

$$\|x\|_X := \sup\{\|x(t)\| : t \in [a, \sigma^2(b)]\}.$$

Define $T : X \rightarrow X$ by

$$Ty(t) = \phi(t) + \int_a^{\sigma(b)} G(t, s)f(s, y^\sigma(s))\Delta s,$$

$t \in [a, \sigma^2(b)]$. Then the BVP (1.1), (1.2) has a solution iff T has a fixed point. Using the Ascoli–Arzela theorem (Theorem 8.26, [9]) it can be shown that $T : X \rightarrow X$ is compact. Let

$$S := \{x \in X : x = \lambda Tx, \lambda \in [0, 1)\}.$$

Since $x = \lambda Tx$ iff the BVP (2.1), (2.2) has a solution x , we have by Lemma 2.1 that if R is chosen so that (2.4) holds then S is bounded by R . But then by Schaefer’s Fixed-Point theorem (Theorem 1.2), T has a fixed point $y \in X$ with $\|y\|_X < R$, which implies that y is a solution of the BVP (1.1), (1.2) with

$$\|y(t)\| < R, \quad t \in [a, \sigma^2(b)].$$

□

Corollary 2.1. *If $f : [a, \sigma^2(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there are constants $\alpha \geq 0$, $K \geq 0$ such that*

$$(2.7) \quad |f(t, p)| \leq 2\alpha pf(t, p) + K$$

for $t \in [a, \sigma^2(b)]$, $p \in \mathbb{R}$, then the scalar BVP

$$\begin{aligned} y^{\Delta\Delta} &= f(t, y^\sigma) \\ y(a) &= A, \quad y(\sigma^2(b)) = B \end{aligned}$$

has a solution $y(t)$. Furthermore if we choose R so that (2.4) holds, where $\beta = \max\{|A|, |B|\}$, then

$$|y(t)| < R, \quad t \in [a, \sigma^2(b)].$$

Example 2.1. Assume $0 \leq a < \sigma^2(b)$ and A and B are constants. Consider the scalar BVP

$$(2.8) \quad y^{\Delta\Delta} = t(y^\sigma)^3$$

$$(2.9) \quad y(a) = A, \quad y(\sigma^2(b)) = B$$

where $0 \leq a < \sigma^2(b)$. Note $f(t, p) = tp^3$ and hence

$$\begin{aligned} |f(t, p)| &\leq t(p^4 + 1) \\ &= pf(t, p) + t \\ &\leq pf(t, p) + \sigma^2(b) \end{aligned}$$

for $t \in [a, \sigma^2(b)]$. Hence (2.7) holds with $\alpha = \frac{1}{2}$, $K = \sigma^2(b)$. Hence by Corollary 2.1 the BVP (2.8), (2.9) has a solution $y(t)$. Furthermore if

$$R > \frac{1}{2}\beta^2 + \beta + \sigma^2(b) \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} |G(t, s)| \Delta s,$$

then

$$(2.10) \quad |y(t)| < R$$

for $t \in [a, \sigma^2(b)]$. In particular, from Remark 2.1 we have that (2.10) holds for $\mathbb{T} = \mathbb{R}$ provided we choose

$$R > \frac{1}{2}\beta^2 + \beta + \frac{b^2(b-a)^2}{8}$$

and (2.10) holds for $\mathbb{T} = \mathbb{Z}$ provided we choose

$$R > \frac{1}{2}\beta^2 + \beta + \frac{(b+2)^2(b+2-a)^2}{8}.$$

REFERENCES

- [1] R. P. Agarwal, M. Bohner, D. O'Regan, and A. Peterson, Dynamic equations on time scales: A survey, *J. Comp. Appl. Math.*, Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, (Preprint in Ulmer Seminare 5) 141(1-2), (2002), 1–26.
- [2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
- [3] M. Bohner and A. Peterson, Editors, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2004.
- [4] L. Erbe and A. Peterson, Riccati equations on a measure chain, In G. S. Ladde, N. G. Medhin, and M. Sambandham, editors, *Proceedings of Dynamic Systems and Applications*, volume 3, pages 193–199, Atlanta, 2001. Dynamic publishers.
- [5] J. Henderson, A. Peterson, and C. C. Tisdell On the Existence and Uniqueness of Solutions to Boundary Value Problems on Time Scales, *Advances in Difference Equations*, submitted.
- [6] J. Henderson and C. C. Tisdell, Topological Transversality and Boundary Value Problems on Time Scales, *J. Math. Anal. Appl.*, 289 (2004), 110–125.
- [7] S. Hilger, Analysis on measure chains — a unified approach to continuous and discrete calculus, *Results Math.* 18, (1990), 18–56.
- [8] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, Second Edition, Harcourt Academic Press, 2001.
- [9] W. Kelley and A. Peterson, *The Theory of Differential Equations: Classical and Qualitative*, Pearson Prentice Hall, 2004.

- [10] N. G. Lloyd, *Degree Theory*, Cambridge University Press, London, 1978.
- [11] Vanessa Spedding, Taming Nature's Numbers, *New Scientist*, July 19, 2003, 28–31.