Oscillation and Nonoscillation of Solutions of Second Order Linear Dynamic Equations with Integrable Coefficients on time scales

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Abstract. We obtain Willett-Wong-type oscillation and nonoscillation theorems for second order linear dynamic equations with integrable coefficients on a time scale. The results obtained extend and are motivated by oscillation and nonoscillation results due to Willett [**20**] and Wong [**21**] for the second order linear differential equation. As applications of the new results obtained, we give the complete classification of oscillation and nonoscillation for the difference equations

$$\Delta^2 x(n) + b\frac{(-1)^n}{t^c}x(n+1) = 0$$

and

$$\Delta^2 x(n) + \left[\frac{a}{t^{c+1}} + b\frac{(-1)^n}{t^c}\right]x(n+1) = 0,$$

for $t \in \mathbb{N}$, $a, b, c \in \mathbb{R}$. We also improve a nonoscillation result of Mingarelli [**17**] and extend an oscillation result of Del Medico and Kong [**7**].
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1. Introduction

In two fundamental papers [**20**], [**21**], Willett and Wong extended and improved oscillation and nonoscillation criteria which had been obtained earlier by many authors for the differential equation

$$x'' + p(t)x = 0. \tag{1.1}$$

Their goal was to be able to handle the difficult cases which arise when $p(t)$ is not eventually of one sign. Their work also surveyed earlier results of Wintner [**19**], Fite [**11**], Hille [**13**], and Hartman [**15**] for the cases when $\int^\infty p(s)ds$ exists. In this paper we obtain 'Willett-Wong-type' criteria for dynamic equations on time scales by means of a 'second-level Riccati equation' (see [**3**] for the discrete case) or what Wong refers to as a new Riccati integral equation in the continuous case. Using this approach, one is able to handle various critical cases. These ideas are of particular importance in treating the case when $P(t) := \int_t^\infty p(s)ds$ is also not eventually of one sign.

Suppose that $\lim_{t\to\infty} \int_{t_0}^t p(s)ds$ exists and let us define $P(t) := \int_t^\infty p(s)ds$. Willett [**20**] and Wong [**21**], respectively, proved the following:
Theorem A. Suppose that

$$\int_t^\infty \bar{P}^2(s)Q_P(s,t)ds \le \frac{1}{4}\bar{P}(t),$$

for large t, where $\bar{P}(t) = \int_t^\infty P^2(s)Q_P(s,t)ds,\ Q_P(s,t) = \exp\left(2\int_t^s P(\tau)d\tau\right)$. Then the equation $x'' + p(t)x(t) = 0$ is nonoscillatory.
Theorem B. If $\bar{P}(t) \not\equiv 0$ satisfies

$$\int_t^\infty \bar{P}^2(s)Q_P(s,t)ds \ge \frac{1+\epsilon}{4}\bar{P}(t),$$

for some $\epsilon > 0$ and all large t. Then the equation $x'' + p(t)x(t) = 0$ is oscillatory.

As applications of Theorem A and Theorem B, Willett [**20**] gives the complete classification of oscillation and nonoscillation for the equation

$$x'' + (\mu t^\eta \sin \nu t)x = 0 \tag{1.2}$$

for $|\frac{\mu}{\nu}| \ne \frac{1}{\sqrt{2}}$, $\mu \ne 0, \nu \ne 0, \eta$ constants, and the equation

$$x'' + (\lambda t^{-2} + \mu t^{-1} \sin \nu t)x = 0 \tag{1.3}$$

for $\lambda \ne \frac{1}{4} - \frac{1}{2}(\frac{\mu}{\nu})^2$, $\mu, \lambda, \nu \ne 0$ constants.

Wong [**21**] established the following theorem.
Theorem C. If there exists a function $\bar{B}(t)$ such that

$$\int_t^\infty (\bar{P}(s) + \bar{B}(s))^2 Q_P(s,t)ds \le \bar{B}(t),$$

for large t, then the equation $x'' + p(t)x = 0$ is nonoscillatory.

As an application of Theorem C, Wong was able to treat the critical cases and showed that equation (1.2) is nonoscillatory, for $|\frac{\mu}{\nu}| = \frac{1}{\sqrt{2}}$ and that equation (1.3) is nonoscillatory, for $\lambda = \frac{1}{4} - \frac{1}{2}(\frac{\mu}{\nu})^2$.

In this paper, we extend Theorems A, B and C to second order dynamic equations on time scales. As applications, we give the complete classification of oscillation and nonoscillation for the difference equations

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0$$

and

$$\Delta^2 x(n) + \left[\frac{a}{n^{c+1}} + b\frac{(-1)^n}{n^c}\right]x(n+1) = 0,$$

for $t = n \in \mathbb{N}$, $a, b, c \in \mathbb{R}$.

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. Since the introduction of time scales calculus by Stefan Hilger [**12**] in 1988, there has been a great deal of interest in extending and unifying the discrete and continuous cases. Consider the second order dynamic equation

$$x^{\Delta\Delta} + p(t)x^{\sigma}(t) = 0, \tag{1.4}$$

where p is a right-dense continuous function on $\mathbb{T}$. We shall assume throughout that $\int_{t_0}^{\infty} p(s)\Delta s$ is convergent.

For completeness, (see [**5**] and [**6**] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \varnothing = \sup \mathbb{T}$, and where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given an interval $[c, d] := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c, d]^{\kappa}$ denotes the interval $[c, d]$ in case $\rho(d) = d$ and denotes the interval $[c, d)$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

We say that $f : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$f^{\Delta}(t) := \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

exists. (Here by $s \to t$ it is understood that s approaches t in the time scale). When f is continuous at t and $\sigma(t) > t$

$$f^{\Delta}(t) := \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$, the delta derivative is just the forward difference operator.

We always have the relation $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{rd}. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable on $[c, d]^{\kappa}$ and whose delta derivative is rd-continous on $[c, d]^{\kappa}$ is denoted by C^1_{rd}. If $F^{\Delta}(t) = f(t)$ for $t \in \mathbb{T}$, then $F(t)$ is said to be a (delta) antiderivative of $f(t)$. If $F(t)$ is a (delta) antiderivative of $f(t)$ then we define the Cauchy (delta) integral of $f(t)$ on $[a, b]$ by

$$\int_a^b f(t)\ \Delta t = F(b) - F(a).$$

See [**5**] for elementary properties of the Cauchy integral. A very basic result ([**5**, Theorem 1.74]) is that if $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$, then it has a (delta) antiderivative on $\mathbb{T}$ and hence the integral $\int_a^b f(t)\ \Delta t$ exists. Therefore, our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales.

We recall that a solution of equation (1.4) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.4) is said to be oscillatory in case all of its solutions are oscillatory.

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is **regressive** provided that

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}.$$

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are right-dense continuous and regressive by $\mathcal{R}$. If $p \in \mathcal{R}$, then we can define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $t \in \mathbb{T}, s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{h=0.} \end{cases}$$

(Here Log denotes the principal logarithm function.)

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is **positively regressive** (denoted by $p \in \mathcal{R}+$) provided that

$$1 + \mu(t)p(t) > 0, \quad t \in \mathbb{T}.$$

2. Notation and Preliminary Lemmas

Lemmas 2.1, 2.3, and 2.4 and the definitions of condition C and condition D were introduced in [**2**].

Let $\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later in Section 4, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following: (see [**8**]).

Condition C : We say that $\mathbb{T}$ satisfies condition C if there exists an $M > 0$ such that

$$\chi(t) \leq M\mu(t), \quad t \in \mathbb{T}.$$

LEMMA 2.1. *Assume that $\mathbb{T}$ satisfies condition C and suppose that equation (1.4) is nonoscillatory. Let $x(t)$ be a solution of (1.4) with $x(t) > 0$ on $[t_0, \infty)$. Then*

$$z(t) := \frac{x^{\Delta}(t)}{x(t)}$$

is a solution of the Riccati equation

$$z^{\Delta} + p(t) + \frac{z^2}{1 + \mu(t)z} = 0.$$

on $[t_0, \infty)$. Moreover, if $\int_{t_0}^{\infty} p(t)\Delta t$ is convergent, then $\int_{t_0}^{\infty} \frac{z^2(s)}{1+\mu(s)z(s)}\Delta s$ is also convergent and $\lim_{t\to\infty} z(t) = 0$.

We will also need below conditions which guarantee that $\int_1^t \frac{1}{s}\Delta s$ does not grow faster than $M \ln t$, for some $M > 0$. For a time scale $\mathbb{T}$, the following example shows that the inequality $\int_1^t \frac{1}{s}\Delta s \leq M \ln t$, for any $M > 1$, does not hold in general without some additional restrictions.

EXAMPLE 2.2. Consider the time scale

$$\mathbb{T} = \{2^{2^k}, k \in \mathbb{N}_0\}.$$

It is easy to see from the definition of the integral that for $t_k = 2^{2^k}$ we have

$$\lim_{k\to\infty} \frac{\int_{t_0}^{t_k} \frac{1}{s}\Delta s}{\ln t_k} = \frac{1}{\ln 2} \lim_{k\to\infty} \frac{1}{2^k} \sum_{j=0}^{k-1} \left(2^{2^j} - 1\right) = \infty.$$

So we shall impose an additional assumption on the time scale $\mathbb{T}$ to establish the oscillation results in Section 4. We note first that if $\mathbb{T}$ satisfies condition C, then the set

$$\check{\mathbb{T}} = \{t \in \mathbb{T} |\ t > 0 \text{ is isolated or right scattered or left scattered}\}$$

is necessarily countable since a bounded real interval can contain only finitely many elements of $\check{\mathbb{T}}$.

We introduce the following

Condition D Suppose that $\mathbb{T}$ satisfies condition C and let

$$\check{\mathbb{T}} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\},$$

where

$$0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots.$$

Then we say $\mathbb{T}$ satisfies **Condition D** if there is a constant $K > 1$ such that

$$\frac{t_{k+1} - t_k}{t_k - t_{k-1}} \leq K, \quad \text{for all } k \geq 1.$$

That is, for right or left-scattered points or for isolated points, condition D says that the ratio $\frac{\mu(t_k)}{\mu(t_{k-1})}$ is uniformly bounded for all k when we consider the time scale $\check{\mathbb{T}}$.

Lemma 2.3. [**2**, Lemma 2.3] *Assume that $\mathbb{T}$ satisfies condition (D) and suppose that $x(t)$ is a solution of (1.4) that satisfies $x(t) > 0$ for $t \geq T = t_k$, for some $k \geq 0$. Then we have, for $t \in \mathbb{T}$, $t \geq T$,*

$$\ln \frac{x(t)}{x(T)} \leq \int_T^t \frac{x^\Delta(s)}{x(s)}\,\Delta s \quad \text{and} \quad \int_T^t \frac{1}{s}\,\Delta s \leq K \ln \frac{t}{T}.$$

Lemma 2.4. [**2**, Lemma 2.4] *Assume that $\int_{t_0}^\infty p(t)\Delta t$ is convergent, $P(t) = \int_t^\infty p(s)\Delta s$, $\mu(t)$ is bounded and that $\mathbb{T}$ satisfies condition (D). If (1.4) is nonoscillatory, then there is a $T \in [t_0, \infty)$ such that*

$$\int_T^\infty P^2(t) e_{\frac{2P}{1-\mu P}}(t, T)\Delta t < \infty.$$

Also, if $x(t) > 0$ is a positive solution of (1.4) on $[T, \infty)$ and $z(t) := \frac{x^\Delta(t)}{x(t)}$, then $z(t)$ is a solution of the Riccati equation

$$z^\Delta + p(t) + \frac{z^2}{1 + \mu(t)z} = 0.$$

on $[T, \infty)$, with $1 + \mu(t)z(t) > 0$ on $[T, \infty)$. Furthermore,

$$w(t) := \int_t^\infty \frac{z^2(s)}{1 + \mu(t)z(s)}\Delta s > 0$$

satisfies the integral equation

$$w(t) = \int_t^\infty e_{\frac{2P}{1-\mu P}}(s, t)\, \frac{P^2(s) + w(s)w(\sigma(s))}{1 - \mu(s)P(s)}\Delta s. \tag{2.1}$$

for large $t \in [T, \infty)$.

The following lemma appears in [**9**].

Lemma 2.5 (Riccati technique). *Equation (1.4) is nonoscillatory if and only if there exists $T \in [t_0, \infty)$ and a function u satisfying the Riccati dynamic inequality*

$$u^\Delta(t) + p(t) + \frac{u^2(t)}{1 + \mu(t)u(t)} \leq 0$$

with $1 + \mu(t)u(t) > 0$ for $t \in [T, \infty)$.

3. Wong-Willett-type nonoscillation theorems

In the results of this section, which give sufficient conditions for nonoscillation, we do not need to assume that $\mu(t)$ is bounded nor that condition D holds, in contrast to the results in Section 4 dealing with oscillation criteria. The first result may be regarded as an extension of Theorem C above.

THEOREM 3.1. *Assume that* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent and define* $P(t) = \int_t^{\infty} p(s)\Delta s$, *and let* $T \in [t_0, \infty)$ *be such that* $1 \pm \mu(t)P(t) > 0$, *for* $t \geq T$. *If* $\int_T^{\infty} P^2(t) e_{\frac{2P}{1-\mu P}}(t,T)\Delta t$ *converges and there exists a function* $w(t) > 0$, *for large* $t \geq T$, *satisfying*

$$\int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t) \frac{P^2(s) + w(s)w(\sigma(s))}{1 - \mu(s)P(s)} \Delta s \leq w(t) \tag{3.1}$$

for large $t \geq T$, *then equation* (1.4) *is nonoscillatory.*

PROOF. Let

$$u(t) := P(t) + \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t) \frac{P^2(s) + w(s)w(\sigma(s))}{1 - \mu(s)P(s)} \Delta s. \tag{3.2}$$

for large t. We recall the following facts dealing with elementary properties of the exponential functions ([**5**, Theorem 2.36]):

$$e_P(\sigma(t), T) = (1+\mu(t)P(t))e_P(t,T), \quad e_{-P}(\sigma(t), T) = (1-\mu(t)P(t))e_{-P}(t,T),$$

$$e_{\frac{2P}{1-\mu P}}(s,t) = e_{\frac{-2P}{1+\mu P}}(t,T) e_{\frac{2P}{1-\mu P}}(s,T),$$

$$e_{\frac{-2P}{1+\mu P}}(t,T) = \frac{e_{-P}(t,T)}{e_P(t,T)}, \quad e_{\frac{2P}{1-\mu P}}(s,T) = \frac{e_P(s,T)}{e_{-P}(s,T)}.$$

From these relations we obtain after some manipulations

$$\begin{aligned}
u^{\Delta}(t) &= -p(t) - \frac{2P(t)}{1+\mu(t)P(t)} \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t) \frac{P^2(s) + w(s)w(\sigma(s))}{1 - \mu(s)P(s)} \Delta s \\
&- \frac{P^2(t) + w(t)w(\sigma(t))}{1 + \mu(t)P(t)} \\
&= -p(t) - \frac{2P(t)}{(1+\mu(t)P(t))}[u(t) - P(t)] \\
&- \frac{1}{1+\mu(t)P(t)}[P^2(t) + w(t)w(\sigma(t))] \\
&= -p(t) + \frac{P^2(t) - 2P(t)u(t) - w(t)w(\sigma(t))}{1 + \mu(t)P(t)}.
\end{aligned}$$

From $1 \pm \mu(t)P(t) > 0$, $w(t) > 0$, (3.1) and (3.2), we have that $w(t) \geq u(t) - P(t) \geq 0$, for large t. Hence, we have

$$u^{\Delta}(t) \leq -p(t) + \frac{P^2(t) - 2P(t)u(t) - (u(t) - P(t))(u(\sigma(t)) - P(\sigma(t)))}{1 + \mu(t)P(t)}. \tag{3.3}$$

Then from (3.3), noting that

$$P(\sigma(t)) = P(t) - \mu(t)p(t), \quad u(\sigma(t)) = u(t) + \mu(t)u^{\Delta}(t),$$

we obtain after some manipulations (suppressing arguments)

$$u^{\Delta} \leq -p - \frac{u^2 + \mu u u^{\Delta} - \mu p P + \mu p u - P u^{\Delta} \mu}{1 + \mu P}.$$

So multiplying both sides by $1+\mu P$, and simplifying we get

$$(1+\mu u)u^{\Delta} \le -p(1+\mu u) - u^2. \tag{3.4}$$

From (3.2), we have $u(t) - P(t) \ge 0$, so $1+\mu(t)u(t) \ge 1+\mu(t)P(t) > 0$, for large t. Hence solving (3.4) for u^{Δ} we obtain

$$u^{\Delta}(t) \le -p(t) - \frac{u^2(t)}{1+\mu(t)u(t)} \tag{3.5}$$

for large t. From (3.5) and Lemma 2.5, it follows that (1.4) is nonoscillatory. This completes the proof. □

THEOREM 3.2. *Assume that* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent, and suppose that* $\mu(t)$ *is bounded. If* $\int_T^{\infty} P^2(t)e_{\frac{2P}{1-\mu P}}(t,T)\Delta t$ *is convergent for sufficiently large* T *and*

$$\bar{P}(t) := \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s \tag{3.6}$$

satisfies

$$\int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s \le \frac{1}{4}\bar{P}(t) \tag{3.7}$$

for large t*, then equation* (1.4) *is nonoscillatory.*

PROOF. From (3.7), one has immediately that $w(t) = 2\bar{P}(t)$ satisfies (3.1), so the result is a consequence of Theorem 3.1. □

For $\mathbb{T} = \mathbb{Z}$, Theorem 3.2 yields the following

COROLLARY 3.3 (Chen and Erbe [**3**]). *Suppose that* $\sum p_j$ *and* $\sum \frac{P_n^2}{q_n}$ *converge, where* $P_n = \sum_{j=n}^{\infty} p_j$.

Let $N \ge 0$ *be so large that* $|P_n| < 1$*, for* $n \ge N$*. Define*

$$g(j;n) := \frac{q_n}{q_{j+1}(1+P_j)} \quad \text{for} \quad j \ge n \ge N,$$

$$q_n := \prod_{j=N}^{n-1} \frac{1-P_j}{1+P_j}, \quad n \ge N+1, \quad \bar{P}_n := \sum_{j=n}^{\infty} g(j;n)P_j^2.$$

If $\bar{P}$ *satisfies*

$$\sum_{j=n}^{\infty} g(j;n)\bar{P}_j\bar{P}_{j+1} \le \frac{1}{4}\bar{P}_n, \ n \ge N, \text{ for some } N \ge 0,$$

then the difference equation $\Delta^2 x_n + p_n x_{n+1} = 0$ *is nonoscillatory.*

4. Willett-Wong-type oscillation theorems for dynamic equations

In order to obtain the desired oscillation analogues of Theorem B above, we need to place additional restrictions on the time scale $\mathbb{T}$.

THEOREM 4.1. *Assume that* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent,* $t_0 > 0$*, and let* $P(t) := \int_t^{\infty} p(s)\Delta s$*. Assume that* $\mu(t)$ *is bounded and that* $\mathbb{T}$ *satisfies condition D. Let*

$$\bar{P}(t) := \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s. \tag{4.1}$$

If $\bar{P}(t) \not\equiv 0$ *satisfies*

$$\bar{P}(t) = \infty, \quad \text{for all large } t, \tag{4.2}$$

or

$$\int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s \geq \frac{1+\epsilon}{4}\bar{P}(t). \tag{4.3}$$

for some $\epsilon > 0$ *and all large* t*, then equation* (1.4) *is oscillatory.*

We shall first establish the following lemma.

LEMMA 4.2. *Assume that* $\mu(t)$ *is bounded, the improper integral* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent and let* $P(t) := \int_t^{\infty} p(s)\Delta s$*. Assume that* $Q(t)$ *is a nonnegative continuous function defined for* $T \leq t < \infty$*. If there exists* $\epsilon > 0$ *such that*

$$\int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{Q(s)Q(\sigma(s))}{1-\mu(s)P(s)}\Delta s \geq \frac{1+\epsilon}{4}Q(t),\ t \geq T, \tag{4.4}$$

then the inequality

$$v(t) \geq Q(t) + \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{v(s)v(\sigma(s))}{1-\mu(s)P(s)}\Delta s,\ t \geq T, \tag{4.5}$$

does not have a continuous nonnegative solution $v(t)$*.*

PROOF. Assume that $v(t)$ is a continuous nonnegative function which satisfies (4.5). Then $v(t) \geq Q(t) \geq 0$ implies $v(\sigma(t)) \geq Q(\sigma(t)) \geq 0$, which implies in turn that

$$\begin{aligned} v(t) &\geq Q(t) + \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{Q(s)Q(\sigma(s))}{1-\mu(s)P(s)}\Delta s \\ &\geq \left(1+\frac{1+\epsilon}{4}\right)Q(t),\ t \geq T. \end{aligned}$$

Continuing in this manner, we obtain

$$v(t) \geq \left[1+\left(1+\frac{1+\epsilon}{4}\right)^2\left(\frac{1+\epsilon}{4}\right)\right]Q(t).$$

This gives

$$v(t) \geq a_n Q(t),$$

where $1 = a_0 < a_1 < a_2 < \cdots < a_n \to \infty$, as $n \to \infty$, which is a contradiction. $\square$

Proof of Theorem 4.1: Assume (1.4) is nonoscillatory. By Lemma 2.4, there exists a function $v(t) > 0$ which satisfies

$$v(t) = \int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s) + v(s)v(\sigma(s))}{1-\mu(s)P(s)}\Delta s \tag{4.6}$$

for large t. If $\bar{P}(t) = \infty$, then it follows that (4.6) cannot hold, which is a contradiction.

If $\bar{P}(t) < \infty$ then from (4.1) and (4.6), we have

$$v(t) = \int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{v(s)v(\sigma(s))}{1-\mu(s)P(s)}\Delta s + \bar{P}(t). \tag{4.7}$$

However, Lemma 4.2 and (4.3) imply that no continuous nonnegative function $u(t)$ can satisfy (4.7) for all $t \geq T$. From this contradiction, we conclude that (1.4) is oscillatory.

When $\mathbb{T} = \mathbb{N}$, we get the following new oscillation counterpart to Corollary 3.3

Corollary 4.3. *Assume that $\sum p_i$ is convergent, $P_n = \sum_{i=n}^\infty p_i$. Let $N \geq 0$ be so large that $|P_n| < 1$, for $n \geq N$. We define*

$$q_n := \prod_{j=N}^{n-1} \frac{1-P_j}{1+P_j}, \quad g(j;n) := \frac{q_n}{q_{j+1}(1+P_j)},$$

$$\bar{P}_n = \sum_{j=n}^\infty g(j;n)P_j^2, \quad \text{for } j \geq n \geq N.$$

If $\bar{P}_n \not\equiv 0$ satisfies

$$\bar{P}_n = \infty, \tag{4.8}$$

or

$$\sum_{j=n}^\infty g(j;n)\bar{P}_j\bar{P}_{j+1} \geq \frac{1+\epsilon}{4}\bar{P}_n,$$

for some ϵ and large $n \geq N$, then the difference equation $\Delta^2 x_n + p_n x_{n+1} = 0$ is oscillatory.

5. Example 1.

Let

$$p(t) := \frac{b(-1)^t}{t}, \quad t \in \mathbb{T} = \mathbb{N}, \quad b \in \mathbb{R}, \ \ b \neq 0.$$

Let us set $P_j = P(j)$. We have

$$\begin{aligned} P_{2k} &= \sum_{j=2k}^{\infty} p(j) \\ &= b\left(\frac{1}{2k} - \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{2k+3} + \cdots\right) \\ &= b\left(\frac{1}{2k(2k+1)} + \frac{1}{(2k+2)(2k+3)} + \cdots\right). \end{aligned}$$

It follows that

$$\frac{|b|}{4}\sum_{j=k}^{\infty}\frac{1}{j(j+1)} \le |P_{2k}| \le \frac{|b|}{4}\sum_{j=k}^{\infty}\frac{1}{j^2}$$

and hence we have

$$P_{2k} \sim \frac{b}{4k} = \frac{b}{2}\frac{1}{2k}.$$

Similarly we have

$$P_{2k+1} \sim -\frac{b}{2}\frac{1}{2k+1}.$$

So

$$P_n \sim (-1)^n \frac{b}{2}\frac{1}{n}. \tag{5.1}$$

Therefore the series $\sum_{k=n}^{\infty} P_k$ converges.

Since $\sum \frac{1}{k^2}$ converges, we have that $\sum_{k=n}^{\infty} P_k^2$ converges. Using $\ln(1+x) = x - \frac{1}{2}x^2 + \mathrm{o}(x^2)$ as $x \to \infty$, we have for large j

$$\ln\left(1 - \frac{2P_j}{1+P_j}\right) = -\frac{2P_j}{1+P_j} - \frac{1}{2}\left(\frac{2P_j}{1+P_j}\right)^2 + \mathrm{o}\left(\left(\frac{2P_j}{1+P_j}\right)^2\right) \tag{5.2}$$

as $j \to \infty$. Also, we have

$$\frac{P_j}{1+P_j} = P_j(1 - P_j + O(P_j^2)).$$

So from (5.2) we have

$$\sum_{j=N}^{\infty} \ln\left(\frac{1-P_j}{1+P_j}\right) = \sum_{j=N}^{\infty} \ln\left(1 - \frac{2P_j}{1+P_j}\right)$$

is convergent. So

$$q_n = \prod_{N}^{n-1} \frac{1-P_j}{1+P_j} = \exp\left(\sum_{j=N}^{n-1} \ln\left(1 - \frac{2P_j}{1+P_j}\right)\right) < 1 + \epsilon, \text{for large } N.$$

We also have

$$g(j,n) = \frac{q_n}{q_{j+1}(1+P_j)} \le 1 + \epsilon_1, \quad j \ge n \ge N,$$

where we used $P_j \to 0, q_j \to 1$. So we get that

$$\sum_{j=n}^{\infty} g(j,n)\bar{P}_j\bar{P}_{j+1} \le (1+\epsilon_1)\sum_{j=n}^{\infty}\bar{P}_j\bar{P}_{j+1}, \tag{5.3}$$

where $\bar{P}_n = \sum_{j=n}^{\infty} g(j,n)P_j^2$. By (5.1), we get that

$$\begin{aligned}
\bar{P}_n &= \sum_{j=n}^{\infty} g(j,n)P_j^2 \le (1+\epsilon_1)\sum_{j=n}^{\infty} P_j^2 \\
&\le (1+\epsilon_1)(1+\epsilon_2)\left(\frac{b}{2}\right)^2 \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right) \\
&\le (1+\epsilon_1)(1+\epsilon_2)\left(\frac{b}{2}\right)^2 (1+\epsilon_3)\frac{1}{n}.
\end{aligned}$$

From (5.3), we get

$$\begin{aligned}
&\sum_{j=n}^{\infty} g(j,n)\bar{P}_j\bar{P}_{j+1} \\
&\le (1+\epsilon_1)[(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)]^2\left(\frac{b}{2}\right)^4 \\
&\quad \cdot \left[\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right] \\
&= (1+\epsilon_1)[(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)]^2\left(\frac{b}{2}\right)^4 \frac{1}{n}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\bar{P}_n &\ge (1-\epsilon_1)(1-\epsilon_4)\left(\frac{b}{2}\right)^2 \sum_{j=n}^{\infty}\frac{1}{k^2} \\
&\ge (1-\epsilon_1)(1-\epsilon_4)(1-\epsilon_5)\left(\frac{b}{2}\right)^2 \frac{1}{n}.
\end{aligned}$$

We note that if $(\frac{b}{2})^4 < \frac{1}{4}\left(\frac{b}{2}\right)^2$. (i.e., if $|b| < 1$, $b \neq 0$), then Corollary 3.3 shows that

$$\Delta^2 x(n) + b\frac{(-1)^n}{n}x(n+1) = 0, \tag{5.4}$$

is nonoscillatory.

Similarly, when $|b| > 1$, by Corollary 4.3, it follows that (5.4) is oscillatory. In the same way, we can show that

$$x^{\Delta\Delta} + b\frac{(-1)^n}{n+1}x(n+1) = 0, \tag{5.5}$$

is nonoscillatory, for $|b| < 1$.

REMARK 5.1. In [**17**] Mingarelli showed that equation (5.5) is nonoscillatory if $|b| \leq \frac{1}{4}$. Therefore, the above example improves the nonoscillation result in [**17**] by replacing the constant $\frac{1}{4}$ by 1, which is sharp, which will be noted below in Example 3.

6. Example 2.

Let

$$p(t) := \frac{(-1)^t}{t^c}, \quad t \in \mathbb{T} = \mathbb{N}, \quad c < 1.$$

We need the following useful comparison theorem [**9**, Theorem 9].

THEOREM 6.1. *Assume* $a \in C^1_{rd}$ *and*
(i) $a(t) \geq 1$,
(ii) $\mu(t)a^{\Delta}(t) \geq 0$
(iii) $a^{\Delta\Delta}(t) \leq 0$.
Then $x^{\Delta\Delta} + p(t)x^{\sigma} = 0$ *is oscillatory on* $[t_0, \infty)$ *implies* $x^{\Delta\Delta} + a(t)p(t)x^{\sigma} = 0$ *is oscillatory on* $[t_0, \infty)$.

Take $b = \pm 2$. From Example 1, we know that the equation

$$\Delta^2 x(n) \pm 2\frac{(-1)^n}{n}x(n+1) = 0, \tag{6.1}$$

is oscillatory.

Let $a(n) = An^{\alpha}$, $A > 0$, $0 < \alpha < 1$, then we have $\Delta a(n) \geq 0$, $\Delta^2 a(n) \leq 0$ for large n. Using Theorem 6.1 repeatedly and (6.1) is oscillatory, we get that

$$\Delta^2 x(n) \pm 2Bn^{\beta}\frac{(-1)^n}{n}x(n+1) = 0,$$

is oscillatory, for all $B > 0$, $\beta > 0$. So the equation

$$\Delta^2 x(n) \pm 2B\frac{(-1)^n}{n^{1-\beta}}x(n+1) = 0,$$

is oscillatory, for all $B > 0$, $\beta > 0$. This means that the equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0, \tag{6.2}$$

is oscillatory, for all $b \neq 0$, $c < 1$.

Similarly, we also have

$$\Delta^2 x(n) + b\frac{(-1)^n}{(n+1)^c}x(n+1) = 0, \tag{6.3}$$

is oscillatory, for $b \neq 0$, $c < 1$.

REMARK 6.2. In [**7**], Del Medico and Kong prove that the equation

$$\Delta^2 x(n) + \frac{(-1)^n}{\sqrt{n+1}}x(n+1) = 0$$

is oscillatory. Their result is a special case of (6.3).

7. Example 3–Critical Case $c = 1$, $|b| = 1$.

The following theorem is a time scales version of Wong's Theorem 5 in [**21**].

THEOREM 7.1. *Assume that* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent,* $P(t) = \int_t^{\infty} p(s)\Delta s$, *and let* $T \in [t_0, \infty)$ *be such that* $1 \pm \mu(t)P(t) > 0$, *for* $t \geq T$. *If*

$$\int_T^{\infty} P^2(t) e_{\frac{2P}{1-\mu P}}(t, T)\Delta t$$

converges and there exists a function $\bar{B}(t) > 0$, *such that*

$$\int_t^{\infty} Q_P(s,t)\left[\bar{P}(s) + \bar{B}(s)\right]\left[\bar{P}(\sigma(s)) + \bar{B}(\sigma(s))\right]\Delta t \leq \bar{B}(t) \tag{7.1}$$

for large t, *where* $Q_P(s,t) := \frac{1}{1-\mu(s)P(s)} e_{\frac{2P}{1-\mu P}}(s,t)$, *and where* $\bar{P}(s)$ *is defined in* (4.1). *Then* (1.4) *is nonoscillatory.*

PROOF. By Theorem 3.1, it is sufficient to show that (7.1) implies the existence of a solution to equation (3.1). Define $u_0(t) = \bar{P}(t) + \bar{B}(t)$ and inductively, for $n = 1, 2, \cdots$,

$$u_n(t) = \bar{P}(t) + \int_t^{\infty} Q_P(s,t) u_{n-1}(s) u_{n-1}(\sigma(s))\Delta s. \tag{7.2}$$

From (7.1) and (7.2), it is easy to show by induction that

$$0 \leq \bar{P}(t) \leq u_n(t) \leq u_{n-1}(t) \leq \bar{P}(t) + \bar{B}(t).$$

Thus the sequence of functions $\{u_n(t)\}$ has a pointwise limit function $\bar{u}(t) = \lim_{n\to\infty} u_n(t)$. Since the integrand in (7.2) is nonnegative, it follows from the Monotone Convergence Theorem [**1**, Theorem 4.1] that $\bar{u}(t)$ is a solution of (3.1). □

Example 3. We claim that the critical case ($c = 1$, $|b| = 1$ in equation (6.2))

$$\Delta^2 x(n) \pm \frac{(-1)^n}{n} x(n+1) = 0 \tag{7.3}$$

is nonoscillatory. We only show (7.3) is nonoscillatory for the case $p_n = \frac{(-1)^n}{n}$ as the case $p_n = -\frac{(-1)^n}{n}$ is similar. In this example we need to make more precise calculations than those given in Example 1. First we will show $g(j,n) = \frac{q_n}{q_{j+1}(1+P_j)} = 1 + \mathrm{O}(\frac{1}{n})$. Note that

$$\frac{q_n}{q_{j+1}} = \exp\left[(-1)\sum_{i=n}^{j} \ln\left(1 - \frac{2P_i}{1+P_i}\right)\right]. \tag{7.4}$$

By the Taylor expansion, we have

$$\ln\left(1 - \frac{2P_i}{1+P_i}\right) = -\frac{2P_i}{1+P_i} - \frac{1}{2}\left(\frac{2P_i}{1+P_i}\right)^2 + \mathrm{o}\left(\left(\frac{2P_i}{1+P_i}\right)^2\right). \tag{7.5}$$

Note that

$$\int_k^\infty \frac{dt}{2t(2t+1)} \le \frac{1}{2k(2k+1)} + \frac{1}{(2k+2)(2k+3)} + \cdots \le \int_{k-1}^\infty \frac{dt}{2t(2t+1)}.$$

Therefore, we have

$$P_{2k} = \frac{1}{4k} + O\left(\left(\frac{1}{2k}\right)^2\right).$$

Similarly,

$$P_{2k+1} = -\frac{1}{2(2k+1)} + O\left(\left(\frac{1}{2k+1}\right)^2\right).$$

So we have

$$P_n = (-1)^n \frac{1}{2n} + O\left(\left(\frac{1}{n}\right)^2\right). \tag{7.6}$$

Again, by the Taylor expansion, we have

$$\frac{1}{1+P_i} = 1 - P_i + \mathrm{O}(P_i^2) = 1 - (-1)^i \frac{1}{2i} + O\left(\left(\frac{1}{i}\right)^2\right). \tag{7.7}$$

From (7.6), (7.7), we have

$$\frac{P_i}{1+P_i} = (-1)^i \frac{1}{2i} + O\left(\frac{1}{i^2}\right).$$

So, from (5.1), we get that for large n

$$\left|\sum_{i=n}^{j} \frac{P_i}{1+P_i}\right| \le \left|\sum_{i=n}^{\infty} \frac{(-1)^i}{2i}\right| + \left|\sum_{i=j+1}^{\infty} \frac{(-1)^i}{2i}\right| + \left|\sum_{i=n}^{\infty} O\left(\frac{1}{i^2}\right)\right| \le \frac{3}{n} \tag{7.8}$$

$$\sum_{i=n}^{j} \left(\frac{2P_i}{1+P_i}\right)^2 \le C_1 \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right) \le C_1(1+\epsilon_1)\frac{1}{n}. \tag{7.9}$$

Since $e^x = 1 + \mathrm{O}(x)$, for small x, we have from (7.4)–(7.9), that

$$\frac{q_n}{q_{j+1}} = 1 + \mathrm{O}\left(\frac{1}{n}\right). \tag{7.10}$$

From (7.7) and (7.10), we get

$$g(j,n) = \frac{q_n}{q_{j+1}(1+P_j)} \le 1 + \frac{L}{n}, \tag{7.11}$$

for some constant $L > 0$. From (7.6), we get that

$$|P(n)| \le \frac{1}{2n} + \frac{M_1}{n^2}, \tag{7.12}$$

for some constant $M_1 > 0$. Therefore

$$\bar{P}_n = \sum_{j=n}^{\infty} g(j,n) P_j^2$$

$$= \left[1 + \mathrm{O}\left(\frac{1}{n}\right)\right] \sum_{j=n}^{\infty} \left[\frac{(-1)^j}{2j} + \mathrm{O}\left(\frac{1}{j^2}\right)\right]^2$$

$$= \left[1 + \mathrm{O}\left(\frac{1}{n}\right)\right] \sum_{j=n}^{\infty} \left[\frac{1}{4j^2} + \mathrm{O}\left(\frac{1}{j^3}\right)\right] \tag{7.13}$$

Note that $\frac{1}{n} = \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots$. Therefore, we have

$$\sum_{j=n}^{\infty} \frac{1}{j^2} - \frac{1}{n} = \frac{1}{n^2(n+1)} + \frac{1}{(n+1)^2(n+2)} \cdots = \mathrm{O}\left(\frac{1}{n^2}\right). \tag{7.14}$$

So from (7.13), we get

$$\bar{P}_n = \left[1 + \mathrm{O}\left(\frac{1}{n}\right)\right] \left[\frac{1}{4n} + \mathrm{O}\left(\frac{1}{n^2}\right)\right] = \frac{1}{4n} + \mathrm{O}\left(\frac{1}{n^2}\right). \tag{7.15}$$

Let $K > 0$ be the constant such that $\bar{P}(n) \le \frac{1}{4n} + \frac{K}{n^2}$. We choose $\bar{B}(n) = \frac{1}{4n} + \frac{M}{n^2}$, where M is to be determined in terms of K and L. Note that

$$\int_n^{\infty} \frac{1}{t(t+1)^2} dt \le \sum_{k=n}^{\infty} \frac{1}{k(k+1)^2} \le \int_{n-1}^{\infty} \frac{1}{t(t+1)^2} dt.$$

It is easy to get that

$$\sum_{k=n}^{\infty} \frac{1}{k(k+1)^2} = \frac{1}{2n^2} + \mathrm{O}\left(\frac{1}{n^3}\right), \quad \sum_{k=n}^{\infty} \frac{1}{k^2(k+1)} = \frac{1}{2n^2} + \mathrm{O}\left(\frac{1}{n^3}\right).$$

Considering (7.1) in this case and noting that $Q_P(s,n) = g(s,n)$, we have

$$\int_n^{\infty} Q_P(s,n) \left[\bar{P}(s) + \bar{B}(s)\right] \left[\bar{P}(\sigma(s)) + \bar{B}(\sigma(s))\right] \Delta s$$

$$\le \left(1 + \frac{L}{n}\right) \sum_{k=n}^{\infty} \left[\frac{1}{2k} + \frac{K+M}{k^2}\right] \left[\frac{1}{2(k+1)} + \frac{K+M}{(k+1)^2}\right]$$

$$= \frac{1}{4n} + \frac{2M + 2K + L}{4n^2} + \mathrm{O}\left(\frac{1}{n^3}\right),$$

In order that (7.1) be satisfied, it is sufficient to require: $M > \frac{2K+L}{2}$. So by Theorem 7.1, the equation (7.3) is nonoscillatory.

8. Example 4.

In this section we consider the difference equatuion

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0, \tag{8.1}$$

where $c > 1$.

We will use the following useful comparison theorem for nonoscillation [**9**, Corollary 10].

THEOREM 8.1. *Assume* $(\frac{1}{a})^\Delta \in C^1_{rd}$ *and for large* t

(i) $0 < a(t) \leq 1$,
(ii) $\mu(t)a^\Delta(t) \leq 0$
(iii) $(\frac{1}{a(t)})^{\Delta\Delta} \leq 0$.

Then $x^{\Delta\Delta} + p(t)x(\sigma(t)) = 0$ is nonoscillatory on $[t_0, \infty)$ implies $x^{\Delta\Delta} + a(t)p(t)x(\sigma(t)) = 0$ is nonoscillatory on $[t_0, \infty)$.

Take $b = \pm\frac{1}{2}$. By Example 1, we know that the equation

$$\Delta^2 x(n) \pm \frac{1}{2}\frac{(-1)^n}{n}x(n+1) = 0, \tag{8.2}$$

is nonoscillatory.

Let $a(n) = \frac{A}{n^\alpha}$, $A > 0$, $0 < \alpha < 1$. We have $0 < a(n) \leq 1$, for large n, $\Delta x(n) \leq 0$, $\Delta^2(\frac{1}{a(n)}) \leq 0$. Using Theorem 8.1 repeatedly, we get that

$$\Delta^2 x(n) \pm \frac{B}{2n^\beta} \times \frac{(-1)^n}{n}x(n+1) = 0,$$

is nonoscillatory, for all $B > 0$, $\beta > 0$. So the equation

$$\Delta^2 x(n) \pm B\frac{(-1)^n}{2n^{1+\beta}}x(n+1) = 0,$$

is nonoscillatory, for all $B > 0$, $\beta > 0$.
This means that the equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0,$$

is nonoscillatory for all $b \neq 0$, $c > 1$.
In conclusion from Examples 1-4, we have for the difference equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0, \tag{8.3}$$

the following conclusions.

(I) For $c = 1$, we have

$$\begin{cases} (i) \text{ if } \ |b| \leq 1, \ (8.3) \text{ is nonoscillatory} \\ (ii) \text{ if } \ |b| > 1, \ (8.3) \text{ is oscillatory.} \end{cases}$$

(II) For $c < 1$, $b \neq 0$, we have (8.3) is oscillatory.
(III) For $c > 1$, we have (8.3) is nonoscillatory.

9. Example.

Consider the equation

$$\Delta^2 x(n) + p(n)x(n+1) = 0, \tag{9.1}$$

for $n \in \mathbb{N}$, where $p(n) = \frac{a}{n^2} + b\frac{(-1)^n}{n}$, $a, b, \in \mathbb{R}$.

We will use the following lemma.

LEMMA 9.1. *Assume that $\alpha > 0$. Then we have*

$$\sum_{i=n}^{\infty} \frac{(-1)^i}{i^\alpha} \sim \frac{(-1)^n}{2n^\alpha}. \tag{9.2}$$

PROOF. By an appropriate Taylor expansion, we get that

$$\frac{1}{(2k)^\alpha} - \frac{1}{(2k+1)^\alpha} = \frac{(1+\frac{1}{2k})^\alpha - 1}{(2k+1)^\alpha} = \frac{\frac{\alpha}{2k}[1+o(1)]}{(2k+1)^\alpha}.$$

So

$$\begin{aligned} &\sum_{i=2k}^{\infty} \frac{(-1)^i}{i^\alpha} \\ = &\left[\frac{1}{(2k)^\alpha} - \frac{1}{(2k+1)^\alpha}\right] + \left[\frac{1}{(2k+2)^\alpha} - \frac{1}{(2k+3)^\alpha}\right] + \cdots \\ = &\left[\frac{\frac{\alpha}{2k}(1+o(1))}{(2k+1)^\alpha}\right] + \left[\frac{\frac{\alpha}{2k+2}(1+o(1))}{(2k+3)^\alpha}\right] + \cdots \\ \sim &\frac{\alpha}{(2k)(2k+1)^\alpha} + \frac{\alpha}{(2k+2)(2k+3)^\alpha} + \cdots \\ \sim &\frac{1}{2(2k)^\alpha}. \end{aligned}$$

Similarly, we have

$$\sum_{i=2k+1}^{\infty} \frac{(-1)^i}{i^\alpha} \sim -\frac{1}{2(2k+1)^\alpha},$$

so

$$\sum_{i=n}^{\infty} \frac{(-1)^i}{i^\alpha} \sim \frac{(-1)^n}{2n^\alpha}.$$

This completes the proof. □

The following Lemma may be found in [**10**, page 270].

LEMMA 9.2. *If there is an integer N and a function $u(n) > 0$ for $n \in [N, \infty)$ such that*

$$\sum_{n=N}^{\infty} \{p(n)u^2(n) - [\Delta u(n-1)]^2\} = \infty,$$

then $\Delta^2 x(n) + p(n)x(n+1) = 0$ is oscillatory on $[n_0, \infty)$.

In (9.1), if $a > \frac{1}{4}$, take $u(n) = n^{\frac{1}{2}}$. Then we claim

$$\sum_{n=n_0}^{\infty} \{p(n)u^2(n) - [\Delta u(n-1)]^2\} = \infty, \tag{9.3}$$

and so (9.1) is oscillatory by Lemma 9.2. To see this notice that the expression on the left side of (9.3) is given by

$$\sum_{t=n_0}^{\infty} \left\{ \left(\frac{a}{n^2} + \frac{b(-1)^n}{n} \right) n - [n^{\frac{1}{2}} - (n-1)^{\frac{1}{2}}]^2 \right\}$$

$$= \sum_{n=n_0}^{\infty} \left\{ \frac{a}{n} + b(-1)^n - [n^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]^{-2} \right\}$$

$$= \sum_{n=n_0}^{\infty} \left\{ \frac{a}{n} + b(-1)^n - [n^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]^{-2} \right\} = \infty,$$

for $a > \frac{1}{4}$, since $\lim_{n\to\infty} \frac{n}{[n^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}]^2} = \frac{1}{4}$.

It is easy to see that from (7.14) and (7.6) that

$$\sum_{j=n}^{\infty} \frac{1}{j^2} = \frac{1}{n} + O\left(\frac{1}{n^2}\right), \quad \sum_{j=n}^{\infty} \frac{(-1)^j}{j} = \frac{(-1)^n}{2n} + O\left(\frac{1}{n^2}\right).$$

So

$$P_n = \frac{a}{n} + b\frac{(-1)^n}{2n} + O\left(\frac{1}{n^2}\right). \tag{9.4}$$

Using appropriate Taylor expansions, we have

$$\ln\left(1 - \frac{2P_j}{1+P_j}\right) = -\frac{2P_j}{1+P_j} - \frac{1}{2}\left(\frac{2P_j}{1+P_j}\right)^2 + \mathrm{o}\left(\left(\frac{2P_j}{1+P_j}\right)^2\right) \tag{9.5}$$

and (using (9.4)),

$$\frac{1}{(1+P_i)} = 1 - \frac{a}{i} - b\frac{(-1)^i}{2i} + O\left(\frac{1}{i^2}\right). \tag{9.6}$$

Note that (using (9.4)–(9.6)) the series

$$\sum_{j=N}^{\infty} \ln\left(\frac{1-P_j}{1+P_j}\right) = \sum_{j=N}^{\infty} \ln\left(1 - \frac{2P_j}{1+P_j}\right)$$

is not convergent.

Note also from (7.4) that

$$\frac{q_n}{q_{j+1}} = \exp\left[\sum_{i=n}^{j} (-1) \ln\left(1 - \frac{2P_i}{1+P_i}\right)\right]. \tag{9.7}$$

From (9.7) and the fact that

$$\frac{2P_i}{1+P_i} = \frac{2a}{i} + \mathrm{O}\left(\frac{1}{i^2}\right)$$

we obtain

$$\frac{q_n}{q_{j+1}} = \exp\left\{\sum_{i=n}^{j}\left[\frac{2a}{i} + O\left(\frac{1}{i^2}\right)\right]\right\}.$$

Using the inequality $\sum_{i=n}^{j}\frac{1}{i} \leq \int_{n-1}^{j}\frac{1}{t}dt = \ln(\frac{j}{n-1})$, we have given $0 < \epsilon_1 < 1$, for sufficiently large n,

$$\frac{q_n}{q_{j+1}} \leq (1+\epsilon_1)\exp\left[\sum_{i=n}^{j}\frac{2a}{i}\right] \leq (1+\epsilon_1)\left(\frac{j}{n-1}\right)^{2a}. \tag{9.8}$$

So by (7.11) and (9.6) there is a constant $L_1 > 0$ such that for sufficiently large n

$$g(j,n) \leq (1+\epsilon_1)\left(1+\frac{L_1}{n}\right)\left(\frac{j}{n-1}\right)^{2a}. \tag{9.9}$$

From (9.4), we have

$$P_n^2 = \left(a + (-1)^n\frac{b}{2}\right)^2\frac{1}{n^2} + \mathrm{O}\left(\frac{1}{n^3}\right).$$

Hence

$$\begin{aligned}
\bar{P}_n &= \sum_{j=n}^{\infty} g(j,n)P_j^2 \\
&\leq (1+\epsilon_1)(1+\frac{L_1}{n})(\frac{1}{n-1})^{2a}\sum_{j=n}^{\infty} j^{2a}\left[\left(a+(-1)^j\frac{b}{2}\right)^2\frac{1}{j^2} + \mathrm{O}\left(\frac{1}{j^3}\right)\right] \\
&\leq (1+\epsilon_1)\left(1+\frac{L_1}{n}\right)\left(\frac{1}{n-1}\right)^{2a} \\
&\quad \cdot\sum_{j=n}^{\infty}\left[\frac{a^2+\left(\frac{b}{2}\right)^2}{j^{2-2a}} + \frac{(-1)^j ab}{j^{2-2a}} + \mathrm{O}\left(\frac{1}{j^{3-2a}}\right)\right]. \qquad (9.10)
\end{aligned}$$

Using $\sum_{j=n}^{\infty}\frac{(-1)^j j^{2a}}{j^2} \sim \frac{(-1)^n n^{2a}}{2n^2}$ for $a < 1$ (which follows from Lemma 9.1) and $\sum_{j=n}^{\infty}\frac{1}{j^{2-2a}} \sim \frac{1}{(1-2a)n^{1-2a}}$, for $a < \frac{1}{2}$, we have that given $0 < \epsilon_2 < 1$, for all sufficiently large n,

(9.11)

$$\bar{P}_n \leq (1+\epsilon_1)(1+\frac{L_1}{n})(\frac{1}{n-1})^{2a}\left[\frac{(1+\epsilon_2)}{1-2a}\frac{(a^2+(\frac{b}{2})^2)}{n^{1-2a}} + \mathrm{O}\left(\frac{1}{n^{2-2a}}\right)\right].$$

For $a \geq \frac{1}{2}$, we have by Lemma 9.2, equation (9.1) is oscillatory. Given $0 < \epsilon_3 < 1$, we have for n sufficiently large that

$$\left(\frac{n}{n-1}\right)^{2a} < 1 + \epsilon_3, \quad \text{for large n},$$

so for $a < \frac{1}{2}$

$$(9.12) \qquad \bar{P}_n \leq \left(1 + \frac{L_1}{n}\right) \frac{(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)}{1-2a} \frac{(a^2 + (\frac{b}{2})^2)}{n} + \mathrm{O}\left(\frac{1}{n^2}\right) = \frac{C}{n} + \mathrm{O}\left(\frac{1}{n^2}\right),$$

where $C = (1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)\frac{(a^2+(\frac{b}{2})^2)}{1-2a}$. From (9.9) and (9.12), we get given $0 < \epsilon_4 < 1$ and $0 < \epsilon_5 < 1$, for n sufficiently large,

$$\begin{aligned}
&\sum_{j=n}^{\infty} g(j,n)\bar{P}_j \bar{P}_{j+1} \\
\leq\ & \left(1 + \frac{L_1}{n}\right) \frac{(1+\epsilon_1)}{(n-1)^{2a}} \sum_{j=n}^{\infty} j^{2a} \left[\frac{C}{j} + O\left(\frac{1}{j^2}\right)\right] \left[\frac{C}{j+1} + O\left(\frac{1}{(j+1)^2}\right)\right] \\
\leq\ & \left(1 + \frac{L_1}{n}\right) \frac{(1+\epsilon_1)(1+\epsilon_4)}{(n-1)^{2a}} \left[\frac{C^2}{(1-2a)n^{1-2a}} + O\left(\frac{1}{n^{2-2a}}\right)\right] \\
\leq\ & (1+\epsilon_5)(1+\epsilon_1)\frac{(1+\epsilon_4)C^2}{(1-2a)n} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Similarly, as in (9.12), we have given $0 < \epsilon_6 < 1$, for n large,

$$\bar{P}_n \geq \frac{(1-\epsilon_6)}{n} \frac{(a^2 + (\frac{b}{2})^2)}{1-2a} + \mathrm{O}\left(\frac{1}{n^2}\right).$$

It is easy to see that when $\frac{C^2}{1-2a} < \frac{(a^2+(\frac{b}{2})^2)}{4(1-2a)}$, that is, when $\frac{a^2+(\frac{b}{2})^2}{(1-2a)^2} < \frac{1}{4}$ which is equivalent to $a < \frac{1}{4} - (\frac{b}{2})^2$, then by Corollary 3.3, equation (9.1) is nonoscillatory.

Likewise, when $a > \frac{1}{4} - (\frac{b}{2})^2$, using Corollary 4.3, it follows that equation (9.1) is oscillatory.

10. Example–Critical Case $a = \frac{1}{4} - (\frac{b}{2})^2$

Consider the equation

$$(10.1) \qquad \Delta^2 x(n) + p(n)x(n+1) = 0,$$

for $n \in \mathbb{N}$, where $p(n) = \frac{a}{n^2} + b\frac{(-1)^n}{n}$, $a = \frac{1}{4} - (\frac{b}{2})^2$, $a, b, c \in \mathbb{R}$.

As in Example 3 and Example 9 (see (9.8) and (9.9)), we have there are constants $L > 0$, $L_1 > 0$ such that for all sufficiently large n,

$$\frac{q_n}{q_{j+1}} \leq \left(1 + \frac{L}{n}\right) \left(\frac{j}{n-1}\right)^{2a}$$

and

$$Q_P(j,n) = g(j,n) \le \left(1+\frac{L_1}{n}\right)\left(1+\frac{L}{n}\right)\left(\frac{j}{n-1}\right)^{2a} \tag{10.2}$$

Note that

$$\frac{1}{(1-2a)n^{1-2a}} = \int_n^\infty \frac{1}{t^{2-2a}}dt \le \sum_{j=n}^\infty \frac{1}{j^{2-2a}} \tag{10.3}$$

and

$$\sum_{j=n}^\infty \frac{1}{j^{2-2a}} \le \int_{n-1}^\infty \frac{1}{t^{2-2a}}dt = \frac{1}{(1-2a)(n-1)^{1-2a}}. \tag{10.4}$$

Also

$$\begin{aligned}\frac{1}{(n-1)^{1-2a}} &= \frac{1}{n^{1-2a}}\left(\frac{n-1}{n}\right)^{2a-1}\\ &= \frac{1}{n^{1-2a}}\left[1+\frac{1-2a}{n}+O\left(\frac{1}{n^2}\right)\right] && (10.5)\\ &= \frac{1}{n^{1-2a}}\left[1+O\left(\frac{1}{n}\right)\right] && (10.6)\end{aligned}$$

We have by (10.4) and (10.6) that

$$\sum_{j=n}^\infty \frac{a^2+\left(\frac{b}{2}\right)^2}{j^{2-2a}} = \frac{a^2+\left(\frac{b}{2}\right)^2}{(1-2a)n^{1-2a}} + O\left(\frac{1}{n^{2-2a}}\right). \tag{10.7}$$

Now, similar to the way we went from (9.10) to (9.11) in Example 9, we use the more precise estimate (10.7) to get the result
(10.8)

$$\bar{P}_n \le \left(1+\frac{L_1}{n}\right)\left(1+\frac{L}{n}\right)\left(\frac{1}{n-1}\right)^{2a}\left[\frac{a^2+(\frac{b}{2})^2}{(1-2a)n^{1-2a}}+O\left(\frac{1}{n^{2-2a}}\right)\right].$$

Note that

$$\left(\frac{n}{n-1}\right)^{2a} = \left(1-\frac{1}{n}\right)^{-2a} = 1+\frac{2a}{n}+O\left(\frac{1}{n^2}\right).$$

We have

$$\bar{P}_n \le \frac{C_1}{n} + O\left(\frac{1}{n^2}\right), \tag{10.9}$$

where $C_1 = \frac{a^2+(\frac{b}{2})^2}{1-2a}$.
Below we will use the following results

$$\left(1-\frac{1}{n}\right)^{-2a} = 1+\frac{2a}{n}+O\left(\frac{1}{n^2}\right), \tag{10.10}$$

(10.11)
$$\sum_{j=n}^{\infty}\frac{1}{j^{3-2a}}=\frac{1}{(2-2a)n^{2-2a}}+O\left(\frac{1}{n^{3-2a}}\right),\quad \sum_{j=n}^{\infty}\frac{1}{j^{4-2a}}=O\left(\frac{1}{n^{3-2a}}\right).$$

$$\begin{aligned}\sum_{j=n}^{\infty}\frac{j^{2a}}{j(j+1)} &= \sum_{j=n}^{\infty}\frac{1}{j^{2-2a}}(1+\frac{1}{j})^{-1}\\ &= \sum_{j=n}^{\infty}\frac{1}{j^{2-2a}}\left[1-\frac{1}{j}+O(\frac{1}{j^2})\right] \qquad (10.12)\\ &= \sum_{j=n}^{\infty}\frac{1}{j^{2-2a}}-\sum_{j=n}^{\infty}\frac{1}{j^{3-2a}}+O\left(\frac{1}{n^{3-2a}}\right). \qquad (10.13)\end{aligned}$$

$$\sum_{j=n}^{\infty}\frac{j^{2a}}{j(j+1)^2}=\sum_{j=n}^{\infty}\frac{1}{j^{3-2a}}(1+\frac{1}{j})^{-2}=\sum_{j=n}^{\infty}\frac{1}{j^{3-2a}}+O\left(\frac{1}{n^{3-2a}}\right). \qquad (10.14)$$

$$\sum_{j=n}^{\infty}\frac{j^{2a}}{j^2(j+1)}=\sum_{j=n}^{\infty}\frac{1}{j^{3-2a}}(1+\frac{1}{j})^{-1}=\sum_{j=n}^{\infty}\frac{1}{j^{3-2a}}+O\left(\frac{1}{n^{3-2a}}\right). \qquad (10.15)$$

From (10.4) and (10.5) we have

$$\sum_{j=n}^{\infty}\frac{1}{j^{2-2a}}=\frac{1}{(1-2a))n^{1-2a}}+\frac{1}{n^{2-2a}}+O\left(\frac{1}{n^{3-2a}}\right) \qquad (10.16)$$

Let $K>0$ be a constant such that $\bar{P}_n\le\frac{C_1}{n}+\frac{K}{n^2}$. We choose $\bar{B}_n=\frac{C_1}{n}+\frac{M}{n^2}$, where M is to be determined in terms of K, L and L_1. Next we want to show that (7.1) holds. Using (10.2) and (10.9) we get

$$\begin{aligned}I_n &:= \int_n^{\infty}Q_P(s,n)(\bar{P}(s)+\bar{B}(s))(\bar{P}(\sigma(s))+\bar{B}(\sigma(s)))\Delta s\\ &\le \left(1+\frac{L_1}{n}\right)\left(1+\frac{L}{n}\right)\left(\frac{1}{n-1}\right)^{2a}\\ &\quad\cdot\sum_{j=n}^{\infty}j^{2a}\left[\frac{2C_1}{j}+\frac{M+K}{j^2}\right]\left[\frac{2C_1}{j+1}+\frac{M+K}{(j+1)^2}\right]\\ &= \left(1+\frac{L_1}{n}\right)\left(1+\frac{L}{n}\right)\left(\frac{1}{n-1}\right)^{2a}\\ &\quad\cdot\sum_{j=n}^{\infty}j^{2a}\left[\frac{4C_1^2}{j(j+1)}+\frac{2C_1(M+K)}{j(j+1)^2}+\frac{2C_1(M+K)}{j^2(j+1)}+\frac{(M+K)^2}{j^2(j+1)^2}\right].\end{aligned}$$

Using (10.12)–(10.15) we get that

$$I_n \leq (1+\frac{L_1}{n})(1+\frac{L}{n})(1-\frac{1}{n})^{-2a}\frac{1}{n^{2a}}$$
$$\cdot\left\{\sum_{j=n}^{\infty}\left[\frac{4C_1^2}{j^{2-2a}}-\frac{4C_1^2}{j^{3-2a}}+\frac{4C_1(M+K)}{j^{3-2a}}\right]+O(\frac{1}{n^{3-2a}})\right\}.$$

Using (10.10), (10.11) and (10.16) yields

$$I_n \leq \left(1+\frac{L_1}{n}\right)\left(1+\frac{L}{n}\right)\left[1+\frac{2a}{n}+O\left(\frac{1}{n^2}\right)\right]\frac{1}{n^{2a}}\cdot$$
$$\left[\frac{4C_1^2}{(1-2a)n^{1-2a}}+\frac{4C_1^2}{n^{2-2a}}-\frac{4C_1^2-4C_1(M+K)}{(2-2a)n^{2-2a}}+O\left(\frac{1}{n^{3-2a}}\right)\right]$$
$$= \frac{4C_1^2}{(1-2a)n}+\left[4C_1^2-\frac{4C_1^2-4C_1(M+K)}{2-2a}+\frac{4C_1^2(L_1+L+2a)}{1-2a}\right]\frac{1}{n^2}$$
$$+O\left(\frac{1}{n^3}\right).$$

Note that in this critical case $a=\frac{1}{4}-\left(\frac{b}{2}\right)^2$, $\frac{4C_1^2}{1-2a}=C_1$ and it follows from this that $\frac{4C_1}{2-2a}<1$. So we can take $M>0$ such that

$$M>4C_1^2-\frac{4C_1^2-4C_1(M+K)}{2-2a}+\frac{4C_1^2(L_1+L+2a)}{1-2a},$$

so that (7.1) is satisfied. Hence, by Theorem 7.1, equation (10.1) is nonoscillatory.

11. Classification of $\Delta^2 x(n)+\left[\frac{a}{t^{c+1}}+b\frac{(-1)^n}{t^c}\right]x(n+1)=0$

Consider the equation

$$\Delta^2 x(n)+\left[\frac{a}{t^{c+1}}+b\frac{(-1)^n}{t^c}\right]x(n+1)=0, \tag{11.1}$$

where $n\in\mathbb{N}$, $a,b,c\in\mathbb{R}$. For convenience, we use the notation OSC to mean oscillatory, and NONOSC to mean nonoscillatory. We say that $p(t)$ is OSC or NONOSC in case the equation $x^{\Delta\Delta}+p(t)x(\sigma(t))=0$ is oscillatory or nonoscillatory, respectively.

The following Hille-type Theorem on time scales may be found in [**18**].

THEOREM 11.1. *If*

$$\liminf_{t\to\infty} t\int_t^\infty p(s)\Delta s>\frac{1}{4}$$

then equation (1.4) *is oscillatory. If*

$$-\frac{3}{4}<\liminf_{t\to\infty} t\int_t^\infty p(s)\Delta s\leq\limsup_{t\to\infty} t\int_t^\infty p(s)\Delta s<\frac{1}{4},$$

then equation (1.4) *is nonoscillatory.*

Case (I): $a > 0$.

(1) $c > 1$: From Lemma 9.1, we have

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{k^c} \sim \frac{(-1)^n}{2n^c},$$

and it is also true that $\sum_{k=n}^{\infty} \frac{1}{k^{c+1}} \sim \frac{1}{cn^c}$. So

$$t \int_t^{\infty} p(s)\Delta s = n \sum_{k=n}^{\infty} \left[\frac{a}{k^{c+1}} + \frac{b(-1)^k}{k^c} \right] \to 0.$$

By Theorem 11.1, equation (11.1) is nonoscillatory.

(2) c=1: See Sections 9 and 10 for the classification of (11.1) in this case.

(3) $0 < c < 1$: By Section 8, $\Delta^2 x(n) + \frac{b(-1)^n}{n^c} x(n+1) = 0$ is oscillatory if $b \neq 0$. By the Sturm Theorem, we have

$$\Delta^2 x(n) + \left(\frac{a}{n^{c+1}} + \frac{b(-1)^n}{n^c} \right) x(n+1) = 0$$

is oscillatory.

(4) $c = 0$: In this case, $\int_N^{\infty} p(t)\Delta t = \infty$, so by the Fite–Leighton-Wintner Theorem (see [**5**, Theorem 4.64]), equation (11.1) is oscillatory.

(5) $c < 0, b \neq 0$: By Section 8, $x^{\Delta\Delta}(n) + b(-1)^n x(n+1) = 0$ is oscillatory so by the Sturm Comparison Theorem (see [**5**]),

$$x^{\Delta\Delta}(n) + \left(\frac{a}{n} + b(-1)^n \right) x(n+1) = 0, \tag{11.2}$$

is oscillatory.

Applying Theorem 6.1 to (11.2) repeatedly as in Example 2, it follows that

$$x^{\Delta\Delta}(n) + \left(\frac{a}{n^{1+c}} + \frac{b(-1)^n}{n^c} \right) x(n+1) = 0,$$

is oscillatory for $c < 0$.

(6) $c < 0$, $b = 0$: In this case, $\int_N^{\infty} p(t)\Delta t = \infty$, so by the Fite–Leighton-Wintner Theorem (see [**5**, Theorem 4.64]), equation (11.1) is oscillatory.

Case (II): $a = 0$: See Section 8 for the results for this case.

Case (III): $a < 0$:

(1) $c < 1$:

(i) $b \neq 0$: In this case, choose the real number $M > 0$ sufficiently large so that $aM > \frac{1}{4} - (\frac{bM}{2})^2$, then by Section 9,

$$\Delta^2 x(n) + \left(\frac{aM}{n^2} + \frac{bM(-1)^n}{n} \right) x(n+1) = 0$$

is oscillatory.

Now again by repeated applications of Theorem 6.1,

$$\Delta^2 x(n) + \left(\frac{a}{n^{1+c}} + \frac{b(-1)^n}{n^c} \right) x(n+1) = 0$$

is oscillatory for $c < 1$ and $b \neq 0$.

(ii) b=0:

Note that $x^{\Delta\Delta}(n) = 0$ is nonoscillatory, so by the Sturm Theorem,

$$\Delta^2 x(n) + \frac{a}{n^{1+c}} x(n+1) = 0$$

is nonoscillatory.

(2) $c = 1$: These are the results of Section 9 and Section 10.

(3) $c > 1$: By Section 8, $\Delta^2 x(n) + \frac{b(-1)^n}{n^c} x(n+1) = 0$ is nonoscillatory, so by the Sturm Theorem, $\Delta^2 x(n) + \left(\frac{a}{n^{1+c}} + \frac{b(-1)^n}{n^c} \right) x(n+1) = 0$ is nonoscillatory.

In summary, we get the following complete classification of the difference equation (11.1):

Case (I): $a > 0$:

(1) $c > 1$: NONOSC.

(2) $c = 1$: (i) $a > \frac{1}{4} - (\frac{b}{2})^2$ OSC. (ii) $a \leq \frac{1}{4} - (\frac{b}{2})^2$ NONOSC.

(3) $c < 1$: OSC.

Case (II): $a = 0$: These are the results of Section 8.

Case (III): $a < 0$:

(1) $c < 1$: (i) $b \neq 0$ OSC. (ii) $b = 0$ NONOSC.

(2) $c = 1$: (i) $a > \frac{1}{4} - (\frac{b}{2})^2$ OSC. (ii) $a \leq \frac{1}{4} - (\frac{b}{2})^2$ NONOSC.

(3) $c > 1$: NONOSC.

12. Classification of $x'' + (\frac{a}{t^{1+c}} + \frac{b \sin \lambda t}{t^c})x = 0$

Similarly, we can get the complete classification of the differential equation

$$x'' + \left(\frac{a}{t^{1+c}} + \frac{b \sin \lambda t}{t^c} \right) x = 0.$$

Case (I): $a > 0$:

(1) $c > 1$: NONOSC.

(2) $c = 1$: These are the results of Willett [**20**]and Wong [**21**].

(3) $c < 1$: OSC.

Case (II): $a = 0$: These are results of Willett [**20**] and Wong [**21**].

Case (III): $a < 0$:

(1) $c < 1$: (i) $b \neq 0$ OSC. (ii) $b = 0$ NONOSC.

(2) $c = 1$: These are again results of Willett [**20**] and Wong [**21**].

(3) $c > 1$: NONOSC.

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