Some new comparison results for second order linear dynamic equations

Jia Baoguo, Lynn Erbe and Allan Peterson
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130, U.S.A.
lerbe2@math.unl.edu, apeterso1@math.unl.edu,
Jia Baoguo
School of Mathematics and Computer Science
Zhongshan University
Guangzhou, China, 510275
mcsjbg@mail.sysu.edu.cn

Abstract. We obtain two comparison theorems for second order linear dynamic equations on a time scale. These results improve recent comparison theorems {See: Erbe, Peterson and Řehák, J. Math. Anal. Appl. 275 (2002) 418-438}. The main interest is in the case when the coefficient functions are not of one sign for large t. Several examples are given to show that our results are new even in the continuous case.

Keywords and Phrases: comparison theorems, oscillation, condition $(\hat{A})$, dominant solution, dynamic equations, time scales.
2000 AMS Subject Classification: 34K11, 39A10, 39A99.

1. Introduction

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. We shall be interested in obtaining comparison theorems for the second order linear equations

$$[r(t)x^{\Delta}(t)]^{\Delta} + p(t)x^{\sigma}(t) = 0, \tag{1.1}$$

$$[R(t)x^{\Delta}(t)]^{\Delta} + a(t)P(t)x^{\sigma}(t) = 0, \tag{1.2}$$

where $r(t) > 0, R(t) > 0$ and p, P, a, r, R are right-dense continuous functions on $\mathbb{T}$.

For completeness, we recall some basic results for dynamic equations and the calculus on time scales. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

(where we assume $\inf \emptyset = \sup \mathbb{T}$, and $\sup \emptyset = \inf \mathbb{T}$, and where $\emptyset$ denotes the empty set). If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ and $t \neq \sup \mathbb{T}$, we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given an interval $[c, d] := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c, d]^\kappa$ denotes the interval $[c, d]$ in case $\rho(d) = d$, and denotes the interval $[c, d)$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.
The theory of time scales was introduced by Stefan Hilger in his PhD. thesis in 1988 in order to unify continuous and discrete analysis (see [**12**]). Not only does this unify the theories of differential equations and difference equations, but it also extends these classical situations to cases "in between", e.g., to the so-called $q-$difference equations (see Example 4.5) which are important in quantum theory (see Kac and Cheung [**13**]). For more on q-difference equations see [**1**], [**6**], and [**17**]. Moreover, the theory can be applied to other different types of time scales. We refer to the two books on the subject of time scales by Bohner and Peterson [**3**], [**4**] which summarize and organize much of time scale calculus and applications to dynamic equations. For other interesting applications see the cover story article by V. Spedding in NewScientist [**16**]. Also since the equations we study here are second order linear equations, there are numerous applications of these equations, for example our equations could be thought of as a simple model of a mass-spring problem where Hooke's law does not apply.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{rd}. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable on $[c, d]^\kappa$ and whose delta derivative is rd-continous on $[c, d]^\kappa$ is denoted by C^1_{rd}.

We recall that a solution of equation (1.1) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. The study of the oscillatory and nonoscillatory properties of equation (1.1) and its many generalizations and extensions is voluminous and we refer to [**5**], [**8**] and the references therein. In particular, we refer to the very extensive studies for the case $\mathbb{T} = \mathbb{R}$ of Willett [**18**] and Wong [**20**].

The following condition (A) was introduced in [**9**] for the continuous case in order to obtain some new oscillation and comparison results in the case when the function $p(t)$ takes on both positive and negative values for large t.

Definition 1. We say that a function $g : \mathbb{T} \to \mathbb{R}$ satisfies condition (A) if the following condition holds:

$$\liminf_{t\to\infty} \int_T^t g(s)\Delta s \geq 0 \quad and \quad \not\equiv 0,$$

for all large T.

We wish to extend this notion to a *pair* of functions (p, r), so we introduce the following definition:

Definition 2. We say that the pair (p, r) satisfies condition $(\hat{A})$, if there exists a continuously differentiable function $h : \mathbb{T} \to \mathbb{R}$, such that either $h^\Delta(t)$ is of one sign for all $t \in \mathbb{T}$ or $h^\Delta(t) \equiv 0$, and is such that

$$g(t) := p(t)h^2(\sigma(t)) - r(t)(h^\Delta(t))^2$$

satisfies condition (A).

Remark 1.1. It is obvious that condition (A) implies condition $(\hat{A})$ (take $h(t) \equiv 1$)but the converse is not true. (See Example 1.4 below).

The following two theorems are proved in [**11**]:

Theorem 1.2. *Assume $a \in C^1_{cd}$, $r(t) \leq R(t)$, $P(t) \leq p(t)$ and*
(i) the function p satisfies condition (A)
(ii) $\int_\tau^\infty \frac{1}{r(s)}\Delta s = \infty$,
(iii) $0 < a(t) \leq 1$, $a^\Delta(t) \leq 0$.
Then (1.1) is nonoscillatory on $[\tau, \infty)$ implies (1.2) is nonoscillatory on $[\tau, \infty)$.

Theorem 1.3. *Assume $a \in C^1_{cd}$, $R(t) \leq r(t)$, $p(t) \leq P(t)$ and*
(i) the function aP satisfies condition (A),
(ii) $\int_\tau^\infty \frac{1}{R(s)}\Delta s = \infty$,
(iii) $a(t) \geq 1$, $a^\Delta(t) \geq 0$.
Then (1.1) is oscillatory on $[\tau, \infty)$ implies (1.2) is oscillatory on $[\tau, \infty)$.

In this paper, we prove that if condition (i) of Theorem 1.1 and Theorem 1.2 are replaced by the condition that the pair (p, r) and the pair (aP, r) satisfies condition $(\hat{A})$, respectively, then the conclusions of the above theorems are still valid. In Section 4, we give examples to show that our theorems are improvements since we give a class of equations whose oscillatory or nonoscillatory behavior may not be determined from existing results in the literature.

The following example shows that the class of functions g which do not satisfy condition (A) but such that the pair (g, r) satisfies condition $(\hat{A})$ is nonempty.

EXAMPLE 1.4. Let $\mathbb{T}$ be the real interval $[1, \infty)$, $g(t) = \frac{a}{t^2} + \frac{\sin t}{t}$, where $\frac{1}{4} < a \leq 1$. Then we have the following two results:

(i) $g(t)$ does not satisfy condition (A).
(In fact, one can easily see that $\int_t^\infty \frac{\sin s}{s} ds = \frac{\cos t}{t} + O(\frac{1}{t^2})$ and so clearly $g(t)$ does not satisfy condition (A) if $a < 1$.)

(ii) Let $h(t) = t^{\frac{1}{2}}, r(t) = 1$. Then

$$\int_T^t \{g(s)h^2(s) - r(s)[h'(s)]^2\} ds = (a - \frac{1}{4}) \ln \frac{t}{T} + \int_T^t \sin s \, ds \to \infty, (t \to \infty).$$

So the pair (g, r) satisfies condition $(\hat{A})$.

2. Lemmas

The following Lemmas 2.1 and 2.2 in [**3**] are useful in establishing oscillation, nonoscillation, and comparison results for second order linear dynamic equations on time scales.

LEMMA 2.1. *(Riccati technique). Equation (1) is nonoscillatory if and only if there exists $T \in [\tau, \infty)$ and a function u satisfying the Riccati dynamic inequality*

$$u^\Delta(t) + p(t) + \frac{u^2(t)}{r(t) + \mu(t)u(t)} \leq 0$$

with $p(t) + \mu(t)u(t) > 0$ for $t \in [T, \infty)$.

LEMMA 2.2. *(Sturm-Picone comparison theorem). Consider the equation*

$$[\tilde{r}(t)x^\Delta(t)]^\Delta + \tilde{p}(t)x^\sigma(t) = 0, \tag{2.1}$$

where $\tilde{r}$ and $\tilde{p}$ satisfy the same assumptions as r and p. Suppose that $\tilde{r}(t) \leq r(t)$ and $p(t) \leq \tilde{p}(t)$ on $[T, \infty)$ for all large T. Then (2.1) is nonoscillatory on $[\tau, \infty)$ implies (1.1) is nonoscillatory on $[\tau, \infty)$.

The next two lemmas may also be found in [**3**].

LEMMA 2.3. *Assume $a \in \mathbb{T}$, and let $\omega = \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If equation (1.1) is nonoscillatory on $[a, \omega)$, then there is a solution u, called a recessive solution at ω, such that for any second linearly independent solution v, called a dominant solution at ω, we have*

$$\lim_{t \to \omega^-} \frac{u(t)}{v(t)} = 0, \quad \int_b^\omega \frac{1}{r(t)u(t)u^\sigma(t)} \Delta t = \infty, \quad \textit{and} \int_b^\omega \frac{1}{r(t)v(t)v^\sigma(t)} \Delta t < \infty, \tag{2.2}$$

where $b < \omega$ is sufficiently close. Furthermore,

$$\frac{r(t)v^{\Delta}(t)}{v(t)} > \frac{r(t)u^{\Delta}(t)}{u(t)} \tag{2.3}$$

for $t < \omega$ sufficiently close.

LEMMA 2.4. *(Picone's Identity). Assume $x(t)$ is a positive solution of (1.1) on $[a, \infty)$ and let $z(t) := \frac{r(t)x^{\Delta}(t)}{x(t)}$ on $[a, \infty)$, then $r(t) + \mu(t)z(t) > 0$ on $[a, \infty)$. If we assume $h : \mathbb{T} \to \mathbb{R}$ is a continuously differentiable function, then we have for all $t \in [a, \infty)$,*

$$\begin{aligned}
[zh^2]^{\Delta}(t) &= -p(t)h^2(\sigma(t)) + r(t)[h^{\Delta}(t)]^2 \\
&- \left[\frac{z(t)h^{\sigma}(t)}{\sqrt{r(t) + \mu(t)z(t)}} - \sqrt{r(t) + \mu(t)z(t)}h^{\Delta}(t)\right]^2.
\end{aligned}$$

3. Main Results

Our goal in this section is to show that condition (A) (i.e., condition (i)) in Theorems 1.2 and 1.3 can be weakened to the assumption that condition $(\hat{A})$ holds for the pair (p, r).

THEOREM 3.1. *Assume $a \in C^1_{cd}$, $r(t) \leq R(t)$, $P(t) \leq p(t)$ and*
(i) the pair (p, r) satisfies condition $(\hat{A})$,
(ii) $\int_{\tau}^{\infty} \frac{1}{r(s)} \Delta s = \infty$,
(iii) $0 < a(t) \leq 1$, $a^{\Delta}(t) \leq 0$.
Then (1.1) is nonoscillatory on $[\tau, \infty)$ implies (1.2) is nonoscillatory on $[\tau, \infty)$.

PROOF. Assume (1.1) is nonoscillatory on $[\tau, \infty)$. Then (1.1) is disconjugate in a neighborhood of ∞. By Lemma 2.3, there is a dominant solution $v(t)$ at ∞ satisfying

$$\int_T^{\infty} \frac{1}{r(t)v(t)v^{\sigma}(t)} \Delta t < \infty, \tag{3.1}$$

for $T \in \mathbb{T}$ sufficiently large, and without loss of generality we can assume that $v(t) > 0$ on $[T, \infty)$. We would like to show that $v^{\Delta}(t) > 0$ for all large t.

Assume that there exists $T_1 > T$ such that $v^{\Delta}(T_1) \leq 0$. Make the substitution

$$z(t) = \frac{r(t)v^{\Delta}(t)}{v(t)},$$

for $t \geq T_1$. From condition $(\hat{A})$, we may suppose that T_1 is sufficiently large so that

$$\liminf_{t \to \infty} \int_{T_1}^{t} \{p(s)h^2(\sigma(s)) - r(s)[h^{\Delta}(s)]^2\} \Delta s \geq 0, \tag{3.2}$$

holds and is such that $z(T_1) \leq 0$, (i.e., $v^{\Delta}(T_1) \leq 0$).

Note that

$$0 < \frac{r(t)v^{\sigma}(t)}{v(t)} = \frac{r(t)[v(t) + \mu(t)v^{\Delta}(t)]}{v(t)} = r(t) + \mu(t)z(t).$$

We have from Picone's identity (Lemma 2.4)

$$\begin{aligned}[zh^2]^{\Delta}(t) &= -p(t)h^2(\sigma(t)) + r(t)[h^{\Delta}(t)]^2 \\ &- \left[\frac{z(t)h^{\sigma}(t)}{\sqrt{r(t)+\mu(t)z(t)}} - \sqrt{r(t)+\mu(t)z(t)}h^{\Delta}(t)\right]^2.\end{aligned}$$

Integrating from T_1 to t, we have

$$z(t)h^2(t) - z(T_1)h^2(T_1) = -\int_{T_1}^{t} [p(s)h^2(\sigma(s)) - r(s)[h^{\Delta}(s)]^2]\Delta s$$

$$-\int_{T_1}^{t} \left[\frac{z(s)h^{\sigma}(s)}{\sqrt{r(s)+\mu(s)z(s)}} - \sqrt{r(s)+\mu(s)z(s)}h^{\Delta}(s)\right]^2 \Delta s. \tag{3.3}$$

In the following, we will consider two cases:

Case (i)

$$\frac{z(s)h^{\sigma}(s)}{\sqrt{r(s)+\mu(s)z(s)}} - \sqrt{r(s)+\mu(s)z(s)}h^{\Delta}(s) \equiv 0,$$

for $s \geq T_1$. Then in this case we have

$$z(s)h^{\sigma}(s) = (r(s) + \mu(s)z(s))h^{\Delta}(s).$$

By the formula, $h(\sigma(s)) = h(s) + \mu(s)h^{\Delta}(s)$, we get

$$z(s)h(s) = r(s)h^{\Delta}(s).$$

Therefore

$$\left(\frac{v(s)}{h(s)}\right)^{\Delta} = 0.$$

So $v(s) = Ch(s)$. Without loss of generality we assume that $h(s) > 0$, for $s \in \mathbb{T}$, since the other case is similar. Therefore, we must have $C > 0$.

(i) If $h^{\Delta}(s) > 0$, for $s \in [T_1, \infty)$, we have $v^{\Delta}(t) > 0$, which contradicts the fact that $v^{\Delta}(T_1) \leq 0$.

(ii) If $h^{\Delta}(s) \equiv 0$, then $v^{\Delta}(s) \equiv 0$ and so from equation (1.1), we have $p(s) \equiv 0$, which contradicts the fact that the pair (p, r) satisfies condition $(\hat{A})$.

(iii) If $h^{\Delta}(T_2) < 0$, for some $T_2 \in [T_1, \infty)$, we have $v^{\Delta}(t) < 0$ for $t \in [T_2, \infty)$. Hence, in this case, $v(t)$ is a positive decreasing function on $[T_2, \infty)$. But then we get that

$$\int_{T_2}^{\infty} \frac{1}{r(t)v(t)v^{\sigma}(t)}\Delta t \geq \frac{1}{v^2(T_2)} \int_{T_2}^{\infty} \frac{1}{r(t)}\Delta t = \infty,$$

which contradicts (3.1).

Case (ii)

$$\frac{z(s)h^{\sigma}(s)}{\sqrt{r(s)+\mu(s)z(s)}} - \sqrt{r(s)+\mu(s)z(s)}h^{\Delta}(s) \not\equiv 0,$$

for $t \geq T_1$. There exists $\epsilon > 0$ and $T_2 > T_1$ such that for $t \geq T_2$,

$$\int_{T_1}^{t} \left[\frac{z(s)h^{\sigma}(s)}{\sqrt{r(s)+\mu(s)z(s)}} - \sqrt{r(s)+\mu(s)z(s)}h^{\Delta}(s) \right]^2 \Delta s > \epsilon.$$

By (3.2), there exists $T_3 > T_2$ such that for $t \geq T_3$,

$$\int_{T_1}^{t} \{p(s)h^2(\sigma(s)) - r(s)[h^{\Delta}(s)]^2\}\Delta s \geq -\frac{\epsilon}{2}.$$

So by (3.3), when $t > T_3$, we have

$$z(t)h^2(t) < z(T_1)h^2(T_1) + \frac{\epsilon}{2} - \epsilon < 0,$$

which implies that $v^{\Delta}(t) < 0$ for all large $t > T_3$. Hence $v(t)$ is a positive decreasing function on $[T_3, \infty)$. But then again we get that

$$\int_{T_2}^{\infty} \frac{1}{r(t)v(t)v^{\sigma}(t)}\Delta t \geq \frac{1}{v^2(T_2)} \int_{T_2}^{\infty} \frac{1}{r(t)}\Delta t = \infty,$$

which contradicts (3.1). Hence we can suppose that $v(t) > 0, v^{\Delta}(t) > 0$ on $[T_3, \infty)$. Therefore, the function $z(t) = \frac{r(t)v^{\Delta}(t)}{v(t)} > 0$, for $t \geq T_3$ and

$$z^{\Delta}(t) = -p(t) - \frac{v(t)z^2(t)}{r(t)v^{\sigma}(t)}. \tag{3.4}$$

Note that

$$0 < \frac{r(t)v^{\sigma}(t)}{v(t)} = r(t) + \mu(t)z(t).$$

Substituting this in (3.2), we have

$$z^{\Delta}(t) = -p(t) - \frac{z^2(t)}{r(t)+\mu(t)z(t)}.$$

Now, multiplying by $a(t)$ and moving the terms on the right side to the left, we have from the properties of the function a that

$$0 = z^{\Delta}(t)a(t) + p(t)a(t) + \frac{(z(t)a(t))^2}{a(t)r(t)+a(t)\mu(t)z(t)}.$$

$$\geq z^{\Delta}(t)a(t) + z^{\sigma}(t)a^{\Delta}(t) + P(t)a(t) + \frac{(z(t)a(t))^2}{a(t)r(t)+a(t)\mu(t)z(t)}.$$

$$= [z(t)a(t)]^{\Delta} + P(t)a(t) + \frac{(z(t)a(t))^2}{a(t)r(t)+a(t)\mu(t)z(t)}$$

for $t \in [T_3, \infty)$. Hence the function $\varphi(t) = z(t)a(t)$ satisfies the Riccati inequality.

$$\varphi^{\Delta} + P(t)a(t) + \frac{\varphi^2}{a(t)r(t) + \mu(t)\varphi(t)} \le 0$$

for $t \in [T_3, \infty)$. Therefore the equation

$$[a(t)r(t)x^{\Delta}(t)]^{\Delta} + a(t)P(t)x^{\sigma}(t) = 0$$

is nonoscillatory by Lemma 2.1 and so equation (1.2) is nonoscillatory by Lemma 2.2 since $a(t)r(t) \le r(t) \le R(t)$.

□

The corresponding "oscillation" result is

THEOREM 3.2. *Assume* $a \in C^1_{cd}$, $R(t) \le r(t)$, $p(t) \le P(t)$ *and*

(i) *the pair* (aP, r) *satisfies condition* $(\hat{A})$,

(ii) $\int_{\tau}^{\infty} \frac{1}{R(s)} \Delta s = \infty$,

(iii) $a(t) \ge 1$, $a^{\Delta}(t) \ge 0$.

Then (1.1) is oscillatory on $[\tau, \infty)$ *implies (1.2) is oscillatory on* $[\tau, \infty)$.

PROOF. The proof of Theorem 3.2 follows from Theorem 3.1. If we let $b(t) = \frac{1}{a(t)}$, then $b(t) \le 1$ and $b^{\Delta}(t) \le 0$. Therefore, if

$$(R(t)x^{\Delta}(t))^{\Delta} + a(t)P(t)x^{\sigma}(t) = 0$$

is nonoscillatory, then from Theorem 3.1, it follows that

$$(R(t)x^{\Delta}(t))^{\Delta} + b(t)a(t)P(t)x^{\sigma}(t) = 0$$

is also nonoscillatory. That is,

$$(R(t)x^{\Delta}(t))^{\Delta} + P(t)x^{\sigma}(t) = 0$$

is nonoscillatory. But then since $P(t) \ge p(t)$ and $R(t) \le r(t)$, Lemma 2.2 (the Sturm-Picone comparison theorem) implies that equation (1.1) is also nonoscillatory. This is a condtradiction and completes the proof.

□

4. Examples

In this section, we will give several examples to illustrate Theorems 3.1 and 3.2. Since Example 4.1 is somewhat involved, we give the basic idea of its construction.

The following theorem may be found in [**14**], Theorem 5.81. (See also earlier references [**15**], [**19**], [**5**].)

Theorem. The equation $(r(t)x'(t))' + p(t)x = 0$ is oscillatory on the interval $[t_0, \infty)$, if $\int_{t_0}^{\infty} \frac{1}{r(t)} dt = \infty$ and there exists a continuously differentiable function $u(t) > 0$ such that

$$\int_{t_0}^{\infty} [p(t)u^2(t) - r(t)(u'(t))^2] dt = +\infty.$$

Analogous to the above theorem, we may obtain a corresponding time scales version which we state as follows:

Proposition. Assume that $\int_T^\infty \frac{1}{r(t)}\Delta t = \infty$. If there exists a real number α such that

$$\int_T^\infty \{\sigma^{2\alpha}(t)p(t) - [(t^\alpha)^\Delta]^2 r(t)\}\Delta t = +\infty. \tag{4.1}$$

Then all solutions of (1.1) are oscillatory.

Proof. Assume (1.1) is nonoscillatory on $[\tau, \infty)$. Then (1.1) is disconjugate in a neighborhood of ∞. By Lemma 2.3 there is a dominant solution $v(t)$ at ∞. Let $h(t) = t^\alpha$, then we get that the pair (p, r) satisfies condition $(\hat{A})$ by (4.1). In view of the proof of Theorem 3.1, we may then suppose also that $v(t) > 0$, $v^\Delta(t) > 0$ for $t \geq T \geq T_0$.

Make the Riccati substitution $z(t) = \frac{r(t)v^\Delta(t)}{v(t)}$, for $t \geq T$. We have from Picone's Identity (Lemma 2.4)

$$\begin{aligned} [t^{2\alpha}z(t)]^\Delta &= -p(t)(\sigma(t))^{2\alpha} + r(t)[(t^\alpha)^\Delta]^2 \\ &- \left[\frac{z(t)(\sigma(t))^\alpha}{\sqrt{r(t)+\mu(t)z(t)}} - (t^\alpha)^\Delta\sqrt{r(t)+\mu(t)z(t)}\right]^2 \\ &\geq p(t)(\sigma(t))^{2\alpha} - r(t)[(t^\alpha)^\Delta]^2. \end{aligned}$$

Integrating from T to t gives

$$\begin{aligned} &\int_T^t \{\sigma^{2\alpha}(s)p(s) - [(s^\alpha)^\Delta]^2 r(s)\}\Delta s \\ \leq\ & \frac{r(T)v^\Delta(T)T^{2\alpha}}{v(T)} - \frac{r(t)v^\Delta(t)t^{2\alpha}}{v(t)} \leq \frac{r(T)v^\Delta(T)T^{2\alpha}}{v(T)}. \end{aligned}$$

But now the left side is unbounded and the right side is bounded. This contradiction completes the proof of the proposition. □

Example 4.1. Let $\mathbb{T}$ be the real interval $[1, \infty)$. Let us consider functions $p(t)$ of the form $p(t) = \frac{a}{t^2} + \frac{b\sin t}{t}$, $a > 0, b > 0$. It is easy to observe that if $a > b$ then $p(t)$ satisfies condition (A). So we seek to find conditions on a and b such that $p(t)$ satisfies the conditions of Theorem 3.1 but does not satisfy condition (A). For simplicity, we consider the case $r(t) \equiv 1$ so that (1.1) becomes

$$x'' + p(t)x = 0. \tag{4.2}$$

Let $h(t) = t^\gamma$, $\gamma < \frac{1}{2}$. Denote

$$I(T) = \liminf_{t\to\infty} \int_T^t [p(s)h^2(s) - (h'(s))^2]ds$$

$$= T^{2\gamma-1}\left\{\frac{a-\gamma^2}{1-2\gamma} + T^{1-2\gamma}\int_T^\infty \frac{b\sin t}{t^{1-2\gamma}}dt\right\}. \tag{4.3}$$

The basic idea of constructing Example 4.1 is based on the following steps (i)-(iv):

(i) By the above Proposition, when $I(T) = +\infty$, (4.2) is oscillatory. Therefore, in order that (4.2) be nonoscillatory, we choose $\gamma < \frac{1}{2}$.

(ii) Since

$$\int_T^\infty p(t)dt = \frac{1}{T}\left[a + bT\int_T^\infty \frac{\sin t}{t}dt\right],$$

it follows that $p(t)$ does not satisfy condition (A) if $a < b$.

(iii) Also we have

$$\limsup_{t\to\infty} t\int_t^\infty p(s)ds = a + b,$$
$$\liminf_{t\to\infty} t\int_t^\infty p(s)ds = a - b.$$

By Hille's Theorem ([**7**]), if

$$-\frac{3}{4} < \liminf_{t\to\infty} t\int_t^\infty p(s)ds \le \limsup_{t\to\infty} t\int_t^\infty p(s)ds < \frac{1}{4},$$

then equation (4.2) is nonoscillatory. Note that $a \ge 0, b \ge 0$, $a + b < \frac{1}{4}$ implies $a - b > -\frac{3}{4}$. Therefore, Hille's condition holds if we choose $a \ge 0, b \ge 0$ and $a + b < \frac{1}{4}$. That is, equation (4.2) is nonoscillatory.

(iv) From (4.3), we see that $(p, 1)$ satisfies condition $(\hat{A})$, if we take $\frac{a-\gamma^2}{1-2\gamma} > b$.

Therefore, from (i)-(iv), if we choose $0 < a < b$ with $a + b < \frac{1}{4}$, and $0 < \gamma < 1/2$ with $\frac{a-\gamma^2}{1-2\gamma} > b$ then it follows that the pair $(p, 1)$ satisfies condition $(\hat{A})$. In particular, if we take $a = \frac{1}{64}$, $b = \frac{1}{63}$, $\gamma = \frac{1}{64}$. it follows that $p(t) = \frac{1}{64t^2} + \frac{\sin t}{63t}$ is such that equation (4.2) is nonoscillatory.

Now if we set $a(t) := ct^{-\alpha}(\log t)^\beta$, $\alpha > 0, \beta \in \mathbb{R}$, then we have $0 < a(t) \le 1$, $a'(t) \le 0$, for large t. So by Theorem 3.1, the equation

$$x'' + \left(\frac{c(\log t)^\beta}{64t^{2+\alpha}} + \frac{c(\log t)^\beta \sin t}{63t^{1+\alpha}}\right)x = 0$$

is nonoscillatory on $(2, \infty)$ for all $c > 0, \alpha > 0, \beta \in \mathbb{R}$. The authors are unaware of existing results in the literature which yield this conclusion.

Remark 4.2. To continue with the equation in the previous example, suppose that $0 < a < \frac{1}{8}$, and choose b such that $a < b < \frac{1}{2} - \frac{\sqrt{1-4a}}{2} :=$ γ_0. With $f(\gamma) := \frac{a-\gamma^2}{1-2\gamma}$, we note that f assumes a local maximum at γ_0

and we have $f(\gamma_0) = \gamma_0 > b$. Now the equation $f(\gamma) = b$ has two roots $\gamma_{1,2} = b \pm \sqrt{b^2 - b + a}$ and $f(\gamma) > b$ on (γ_1, γ_2). For example, if $a = .1$, then $\gamma_0 \approx .1127$ so if $.1 < b < .1127$, then condition $(\hat{A})$ holds and equation (4.2) is nonoscillatory using the nonoscillation criterion of Hille [**7**].
Moreover, we can use a nonoscillation criterion due to Willett [**18**] (see also Wong [**20**] for further extensions of this to several critical cases) which shows that the equation $x'' + p(t)x = 0$ is nonoscillatory where

$$p(t) = \frac{a}{t^2} + \frac{b \sin t}{t},$$

provided

$$a < \frac{1}{4} - \frac{b^2}{2}.$$

We can therefore determine a convex domain R in the $(a, b)-$plane for which $x'' + \left(\frac{a}{t^2} + \frac{b \sin t}{t}\right) x = 0$ is nonoscillatory and is such that condition $(\hat{A})$ holds (but not condition (A)) and for which the nonoscillation criterion of Hille (see [**7**]) is also not applicable.
Specifically, we define the region R determined by the inequalities

$$R: \quad 0 < a < b, \quad a > b - b^2, \quad a + b > \frac{1}{4}, \quad a < \frac{1}{4} - \frac{b^2}{2}.$$

Then $a < b$ implies that condition (A) does not hold and $a < \frac{1}{4} - \frac{b^2}{2}$ guarantees nonoscillation by the result of Willett [**18**] and Wong [**20**]. The condition $a > b - b^2$ implies that $b < \gamma_0 = \frac{1}{2} - \frac{\sqrt{1-4a}}{2}$ so that condition $(\hat{A})$ holds for the pair $(p, 1)$ with $h(t) = t^{\gamma_0}$. Finally, the condition $a + b > \frac{1}{4}$ shows that the criterion of Hille is also not applicable. Thus, we can choose, as a particular example, $a = .2$, $b = .22$ and with this choice it follows that the equation

$$x'' + \frac{c(\log t)^\beta}{t^\alpha} \left(\frac{a}{t^2} + \frac{b \sin t}{t} \right) x = 0$$

is nonoscillatory for all $\alpha > 0$, $\beta \in \mathbb{R}$, and for all $c > 0$ by Theorem 3.1.

As a second illustration of the applicability of our results we give the following example.

EXAMPLE 4.3. Let $\mathbb{T}$ be the real interval $[1, \infty)$. Let

$$P(t) = \frac{a}{t^{1+b+c}} + \frac{\sin t}{t^{b+c}}, \quad a(t) = t^c,$$

where $0 < a < b < 1$, $c = \frac{1-b}{2}$.

We have

$$\int_T^\infty a(s)P(s)ds = T^{-b} \left(\frac{a}{b} + T^b \int_T^\infty \frac{\sin t}{t^b} dt \right).$$

With $h(t) = t^{\frac{b}{2}}$, and $r(t) = 1$, then we have

$$\int_T^t \{a(s)P(s)h^2(s) - r(s)[h'(s)]^2\}ds$$

$$= a\ln\left(\frac{t}{T}\right) + \int_T^t \sin s\, ds + \frac{b^2}{4(b-1)}\left(\frac{t}{T}\right)^{b-1} \to \infty, \quad \text{as} \quad t \to \infty.$$

Since $0 < a < b < 1$, $a(t)P(t)$ does not satisfy condition (A), but the pair $(aP, 1)$ does satisfy condition $(\hat{A})$. Moreover, since we also have $0 < a < b + c < 1$, by the above Proposition, it follows that $x'' + P(t)x = 0$ is oscillatory. Since $a(t) = t^c \geq 1$, for $t \geq 1$ and $a'(t) \geq 0$, it follows by Theorem 3.2 that $x'' + a(t)P(t)x = 0$ is oscillatory.

Example 4.4. Let $\mathbb{T} = \mathbb{Z}$, $p(t) = \frac{\gamma}{t(t+1)} + \frac{\lambda(-1)^t}{t}$, $r(t) = 1$, $\gamma > 0$, $\lambda > 0$. We have

$$\begin{aligned} \int_t^\infty p(s)\Delta s &= \sum_{k=n}^{\infty}\left[\frac{\gamma}{k(k+1)} + \lambda\frac{(-1)^k}{k}\right] \\ &= \frac{\gamma}{n} + \sum_{k=n}^{\infty}\lambda\frac{(-1)^k}{k}. \end{aligned}$$

Now consider

$$\begin{aligned} & 2m\sum_{k=2m}^{\infty}\frac{(-1)^k}{k} \\ = \; & 2m\left\{\frac{1}{2m} - \left[\frac{1}{(2m+1)(2m+2)} + \frac{1}{(2m+3)(2m+4)} + \cdots\right]\right\} \\ \to \; & 1 - \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx = \frac{1}{2}, \ (n \to \infty). \end{aligned}$$

Similarly, we have

$$(2m+1)\sum_{k=2m+1}^{\infty}\frac{(-1)^k}{k} \to -1 + \frac{1}{2}\int_0^\infty \frac{1}{(1+x)^2}dx = -\frac{1}{2}$$

So, in this case, if $\gamma < \frac{\lambda}{2}$, $p(t)$ does not satisfy condition (A).

Also we have

$$\limsup_{t\to\infty} t\int_t^\infty p(s)ds = \gamma + \frac{\lambda}{2},$$

$$\liminf_{t\to\infty} t\int_t^\infty p(s)ds = \gamma - \frac{\lambda}{2}.$$

By Hille's Theorem [6], if we choose $\gamma + \frac{\lambda}{2} < \frac{1}{4}$, then the equation $x^{\Delta\Delta} + p(t)x^\sigma = 0$ is nonoscillatory.

Let $h(t) = t^\alpha$, $\alpha < \frac{1}{2}$. Denote

$$I(n) = \liminf_{t\to\infty} \int_n^t \left[p(s)h^2(\sigma(s)) - (h^\Delta(s))^2\right] \Delta s$$
$$= \sum_{k=n}^{\infty} \left[\frac{\gamma}{k(k+1)^{1-2\alpha}} + \frac{\lambda(-1)^k}{k^{1-2\alpha}} - [(k+1)^\alpha - k^\alpha]^2\right].$$

Note that

$$\sum_{k=n}^{\infty} \left[\frac{1}{k(k+1)^{1-2\alpha}}\right] \sim \frac{1}{(1-2\alpha)n^{1-2\alpha}}.$$

Since $(k+1)^\alpha - k^\alpha \sim \frac{\alpha}{k^{1-\alpha}}$, large k, we have

$$[(k+1)^\alpha - k^\alpha]^2 = \frac{\alpha^2}{k^{2-2\alpha}} + \frac{2\alpha}{k^{2-2\alpha}}\, o(1).$$

So

$$\lim_{n\to\infty} n^{1-2\alpha} \sum_{k=n}^{\infty} [(k+1)^\alpha - k^\alpha]^2$$
$$= \lim_{n\to\infty} n^{1-2\alpha} \sum_{k=n}^{\infty} \left[\frac{\alpha^2}{k^{(2-2\alpha)}}\right]$$
$$= \frac{\alpha^2}{1-2\alpha}.$$

Therefore

$$\sum_{k=n}^{\infty} [(k+1)^\alpha - k^\alpha]^2 \sim \frac{\alpha^2}{(1-2\alpha)n^{1-2\alpha}}.$$

So the pair $(p(t), 1)$ will satisfy condition $(\hat{A})$ if we take $\frac{\gamma-\alpha^2}{1-2\alpha} > \frac{\lambda}{2}$.

Therefore, choosing $0 < \alpha < \frac{1}{2}$, $0 < \gamma < \frac{\lambda}{2}$ with $\gamma + \frac{\lambda}{2} < \frac{1}{4}$, $\frac{\gamma-\alpha^2}{1-2\alpha} > \frac{\lambda}{2}$, then $p(t) = \frac{\gamma}{t(t+1)} + \frac{\lambda(-1)^t}{t}$ satisfies all the requirements of Theorem 3.1. In particular, if we take $\gamma = \frac{1}{64}$, $\lambda = \frac{2}{63}$, $\alpha = \frac{1}{64}$, it follows that the difference equation

$$\Delta^2 x(t) + \left(\frac{1}{64t(t+1)} + \frac{2(-1)^t}{63t}\right) x(t+1) = 0$$

is nonoscillatory.

Example 4.5. Let $q > 1$. Consider the time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$. In this case, $\sigma(t) = qt$, $\mu(t) = (q-1)t$ for all $t \in \mathbb{T}$. (Recall that any dynamic equation on the time scale $q^{\mathbb{N}_0}$ is called a q-difference equation.) Let

$$p(t) = \frac{\lambda}{t\sigma(t)^{b+c}} + \frac{\beta(-1)^n}{t\sigma(t)^{b+c}}, \quad a(t) = t^c,$$

where $n = \frac{\ln t}{\ln q}$, $\lambda > 0$, $0 < b < 1$, $c = \frac{1-b}{2}$. Let

$$m := \frac{q^b - 1}{q^b + 1},$$

and assume further that $0 < \lambda < m\beta$. Then we have, for $t_n = q^n$,

$$\begin{aligned}\int_{t_n}^{\infty} a(s)p(s)\Delta s &= \frac{1}{q^{b+c}} \sum_{k=n}^{\infty} \frac{1}{q^{k(1+b)}} [\lambda + \beta(-1)^k](q-1)q^k \\ &= \frac{(q-1)}{q^{nb+c}} \left(\frac{\lambda}{q^b - 1} + \frac{\beta(-1)^n}{q^b + 1} \right) \\ &= \frac{(q-1)}{q^{nb+c}} \times \frac{1}{q^b - 1} (\lambda + m\beta(-1)^n).\end{aligned}$$

Notice that this last expression may be negative, for large n, since $0 < \lambda < m\beta$. Hence, $a(t)p(t)$ does not satisfy condition (A).

Take $h(t) = t^{\frac{b}{2}}$, $r(t) = 1$. Then we have, for $t = q^n$

$$\int_1^t \{a(s)p(s)h^2(\sigma(s)) - r(s)[h^{\Delta}(s)]^2\}\Delta s$$

$$= \int_1^t \left\{ \frac{\lambda}{q^c s} + \frac{\beta(-1)^{\frac{\ln s}{\ln q}}}{q^c s} - \left[\frac{q^{\frac{b}{2}} - 1}{q - 1} \right]^2 \frac{1}{s^{2-b}} \right\} \Delta s \to \infty.$$

So $(ap, 1)$ satisfies condition $(\hat{A})$.

Take $h_1(t) = t^{\frac{b+c}{2}}$, $r(t) = 1$. Then we have, for $t = q^n$

$$\int_1^t \{p(s)h_1^2(\sigma(s)) - r(s)[h_1^{\Delta}(s)]^2\}\Delta s$$

$$= \int_1^t \left\{ \frac{\lambda}{s} + \frac{\beta(-1)^{\frac{\ln s}{\ln q}}}{s} - \left[\frac{q^{\frac{b+c}{2}} - 1}{q - 1} \right]^2 \frac{1}{s^{2-b-c}} \right\} \Delta s \to \infty.$$

So $(p, 1)$ satisfies condition $(\hat{A})$. By the above Proposition, $x^{\Delta\Delta} + p(t)x^{\sigma} = 0$ is oscillatory. Since $a(t) = t^c \geq 1$, for $t \geq 1$ and $a^{\Delta}(t) \geq 0$, by Theorem 3.2, $x^{\Delta\Delta} + a(t)p(t)x = 0$ is oscillatory.

The interested reader may also easily construct similar examples on other time scales to illustrate the results of the above theorems.

References

[1] J. P. Bézivin, Sur les équations fonctionnelles aux q-différences, Aequationes Math., 43 (1993) 159–176.

[2] Martin Bohner, Lynn Erbe, and Allan Peterson, Oscillation for nonlinear second order dynamic equation on a time scale, J. Math. Anal. Appl. 301 (2005) 491–507.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[4] M. Bohner and A. Peterson, Editors, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[5] W. A. Coppel, Disconjugacy, Lecture Notes In Mathematics, Springer-Verlag, No. 220, 1971.
[6] G. Derfel, E. Romanenko, and A. Sharkovsky, Long-time properties of solutions of simplest nonlinear q-difference equations, J. Difference Equs. Appl., 6 (2000) 485–511.
[7] O. Došlý and P. Řehák, Half-linear Differential Equation, North-Holland Mathematics Studies 202 Elsevier, Amsterdam, 2005.
[8] L. Erbe, Oscillation criteria for second order linear equations on a time scale, Canad. Appl. Math. Quart., 9 (2001) 1–31.
[9] Lynn Erbe, Oscillation Theorems for second order linear differential equation, Pacific J. Math., 35 (1970) 337–343.
[10] Lynn Erbe, Oscillation criteria for second order nonlinear differential equations, Annali di Mathematica Pura ed Applicata (IV), Vol. XCIV, 257–268.
[11] Lynn Erbe, Allan Peterson, and Pavel Řehák, Comparison theorems for linear dynamic equations on time scales, J. Math. Anal. Appl. 275 (2002) 418–438.
[12] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, Results Math., 18 (1990) 18–56.
[13] V. Kac and P. Cheung, Quantum Calculus, Universitext, Springer, New York, 2000.
[14] Walter Kelley and Allan Peterson, The Theory of Differential Equations Classical and Qualitative, Pearson Education, Inc., Upper Saddle River, New Jersey, 2004.
[15] Miloš Ráb, Kriterien für die 0szillation der lösungen der differentialgleichung $[p(x)y']' + q(x)y = 0$, Časopis Pěst. 84 (1959), 335-370 and 85 (1960) 91.
[16] V. Spedding, Taming nature's numbers, NewScientist (July 19, 2003) 28–31.
[17] W. J. Trijzinsky, Analytic theory of linear q-difference equations, Acta Math. 61 (1933) 1–38.
[18] D. Willett, On the oscillatory behavior of the solutions of second order linear differential equations, Ann. Polon. Math., 21 (1969), 175-194.
[19] Aurel Wintner, A criterion of oscillatory stability, Quart. Appl. Math. 7 (1949) 115–117.
[20] James S. W. Wong, Oscillation and nonscillation of solutions of second order linear differential equations with integrable coefficients, Trans. Amer. Math. Soc. 144 (1969), 197-215.