Computers and Mathematics with Applications xx (xxxx) xxx–xxx

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales

Jia Baoguo [a,b], Lynn Erbe [a], Allan Peterson [a,*]

[a] Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA
[b] School of Mathematics and Computer Science, Zhongshan University, Guangzhou, 510275, China

ARTICLE INFO

Article history:
Received 28 March 2008
Accepted 27 May 2008

Keywords:
Half-linear dynamic equation
Condition (A)
Condition $(\hat{A})$
Condition (B)

ABSTRACT

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = +\infty$. Consider the second-order half-linear dynamic equation

$$(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + p(t)x^{\alpha}(\sigma(t)) = 0,$$

where $r(t) > 0$, $p(t)$ are continuous, $\int_{t_0}^{\infty}(r(t))^{-\frac{1}{\alpha}}\Delta t = \infty$, α is a quotient of odd positive integers. In particular, no explicit sign assumptions are made with respect to the coefficient $p(t)$. We give conditions under which every positive solution of the equations is strictly increasing. For $\alpha = 1$, $\mathbb{T} = \mathbb{R}$, the result improves the original theorem [see: [Lynn Erbe, Oscillation theorems for second-order linear differential equation, Pacific J. Math. 35 (2) (1970) 337–343]]. As applications, we get two comparison theorems and an oscillation theorem for half-linear dynamic equations which improve and extend earlier results. Some examples are given to illustrate our theorems.

1. Introduction

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. Consider the second-order half-linear dynamic equation

$$(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + p(t)x^{\alpha}(\sigma(t)) = 0, \tag{1.1}$$

where $r(t) > 0$, $p(t)$ are continuous, $\int_{t_0}^{\infty}(r(t))^{-\frac{1}{\alpha}}\Delta t = \infty$, α is a quotient of odd positive integers. We emphasize that no explicit sign assumptions are made with respect to the coefficient $p(t)$.

For completeness, we recall some basic results for dynamic equations and the calculus on time scales. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale

* Corresponding author.
E-mail addresses: mcsjbg@mail.sysu.edu.cn (B. Jia), lerbe2@math.unl.edu (L. Erbe), apeterso1@math.unl.edu (A. Peterson).

0898-1221/$ – see front matter © 2008 Published by Elsevier Ltd
doi:10.1016/j.camwa.2008.05.014

interval $[c, d] := \{t \in \mathbb{T} : c \le t \le d\}$ in $\mathbb{T}$ the notation $[c, d]^\kappa$ denotes the interval $[c, d]$ in case $\rho(d) = d$ and denotes the interval $[c, d)$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

The theory of time scales was introduced by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify continuous and discrete analysis (see [1]). Not only does this unify the theories of differential equations and difference equations, but it also extends these classical situations to cases "in between"— e.g., to the so-called q-difference equations which are important in the theory of orthogonal polynomials. Moreover, the theory can be applied to numerous other time scales. We refer to the two books on the subject of time scales by Bohner and Peterson [2,3] which summarize and organize much of time scale calculus and applications to dynamic equations.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{rd}. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable on $[c, d]^\kappa$ and whose delta derivative is rd-continuous on $[c, d]^\kappa$ is denoted by C^1_{rd}.

We recall that a solution of Eq. (1.1) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Eq. (1.1) is said to be oscillatory in case all of its solutions are oscillatory. The study of the oscillatory and nonoscillatory properties of Eq. (1.1) and its many generalizations and extensions is voluminous and we refer to [4,5] and the references therein.

The following condition (A) was introduced in [6] for the continuous case in order to obtain some new oscillation and comparison results for the linear homogeneous differential equation in the case when the function $p(t)$ can take on both positive and negative values for large t.

Definition 1. We say that a function $g : \mathbb{T} \to \mathbb{R}$ satisfies condition (A) if the following condition holds:

$$\liminf_{t\to\infty} \int_T^t g(s)\Delta s \ge 0 \quad \text{and} \quad \not\equiv 0,$$

for all large T.

We wish to extend this notion to a *triple* of functions (α, p, r), so we introduce the following definition:

Definition 2. We say that the triple (α, p, r) satisfies condition $(\hat{A})$, if there exists a continuously differentiable function $h : \mathbb{T} \to \mathbb{R}$, such that either $h^\Delta(t)$ is of one sign for all $t \in \mathbb{T}$ or $h^\Delta(t) \equiv 0$ and is such that $p(t)h^{\alpha+1}(\sigma(t)) - r(t)(h^\Delta(t))^{\alpha+1}$ satisfies condition (A).

Notice that if $h(t) = 1$, $\alpha = 1$, then this means that $p(t)$ satisfies condition (A).

A continuous version of the following definition appeared in [7], Page 814.

Definition 3. We say that a function $p : \mathbb{T} \to \mathbb{R}$ satisfies condition (B) in case there exists a sequence $\{\tau_n\} \subset \mathbb{T}$, $\tau_n \to \infty$, such that $\int_{\tau_n}^t p(s)\Delta s \ge 0$, for $t \ge \tau_n$.

It is obvious that condition (A) implies both condition $(\hat{A})$ and condition (B) (see [6]), but the converse is not true (see Examples 1.1 and 1.2). In Section 2, we prove that if $p(t)$ satisfies condition (B) and the triple (α, p, r) satisfies condition $(\hat{A})$, then positive solutions of (1.1) are strictly increasing. This improves and extends a result of [6].

In Sections 3 and 4, we prove two comparison theorems that improve two main results of [8] and give two examples to illustrate that our theorems are new.

In Section 5, we obtain an oscillation theorem that extends the results of [4,9,10] and give several examples to illustrate our theorem.

The following examples show that the class of functions which satisfy condition $(\hat{A})$ and condition (B) but do not satisfy condition (A) is nonempty.

Example 1.1. Let $q > 1$. Consider the time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$. In this case, $\sigma(t) = qt$, $\mu(t) = (q-1)t$ for all $t \in \mathbb{T}$. (Recall that any dynamic equation on the time scale $q^{\mathbb{N}_0}$ is called a q-difference equation.) Let

$$p(t) = \frac{\lambda}{t(\sigma(t))^b} + \frac{\beta(-1)^n}{t(\sigma(t))^b}, \quad n := \frac{\ln t}{\ln q}$$

where $\lambda > 0$, $0 < b < 3$. Let $\alpha = 3$. Consider the q-difference equation

$$((x^\Delta(t))^3)^\Delta + p(t)x^3(\sigma(t)) = 0.$$

Let

$$m := \frac{q^b - 1}{q^b + 1},$$

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

and assume further that $0 < \lambda < m\beta$. Then we have, for $t_n = q^n$,

$$\begin{aligned}\int_{t_n}^{\infty} p(s)\Delta s &= \frac{1}{q^b}\sum_{k=n}^{\infty}\frac{1}{q^{k(1+b)}}[\lambda+\beta(-1)^k](q-1)q^k \\ &= \frac{(q-1)}{q^{nb}}\left(\frac{\lambda}{q^b-1}+\frac{\beta(-1)^n}{q^b+1}\right) \\ &= \frac{(q-1)}{q^{nb}}\times\frac{1}{q^b-1}(\lambda+m\beta(-1)^n).\end{aligned}$$

Notice that this last expression may be negative, for large n, since $0 < \lambda < m\beta$. Hence, $p(t)$ does not satisfy condition (A).

Take $h(t) = t^{\frac{b}{4}}$, $r(t) = 1$. Then we have, for $t = q^n$

$$\int_1^t \{p(s)h^4(\sigma(s)) - r(s)[h^{\Delta}(s)]^4\}\Delta s = \int_1^t \left\{\frac{\lambda}{s}+\frac{\beta(-1)^{\frac{\ln s}{\ln q}}}{s}-\left[\frac{q^{\frac{b}{4}}-1}{q-1}\right]^4\frac{1}{s^{4-b}}\right\}\Delta s \to \infty.$$

So the triple $(3, p, 1)$ satisfies condition $(\hat{A})$.

Let $\tau_n = q^{2n}$. It is easy to see that $\int_{\tau_n}^t p(s)\Delta s \geq 0$, for $t \geq \tau_n$ and so $p(t)$ satisfies condition (B).

Example 1.2. Let $\mathbb{T}$ be the real interval $[1, \infty)$, $g(t) = 1 + t\sin t$. Then we have

(i) $g(t)$ does not satisfy condition (A), since $\int_T^{\infty} g(t)\mathrm{d}t$ does not converge and $\int_T^{\infty} g(t)\mathrm{d}t \neq \infty$.

(ii) Let $h(t) = t^{-\frac{1}{8}}$, $r(t) = 1$. Then

$$\int_T^t \{g(s)h^2(s) - r(s)[h'(s)]^2\}\mathrm{d}s = t^{\frac{3}{4}}\left(\frac{4}{3}+t^{-\frac{3}{4}}\int_T^t s^{\frac{3}{4}}\sin s\mathrm{d}s\right)+\frac{1}{80}t^{-\frac{5}{4}}-\frac{4}{3}T^{\frac{3}{4}}-\frac{1}{80}T^{-\frac{5}{4}}. \quad (1.2)$$

Integrating by parts twice, it is easy to see that

$$\limsup_{t\to\infty} t^{-\frac{3}{4}}\int_T^t s^{\frac{3}{4}}\sin s\mathrm{d}s = 1, \qquad \liminf_{t\to\infty} t^{-\frac{3}{4}}\int_T^t s^{\frac{3}{4}}\sin s\mathrm{d}s = -1.$$

Take $\epsilon = \frac{1}{9}$. We have

$$-\frac{10}{9} = 1-\epsilon \leq t^{-\frac{3}{4}}\int_T^t s^{\frac{3}{4}}\sin s\,\mathrm{d}s \leq 1+\epsilon = \frac{10}{9},$$

for large t.

Therefore, for large t, we have

$$t^{\frac{3}{4}}\left(\frac{4}{3}+t^{-\frac{3}{4}}\int_T^t s^{\frac{3}{4}}\sin s\,\mathrm{d}s\right) \geq \frac{1}{9}t^{\frac{3}{4}}.$$

Hence by (1.2), we obtain

$$\int_T^{\infty}\{g(s)h^2(s) - r(s)[h'(s)]^2\}\mathrm{d}s = \infty.$$

Similarly, we also can get

$$\int_T^{\infty}\{g(s)h^4(s) - r(s)[h'(s)]^4\}\mathrm{d}s = \infty.$$

Therefore the triple $(1, g, 1)$ and $(3, g, 1)$ satisfy the condition $(\hat{A})$.

(iii) In the following, we show that $g(t)$ satisfies condition (B).

Assume that $0 < t_1 < t_2 < \cdots < t_{2k} < t_{2k+1} < t_{2k+2} < \cdots$ are the positive zero points of $g(t)$. It suffices to prove that

$$\int_{t_{2k}}^{t_{2k+2}} g(s)\mathrm{d}s \geq 0,$$

i.e.,

$$\int_{t_{2k}}^{t_{2k+1}} g(s)\mathrm{d}s \geq -\int_{t_{2k+1}}^{t_{2k+2}} g(s)\mathrm{d}s,$$

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

for large k. That is,

$$\begin{aligned} &t_{2k+1} - t_{2k+1}\cos t_{2k+1} + \sin t_{2k+1} - (t_{2k} - t_{2k}\cos t_{2k} + \sin t_{2k}) \\ &\quad \geq -(t_{2k+2} - t_{2k+2}\cos t_{2k+2} + \sin t_{2k+2}) + (t_{2k+1} - t_{2k+1}\cos t_{2k+1} + \sin t_{2k+1}). \end{aligned} \tag{1.3}$$

Using the fact that $\sin t_j = \frac{-1}{t_j}$ and rearranging, we see that (1.3) is equivalent to

$$t_{2k+2} - \frac{1}{t_{2k+2}} - \sqrt{t_{2k+2}^2 - 1} \geq t_{2k} - \frac{1}{t_{2k}} - \sqrt{t_{2k}^2 - 1}. \tag{1.4}$$

If we set

$$f(x) = x - \frac{1}{x} - \sqrt{x^2 - 1},$$

then it is easy to see that $f'(x) > 0$ for large x and therefore it follows that (1.4) holds for large k. This completes the proof.

2. Lemma

Lemma 2.1. *Let $x(t)$ be a nonoscillatory solution of (1.1) and assume that $p(t)$ satisfies condition (B), the triple (α, p, r) satisfies condition $(\hat{A})$, and $\int_{t_0}^{\infty} (r(t))^{-\frac{1}{\alpha}} \Delta t = \infty$. Then there exists $T \geq t_0$ such that $x(t)x^{\Delta}(t) > 0$, for $t \geq T$.*

Proof. Suppose that x is a nonoscillatory solution of (1.1) and without loss of generality, assume $x(t) > 0$ for $t \geq t_0$. Since $p(t)$ satisfies condition (B), let τ_n be the corresponding sequence with $\int_{\tau_n}^{t} p(s)\Delta s \geq 0$, for $t \geq \tau_n$.

Let us assume, for the sake of contradiction, that $x^{\Delta}(t)$ is not strictly positive for all large t. First consider the case when $x^{\Delta}(t) < 0$ for all large t. Then without loss of generality, we can assume that $x^{\Delta}(t) < 0$ for $t \geq \tau_k \geq t_0$, where k is large and fixed. An integration of Eq. (1.1) for $t > \tau_k$ gives

$$r(t)(x^{\Delta}(t))^{\alpha} + \int_{\tau_k}^{t} p(s)x^{\alpha}(\sigma(s))\Delta s = r(\tau_k)(x^{\Delta}(\tau_k))^{\alpha}. \tag{2.1}$$

Now integrating by parts, we have

$$\int_{\tau_k}^{t} p(s)x^{\alpha}(\sigma(s))\Delta s = x^{\alpha}(t)\int_{\tau_k}^{t} p(s)\Delta s - \int_{\tau_k}^{t} (x^{\alpha}(s))^{\Delta_s}\left(\int_{\tau_k}^{s} p(u)\Delta u\right)\Delta s. \tag{2.2}$$

By the Pötzsche Chain Rule, ([2] Theorem 1.90) we have

$$(x^{\alpha}(t))^{\Delta} = \left\{\int_0^1 \alpha(x(t) + h\mu(t)x^{\Delta}(t))^{\alpha-1}\mathrm{d}h\right\} x^{\Delta}(t) \leq 0,$$

since $(x(t) + h\mu(t)x^{\Delta}(t))^{\alpha-1} \geq 0$ and $x^{\Delta}(t) < 0$. Hence, it follows that

$$\int_{\tau_k}^{t} (x^{\alpha}(t))^{\Delta}\left(\int_{\tau_k}^{s} p(u)\Delta u\right)\Delta s \leq 0,$$

and so from (2.2), we have

$$\int_{\tau_k}^{t} p(s)x^{\alpha}(\sigma(s))\Delta s \geq x^{\alpha}(t)\int_{\tau_k}^{t} p(s)\Delta s \geq 0.$$

Consequently, from (2.1), we have

$$r(t)(x^{\Delta}(t))^{\alpha} \leq r(\tau_k)(x^{\Delta}(\tau_k))^{\alpha}, \quad t \geq \tau_k.$$

Hence

$$x(t) \leq x(\tau_k) + (r(\tau_k))^{\frac{1}{\alpha}}x^{\Delta}(\tau_k)\int_{\tau_k}^{t}\left[\frac{1}{r(s)}\right]^{\frac{1}{\alpha}}\Delta s \to -\infty,$$

as $t \to \infty$, which is a contradiction.

So $x^{\Delta}(t)$ is not negative for all large t and since we are assuming $x^{\Delta}(t)$ is not positive for all large t, it follows that $x^{\Delta}(t)$ must change sign infinitely often.

Make the substitution

$$\omega(t) = r(t)\left[\frac{x^{\Delta}(t)}{x(t)}\right]^{\alpha} h^{\alpha+1}(t),$$

for $t \geq T_1$. We may suppose that T_1 is sufficiently large so that

$$\liminf_{t\to\infty} \int_{T_1}^{t} \{p(s)h^{\alpha+1}(\sigma(s)) - r(s)[h^{\Delta}(s)]^{\alpha+1}\}\Delta s \geq 0, \tag{2.3}$$

holds and is such that $\omega(T_1) \leq 0$, (*i.e.*, $x^{\Delta}(T_1) \leq 0$).

$$\begin{aligned} \omega^{\Delta}(t) &= \left[r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha}\right]^{\Delta} h^{\alpha+1}(\sigma(t)) + r(t)\left[\frac{x^{\Delta}(t)}{x(t)}\right]^{\alpha}(h^{\alpha+1}(t))^{\Delta} \\ &= -p(t)h^{\alpha+1}(\sigma(t)) + r(t)(h^{\Delta}(t))^{\alpha+1} \\ &\quad - r(t)\left[(h^{\Delta}(t))^{\alpha+1} - \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha}(h^{\alpha+1}(t))^{\Delta} + \frac{(x^{\Delta}(t))^{\alpha}(x^{\alpha}(t))^{\Delta}}{x^{\alpha}(t)x^{\alpha}(\sigma(t))}h^{\alpha+1}(\sigma(t))\right]. \end{aligned}$$

If we define (omitting arguments)

$$F(t) := r\left[(h^{\Delta})^{\alpha+1} - \left(\frac{x^{\Delta}}{x}\right)^{\alpha}(h^{\alpha+1})^{\Delta} + \frac{(x^{\Delta})^{\alpha}(x^{\alpha})^{\Delta}}{x^{\alpha}(x^{\sigma})^{\alpha}}(h^{\sigma})^{\alpha+1}\right],$$

then we have

$$\omega^{\Delta}(t) = -p(t)h^{\alpha+1}(\sigma(t)) + r(t)[h^{\Delta}(t)]^{\alpha+1} - F(t). \tag{2.4}$$

(i) Suppose that $t \in \mathbb{T}$ is right-dense. Then $(h^{\alpha+1}(t))^{\Delta} = (\alpha+1)h^{\alpha}(t)h^{\Delta}(t)$, so we have (again omitting arguments)

$$F(t) = (\alpha+1)r\left[\frac{[h^{\Delta}]^{\alpha+1}}{\alpha+1} - h^{\Delta}\left[\frac{x^{\Delta}h}{x}\right]^{\alpha} + \frac{\left[(\frac{x^{\Delta}h}{x})^{\alpha}\right]^{\frac{\alpha+1}{\alpha}}}{\frac{\alpha+1}{\alpha}}\right].$$

We use Young's inequality [11], which says that

$$\frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0, \quad p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1,$$

with equality if and only if $v = u^{\alpha}$, $\alpha := \frac{p}{q}$.

So if we let

$$u = h^{\Delta}(t), \qquad v = \left[\frac{x^{\Delta}(t)h(t)}{x(t)}\right]^{\alpha}, \qquad p = \alpha+1, \qquad q = \frac{\alpha+1}{\alpha},$$

then we have that $F(t) \geq 0$ and

$$F(t) = 0 \quad \text{iff}\ \frac{x^{\Delta}(t)h(t)}{x(t)} = h^{\Delta}(t).$$

(ii) Suppose next that $t \in \mathbb{T}$ is right-scattered. Then $x^{\Delta}(t) = \frac{x(\sigma(t))-x(t)}{\mu(t)}$, $(x^{\alpha}(t))^{\Delta} = \frac{x^{\alpha}(\sigma(t))-x^{\alpha}(t)}{\mu(t)}$, $h^{\Delta}(t) = \frac{h(\sigma(t))-h(t)}{\mu(t)}$, $(h^{\alpha+1}(t))^{\Delta} = \frac{h^{\alpha+1}(\sigma(t))-h^{\alpha+1}(t)}{\mu(t)}$. Let us put $a := \frac{h(\sigma(t))}{h(t)}$, $b := \frac{x(\sigma(t))}{x(t)}$. Then after substituting and rearranging we have

$$F(t) = \frac{r(t)h^{\alpha+1}(t)a^{\alpha+1}}{\mu^{\alpha+1}(t)}f(a,b)$$

where $f(a,b) := (1-a^{-1})^{\alpha+1} - (b-1)^{\alpha}(1-a^{-(\alpha+1)}) + (b-1)^{\alpha}(1-b^{-\alpha})$.

Notice that $f(a,a) = 0$ and

$$\frac{\partial f}{\partial a}(a,b) = \frac{(\alpha+1)a^{-2}}{a^{\alpha}}[(a-1)^{\alpha} - (b-1)^{\alpha}].$$

It follows that if $a > b$, then $\frac{\partial f}{\partial a}(a,b) > 0$, and so $f(a,b) > 0$. Likewise, if $a < b$, then $\frac{\partial f}{\partial a}(a,b) < 0$, and so $f(a,b) > 0$. In other words, $f(a,b) \geq 0$ and

$$f(a,b) = 0 \Leftrightarrow a = b \Leftrightarrow \frac{h(\sigma(t))}{h(t)} = \frac{x(\sigma(t))}{x(t)} \Leftrightarrow \frac{x^{\Delta}(t)}{x(t)} = \frac{h^{\Delta}(t)}{h(t)}.$$

From (i) and (ii), we obtain that $F(t) \geq 0$ and

$$F(t) = 0 \quad \text{iff}\ \frac{h^{\Delta}(t)}{h(t)} = \frac{x^{\Delta}(t)}{x(t)}.$$

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

Integrating both sides of (2.4) from T_1 to t, we have

$$\omega(t)-\omega(T_1)=-\int_{T_1}^{t}\{p(s)h^{\alpha+1}(\sigma(s))-r(s)[h^{\Delta}(s)]^{\alpha+1}\}\Delta s-\int_{T_1}^{t}F(s)\Delta s. \tag{2.5}$$

In the following, we will consider two cases:

Case (I)

$$F(s)\equiv 0,\quad s\geq T_1.$$

We then have

$$\frac{h^{\Delta}(s)}{h(s)}\equiv\frac{x^{\Delta}(s)}{x(s)}.$$

So $x(s)=Ch(s)$. Without loss of generality we assume that $h(s)>0$, for $s\geq T_1$, since the other case is similar. Therefore, we have $C>0$.

(i) If $h^{\Delta}(s)>0$, for $s\in[T_1,\infty)$, we have $x^{\Delta}(t)>0$, which is a contradiction to the assumption that $x^{\Delta}(t)$ changes sign infinitely often.

(ii) If $h^{\Delta}(s)\equiv 0$, we will have $p(s)\equiv 0$, which contradicts the definition of condition $(\hat{A})$.

(iii) If $h^{\Delta}(s)<0$, for $s\in[T_1,\infty)$, we have $x^{\Delta}(t)<0$. which is also a contradiction to the assumption that $x^{\Delta}(t)$ changes sign infinitely often.

Case (II)

$$F(s)\not\equiv 0,$$

for $s\geq T_1$.

In this case we can choose $\epsilon>0$ and $T_2>T_1$ such that for $t\geq T_2$,

$$\int_{T_1}^{t}F(s)\Delta s>\epsilon.$$

By (2.3), there exists $T_3>T_2$ such that for $t\geq T_3$,

$$\int_{T_1}^{t}\{p(s)h^{\alpha+1}(\sigma(s))-r(s)[h^{\Delta}(s)]^{\alpha+1}\}\Delta s\geq-\frac{\epsilon}{2}.$$

So by (2.5), when $t>T_3$, we have

$$\omega(t)\leq\omega(T_1)+\frac{\epsilon}{2}-\epsilon<0,$$

which implies that $x^{\Delta}(t)<0$ for all large $t>T_3$, which is again a contradiction to the assumption that $x^{\Delta}(t)$ changes sign infinitely often. This completes the proof of Lemma 2.1. □

3. Comparison theorems

We are now in a position to obtain some comparison results. Consider the second-order half-linear dynamic equations

$$(r(t)(x^{\Delta}(t))^{\alpha})^{\Delta}+p(t)x^{\alpha}(\sigma(t))=0, \tag{3.1}$$

and

$$(R(t)(x^{\Delta}(t))^{\alpha})^{\Delta}+a(t)P(t)x^{\alpha}(\sigma(t))=0, \tag{3.2}$$

where $r(t)>0$, $R(t)>0$, $p(t)$, $P(t)$ are continuous, $a(t)$ is continuously differentiable, and α is a quotient of odd positive integers.

The following two lemmas from [8] are very useful in establishing oscillation, nonoscillation, and comparison results for second-order linear and half-linear dynamic equations on time scales.

Lemma 3.1 (*Riccati Technique*). *Eq. (3.1) is nonoscillatory if and only if there exists $T\in[t_0,\infty)$ and a continuously differentiable function $\omega:[T,\infty)\to\mathbb{R}$ such that $r^{\frac{1}{\alpha}}(t)+\mu(t)\omega^{\frac{1}{\alpha}}(t)>0$ holds and*

$$\omega^{\Delta}(t)+p(t)+S[\omega,r](t)\leq 0,\quad \textit{for } t\in[T,\infty), \tag{3.3}$$

where

$$S[\omega, r](t) = \begin{cases} \left\{ \dfrac{\alpha \omega^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}} \right\}(t) & \text{at right-dense } t, \\ \left\{ \dfrac{\omega}{\mu} \left(1 - \dfrac{r}{[\mu \omega^{\frac{1}{\alpha}} + r^{\frac{1}{\alpha}}]^{\alpha}} \right) \right\}(t) & \text{at right-scattered } t. \end{cases}$$

If in Lemma 2.1, we let $h(t) \equiv 1$ then it is easy to obtain the expression for $S[\omega, r](t)$ from the expression for $F(t)$.

Lemma 3.2 (*Sturm–Picone Comparison Theorem*). *Consider the equation*

$$[\tilde{r}(t)(x^{\Delta}(t))^{\alpha}]^{\Delta} + \tilde{p}(t)x^{\alpha}(\sigma(t)) = 0, \tag{3.4}$$

where $\tilde{r}$ and $\tilde{p}$ satisfy the same assumptions as r and p. Suppose that $0 < \tilde{r}(t) \le r(t)$ and $p(t) \le \tilde{p}(t)$ on $[T, \infty)$ for all large T. Then (3.4) is nonoscillatory on $[t_0, \infty)$ implies that (3.1) is nonoscillatory on $[t_0, \infty)$.

The proofs of the following two theorems may be found in [8]:

Theorem A. *Assume $a \in C^1_{cd}$, $0 < r(t) \le R(t)$, $P(t) \le p(t)$ for $t \in [t_0, \infty)$ and*

(i) *the function $p(t)$ satisfies condition (A),*
(ii) $\int_{t_0}^{\infty} (r(t))^{-\frac{1}{\alpha}} \Delta t = \infty$,
(iii) $0 < a(t) \le 1$, $a^{\Delta}(t) \le 0$.

Then (3.1) is nonoscillatory on $[t_0, \infty)$ implies that (3.2) is nonoscillatory on $[t_0, \infty)$.

Theorem B. *Assume $a \in C^1_{cd}$, $0 < R(t) \le r(t)$, $p(t) \le P(t)$ for $t \in [t_0, \infty)$ and*

(i) *the function aP satisfies condition (A),*
(ii) $\int_{t_0}^{\infty} (R(t))^{-\frac{1}{\alpha}} \Delta t = \infty$,
(iii) $a(t) \ge 1$, $a^{\Delta}(t) \ge 0$, $t \in [t_0, \infty)$.

Then (3.1) is oscillatory on $[t_0, \infty)$ implies (3.2) is oscillatory on $[t_0, \infty)$.

Our goal in this section is to show that condition (A) (i.e., condition (i)) in Theorems A and B can be weakened to the assumptions that condition (B) and condition $(\hat{A})$ hold for the triple (α, p, r).

Theorem 3.3. *Assume $a \in C^1_{cd}$, $r(t) \le R(t)$, $P(t) \le p(t)$ and*

(i) *$p(t)$ satisfies condition (B), the triple (α, p, r) satisfies condition $(\hat{A})$,*
(ii) $\int_{t_0}^{\infty} (r(t))^{-\frac{1}{\alpha}} \Delta t = \infty$,
(iii) $0 < a(t) \le 1$, $a^{\Delta}(t) \le 0$.

Then (3.1) is nonoscillatory on $[t_0, \infty)$ implies (3.2) is nonoscillatory on $[t_0, \infty)$.

Proof. The assumptions of the theorem imply that there exists a solution x of (3.1) and $T \in \mathbb{T}$ such that $x(t) > 0$ and $x^{\Delta}(t) > 0$ on $[T, \infty)$ by Lemma 2.1. Therefore, the function $\omega(t) = r(t)(\frac{x^{\Delta}(t)}{x(t)})^{\alpha} > 0$ satisfies (3.3) with $r^{\frac{1}{\alpha}}(t) + \mu(t)\omega^{\frac{1}{\alpha}}(t) > 0$. We have $aS[\omega, r] = S[a\omega, ar]$ (see Lemma 3.1).

Now, multiplying (3.3) by $a(t)$, we get

$$\begin{aligned} 0 &\ge \omega^{\Delta} a + pa + S[a\omega, ar](t) \\ &\ge \omega^{\Delta} a + Pa + S[a\omega, ar](t) \\ &\ge \omega^{\Delta} a + \omega a^{\Delta} + Pa + S[a\omega, ar](t) \\ &= (\omega a)^{\Delta} + Pa + S[a\omega, ar](t) \end{aligned}$$

for $t \in [T, \infty)$. Hence the function $\varphi = \omega a$ satisfies the generalized Riccati inequality,

$$\varphi^{\Delta} + P(t)a(t) + S[\varphi, ar](t) \le 0$$

with $(ar)^{\frac{1}{\alpha}}(t) + \mu(t)\varphi^{\frac{1}{\alpha}}(t) > 0$, for $t \in [T, \infty)$. Therefore the equation

$$(a(t)r(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + a(t)P(t)x^{\alpha}(\sigma(t)) = 0,$$

is nonoscillatory by Lemma 3.1 and so Eq. (3.2) is nonoscillatory by Lemma 3.2 since $a(t)r(t) \le r(t) \le R(t)$. □

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

The corresponding "oscillation" result is

Theorem 3.4. *Assume* $a \in C^1_{cd}$, $R(t) \le r(t)$, $p(t) \le P(t)$ *and*

(i) aP *satisfies condition* (B), *the triple* (α, aP, r) *satisfies condition* $(\hat{A})$,
(ii) $\int_{t_0}^{\infty}(R(t))^{-\frac{1}{\alpha}}\Delta t = \infty$,
(iii) $a(t) \ge 1$, $a^{\Delta}(t) \ge 0$.

Then (3.1) *is oscillatory on* $[t_0, \infty)$ *implies* (3.2) *is oscillatory on* $[t_0, \infty)$.

Proof. The proof of Theorem 3.4 follows from Theorem 3.3. If we let $b = \frac{1}{a}$, then $b(t) \le 1$ and $b^{\Delta}(t) \le 0$. Therefore if (3.2) is nonoscillatory, then from Theorem 3.3, it follows that

$$(R(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + b(t)a(t)P(t)x^{\alpha}(\sigma(t)) = 0,$$

is also nonoscillatory. That is,

$$(R(t)(x^{\Delta}(t))^{\alpha})^{\Delta} + P(t)x^{\alpha}(\sigma(t)) = 0,$$

is nonoscillatory. But then since $P(t) \ge p(t)$ and $R(t) \le r(t)$, Lemma 3.2 (the Sturm–Picone comparison theorem) implies that Eq. (3.1) is also nonoscillatory. That is a contradiction and completes the proof. □

4. Examples

In this section, we will give several examples to illustrate Theorems 3.3 and 3.4. Since Example 4.1 is somewhat involved, we give the basic idea of its construction. We would also like to point out that in [12] the linear case ($\alpha = 1$) for the case $\mathbb{T} = \mathbb{R}$ as well as several other illustrative time scales was extensively investigated and a wide class of functions of the form $p(t) = \frac{a}{t^2} + \frac{b\sin t}{t}$ was determined which are such that the triple $(1, p, 1)$ satisfies condition $(\hat{A})$. This was shown to lead to a number of very useful comparison and oscillation results for the linear case. The following examples deal in an analogous way with the case $\alpha \ne 1$.

Example 4.1. Let $\alpha = 3$, and let $\mathbb{T}$ be the real interval. Let us consider a function $p(t)$ of the form $p(t) = \frac{a}{t^4} + \frac{b\sin t}{t^3}$, $a > 0$, $b > 0$. It is easy to observe that if $a > 3b$ then $p(t)$ satisfies condition (A). So we seek to find conditions on a and b such that $p(t)$ satisfies the conditions of Theorem 3.3 but does not satisfy condition (A). For simplicity, We consider the case $r \equiv 1$ so that (1.1) becomes

$$((x')^3)'(t) + p(t)x^3(t) = 0. \tag{4.1}$$

Let $h(t) = t^{\gamma}$, $\gamma < \frac{3}{4}$. Denote

$$\begin{aligned} I(T) &= \liminf_{t\to\infty} \int_T^t [p(t)h^4(t) - (h'(t))^4]\mathrm{d}t \\ &= T^{4\gamma-3}\left\{\frac{a-\gamma^4}{3-4\gamma} + bT^{3-4\gamma}\int_T^{\infty}\frac{\sin t}{t^{3-4\gamma}}\mathrm{d}t\right\}. \end{aligned} \tag{4.2}$$

The basic idea of constructing Example 4.1 is based on the following steps (i)–(iv).

(i) By Theorem 5.2, when $I(T) = +\infty$, (4.1) is oscillatory. Therefore, in order that (4.1) be nonoscillatory, we choose $\gamma < \frac{3}{4}$.

(ii) Since

$$\int_T^{\infty} p(t)\mathrm{d}t = T^{-3}\left[\frac{a}{3} + bT^3\int_T^{\infty}\frac{\sin t}{t^3}\mathrm{d}t\right],$$

it follows that $p(t)$ does not satisfy condition (A), if $\frac{a}{3} < b$.

(iii) Also we have

$$\limsup_{t\to\infty} t^3\int_t^{\infty} p(s)\mathrm{d}s = \frac{a}{3} + b,$$

$$\liminf_{t\to\infty} t^3\int_t^{\infty} p(s)\mathrm{d}s = \frac{a}{3} - b.$$

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

By Hille's Theorem [11], if

$$-\frac{2\alpha+1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha} < \liminf_{t\to\infty} t^3 \int_t^{\infty} p(s)\mathrm{d}s$$

$$\leq \limsup_{t\to\infty} t^3 \int_t^{\infty} p(s)\mathrm{d}s < \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha},$$

then Eq. (4.1) is nonoscillatory. Therefore, if we choose

$$-\frac{7}{4}\times\frac{3^3}{4^3} < \frac{a}{3} - b < \frac{a}{3} + b < \frac{3^3}{4^4},$$

then Eq. (4.1) is nonoscillatory. Note that $a > 0, b > 0, a + b < \frac{3^3}{4^4}$ implies $\frac{a}{3} - b > -\frac{7}{4} \times \frac{3^3}{4^3}$. Therefore, Hille's condition holds if we choose $a > 0$, $b > 0$ and $a + b < \frac{3^3}{4^4}$. That is, Eq. (4.1) is nonoscillatory.

(iv) From (4.2), we see that the triple (α, p, r) satisfies the condition $(\hat{A})$, if we take $\frac{a-\gamma^4}{3-4\gamma} > b$.

Therefore, from (i)–(iv), if we choose $0 < \frac{a}{3} < b$ with $\frac{a}{3} + b < \frac{3^3}{4^4}$, and $\gamma < \frac{3}{4}$ with $\frac{a-\gamma^4}{3-4\gamma} > b$, then it follows that the triple (α, p, r) satisfies condition $(\hat{A})$. In particular, if we take $a = \frac{1}{4}, b = \frac{1}{64}, \gamma = \frac{1}{16}$, it follows that $p(t) = \frac{1}{4t^4} + \frac{\sin t}{64t^3}$ is such that Eq. (4.1) is nonoscillatory.

Now if we set $a(t) := ct^{-d}(\log t)^{\beta}, c > 0, d > 0, \beta \in \mathbb{R}$, then we have $0 < a(t) \leq 1, a'(t) \leq 0$, for large t. So by Theorem 3.3, the equation

$$((x')^3)'(t) + \left(\frac{c(\log t)^{\beta}}{4t^{4+d}} + \frac{c(\log t)^{\beta}\sin t}{64t^{3+d}}\right)x^3(t) = 0$$

is nonoscillatory on $(2, \infty)$ for all $c > 0, d > 0, \beta \in \mathbb{R}$.

Example 4.2. Let $\alpha = 3, \mathbb{T} = [1, \infty)$. Let

$$P(t) = \frac{a}{t^{1+b+c}} + \frac{\sin t}{t^{b+c}}, \qquad a(t) = t^c,$$

where $a > 0, 0 < b < 3, c = \frac{3-b}{2}$.

We have

$$\int_T^{\infty} a(s)P(s)\mathrm{d}s = T^{-b}\left(\frac{a}{b} + T^b \int_T^{\infty} \frac{\sin t}{t^b}\mathrm{d}t\right).$$

With $h(t) = t^{\frac{b}{4}}, r(t) = 1$. Then we have

$$\int_T^t \{a(s)P(s)h^4(s) - r(s)[h'(s)]^4\}\mathrm{d}s$$

$$= a\ln s\,|_T^t + \int_T^t \sin s\mathrm{d}s + \left(\frac{b}{4}\right)^4 \times \frac{1}{b-3}s^{b-3}\,|_T^t \to \infty, \quad (t\to\infty).$$

So when $0 < \frac{a}{b} < 1, 0 < b < 3, a(t)P(t)$ does not satisfy condition (A), but the triple $(3, aP, 1)$ does satisfy condition $(\hat{A})$.

Take $h_1(t) = t^{\frac{b+c}{4}}, r(t) = 1$. Then we have

$$\int_T^t \{P(s)[h_1(s)]^4 - r(s)[h_1'(s)]^4\}\mathrm{d}s \to \infty, \quad (t\to\infty),$$

where $a > 0, 0 < b < 3, c = \frac{3-b}{2}$. By Theorem 5.2, $((x')^3)'(t) + P(t)x^3(t) = 0$ is oscillatory. Since $a(t) = t^c \geq 1$, for $t \geq 1$ and $a'(t) \geq 0$, it follows by Theorem 3.4 that $((x')^3)'(t) + a(t)P(t)x^3(t) = 0$ is oscillatory.

Example 4.3. Let $\mathbb{T} = \mathbb{Z}, \alpha = 3, p(t) = \frac{\gamma}{t(\sigma(t))^3} + \frac{\lambda(-1)^t}{(\sigma(t))^3}, r(t) = 1, \gamma > 0, \lambda > 0$. We have

$$\int_t^{\infty} p(s)\Delta s = \sum_{k=n}^{\infty}\left[\frac{\gamma}{k(k+1)^3} + \lambda\frac{(-1)^k}{(k+1)^3}\right].$$

Note that

$$\sum_{k=n}^{\infty}\left[\frac{\gamma}{k(k+1)^3}\right] \sim \frac{\gamma}{3n^3}, \quad \text{for large } n.$$

Also, we have

$$(2m)^3\sum_{k=2m}^{\infty}\frac{(-1)^k}{(k+1)^3} = (2m)^3\left\{\frac{1}{(2m+1)^3}-\left[\frac{\left(1+\frac{1}{2m+2}\right)^3-1}{(2m+3)^3}+\frac{\left(1+\frac{1}{2m+4}\right)^3-1}{(2m+5)^3}+\cdots\right]\right\}$$

$$= (2m)^3\left\{\frac{1}{(2m+1)^3}-\left[\frac{\frac{3}{2m+2}+\mathrm{o}\left(\frac{1}{2m+2}\right)}{(2m+3)^3}+\frac{\frac{3}{2m+4}+\mathrm{o}\left(\frac{1}{2m+4}\right)}{(2m+5)^3}+\cdots\right]\right\}$$

$$\to 1-\frac{1}{2}\int_0^{\infty}\frac{1}{(1+x)^2}\mathrm{d}x=\frac{1}{2},\quad (n\to\infty).$$

Similarly, we have

$$(2m+1)^3\sum_{k=2m+1}^{\infty}\frac{(-1)^k}{(k+1)^3}\to -1+\frac{1}{2}\int_0^{\infty}\frac{1}{(1+x)^2}\mathrm{d}x=-\frac{1}{2}.$$

So, in this case, if $\frac{\gamma}{3}<\frac{\lambda}{2}$, $p(t)$ does not satisfy condition (A).

Furthermore,

$$\limsup_{t\to\infty} t^3\int_t^{\infty}p(s)\Delta s=\frac{\gamma}{3}+\frac{\lambda}{2},$$

$$\liminf_{t\to\infty} t^3\int_t^{\infty}p(s)\Delta s=\frac{\gamma}{3}-\frac{\lambda}{2}.$$

By Hille's Theorem [11], if we choose $\frac{\gamma}{3}+\frac{\lambda}{2}<\frac{3^3}{4^4}$, then equation

$$((x^{\Delta})^3)^{\Delta}(t)+p(t)x^3(\sigma(t))=0$$

is nonoscillatory.

Let $h(t)=t^{\beta}$, $\beta<\frac{3}{4}$. Denote

$$I(n) = \liminf_{t\to\infty}\int_n^t\left(p(s)h^4(\sigma(s))-(h^{\Delta}(s))^4\right)\Delta s$$

$$= \sum_{k=n}^{\infty}\left[\frac{\gamma}{k(k+1)^{3-4\beta}}+\frac{\lambda(-1)^k}{k^{3-4\beta}}-[(k+1)^{\beta}-k^{\beta}]^4\right].$$

Note that

$$\sum_{k=n}^{\infty}\left[\frac{1}{k(k+1)^{3-4\beta}}\right]\sim\frac{1}{(3-4\beta)n^{3-4\beta}}.$$

Since $(k+1)^{\beta}-k^{\beta}\sim\frac{\beta}{k^{1-\beta}}$, for large k, we have

$$[(k+1)^{\beta}-k^{\beta}]^4=\frac{\beta^4}{k^{4-4\beta}}+\frac{\beta^4}{k^{4-4\beta}}\,\mathrm{o}(1).$$

So

$$\lim_{n\to\infty} n^{3-4\beta}\sum_{k=n}^{\infty}[(k+1)^{\beta}-k^{\beta}]^4 = \lim_{n\to\infty} n^{3-4\beta}\sum_{k=n}^{\infty}\left[\frac{\beta^4}{k^{(4-4\beta)}}\right]$$

$$=\frac{\beta^4}{3-4\beta}.$$

Therefore

$$\sum_{k=n}^{\infty}[(k+1)^{\beta}-k^{\beta}]^4\sim\frac{\beta^4}{(3-4\beta)n^{3-4\beta}}.$$

So the triple $(3, p, 1)$ will satisfy condition $(\hat{A})$ if we take $\frac{\gamma-\beta^4}{3-4\beta}>\frac{\lambda}{2}$.

Therefore, choosing $0<\beta<\frac{3}{4}$, $0<\frac{\gamma}{3}<\frac{\lambda}{2}$ with $\frac{\gamma}{3}+\frac{\lambda}{2}<\frac{3^3}{4^4}$, $\frac{\gamma-\beta^4}{3-4\beta}>\frac{\lambda}{2}$, then $p(t)=\frac{\gamma}{t(t+1)^3}+\frac{\lambda(-1)^t}{(t+1)^3}$ satisfies all the requirements. In particular, if we take $\gamma=\frac{1}{3}$, $\lambda=\frac{3}{4}$, $\beta=\frac{7}{10}$. it follows that $p(t)=\frac{1}{3t(t+1)^3}+\frac{3(-1)^t}{4(t+1)^3}$ is such that Eq. (3.1) is nonoscillatory.

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

5. Oscillation theorem

In this section, by means of Lemma 2.1, we obtain an oscillation theorem which extends some earlier results. The following theorem may be found in [9], Theorem 5.81. (See also [10,4].)

Theorem 5.1. *The equation $(r(t)x'(t))' + p(t)x = 0$ is oscillatory on the interval $[t_0, \infty)$, if $\int_{t_0}^{\infty} r^{-1}(t)\mathrm{d}t = \infty$ and there exists a continuously differentiable function $u(t) > 0$ such that*

$$\int_{t_0}^{\infty} [p(t)u^2(t) - r(t)(u'(t))^2]\mathrm{d}t = +\infty.$$

Analogous to the above theorem, we may obtain a corresponding version for half-linear dynamic equations on time scales which we state as follows:

Theorem 5.2. *Assume that $p(t)$ satisfies condition (B) and assume $\int_{t_0}^{\infty}(r(t))^{-\frac{1}{\alpha}}\Delta t = \infty$. If there exists a continuously differentiable function $h : \mathbb{T} \to \mathbb{R}$, such that either $h^{\Delta}(t)$ is of one sign for all $t \in \mathbb{T}$ or $h^{\Delta}(t) \equiv 0$, and is such that*

$$\int_{t_0}^{\infty} [p(t)h^{\alpha+1}(\sigma(t)) - r(t)(h^{\Delta}(t))^{\alpha+1}]\Delta t = +\infty, \tag{5.1}$$

then all solutions of (1.1) are oscillatory.

Proof. Let us suppose that (1.1) is nonoscillatory and x is a solution of (1.1). To be specific, suppose that $x(t) > 0$ for all large t, since the other case is similar.

By (5.1), we obtain that the triple (α, p, r) satisfies condition $(\hat{A})$. In view of Lemma 2.1, we may then suppose also that $x^{\Delta}(t) > 0$ for $t \geq T$. Make the substitution $\omega(t) = r(t)\left[\frac{x^{\Delta}(t)}{x(t)}\right]^{\alpha}$, for $t \geq T$. By the proof of Lemma 2.1, we have

$$\omega^{\Delta}(t) = -p(t)h^{\alpha+1}(\sigma(t)) + r(t)[h^{\Delta}(t)]^{\alpha+1} - F(t)$$

where $F(t) \geq 0$. So

$$\omega^{\Delta}(t) \leq -p(t)h^{\alpha+1}(\sigma(t)) + r(t)[h^{\Delta}(t)]^{\alpha+1}.$$

Integrating from T to t gives

$$\int_{T}^{t} \{p(t)h^{\alpha+1} - r(t)[h^{\Delta}(t)]^{\alpha+1}\}\Delta t \leq (\omega h^{\alpha+1})(T) - (\omega h^{\alpha+1})(t) \leq (\omega h^{\alpha+1})(T).$$

But now the left-hand side is unbounded and the right-hand side is bounded. this contradiction proves the theorem. □

For $\mathbb{T} = \mathbb{R}$, we proved in Section 1 that $p(t) = 1 + t\sin t$ satisfies condition (B) and the triple $(3, p(t), 1)$ satisfies condition $(\hat{A})$. So by Theorem 5.2 all solutions of

$$[(x')^3]'(t) + (1 + t\sin t)x^3(t) = 0,$$

are oscillatory.

Let $q > 1$. Consider the time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$. Let

$$p(t) = \frac{\lambda}{t\sigma(t)^b} + \frac{\beta(-1)^n}{t\sigma(t)^b}$$

where $\lambda > 0$, $0 < b < 3$. Let $\alpha = 3$. Consider the q-difference equation

$$((x^{\Delta}(t))^3)^{\Delta} + p(t)x^3(\sigma(t)) = 0. \tag{5.2}$$

In Section 1, we have proved that $p(t)$ satisfies condition (B) and for $h(t) = t^{\frac{b}{4}}$, $r(t) = 1$, $t = q^n$

$$\int_{1}^{t} \{p(s)h^4(\sigma(s)) - r(s)[h^{\Delta}(s)]^4\}\Delta s \to \infty.$$

So by Theorem 5.1, all solutions of (5.2) are oscillatory.

Example 5.1. Let $\mathbb{T} = \mathbb{R}$, $\alpha = 3$, $r(t) = 1$ and $p(t) = \frac{a}{t^{1+b}} + \frac{c\sin t}{t^b}$, where $0 < b < 3$, $a > 0$, $c \in \mathbb{R}$. It is easy to see that $p(t)$ satisfies condition (B). Take $h(t) = t^{\frac{b}{4}}$. We have

$$\int_{T}^{\infty} \{p(s)[h(s)]^4 - r(s)[h'(s)]^4\}\mathrm{d}s = \infty.$$

Please cite this article in press as: B. Jia, et al., New comparison and oscillation theorems for second-order half-linear dynamic equations on time scales, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2008.05.014

So by Theorem 5.2, all solutions of the second-order half-linear differential equations

$$((x')^3)'(t) + \left(\frac{a}{t^{1+b}} + \frac{c\sin t}{t^b}\right) x^3(t) = 0,$$

are oscillatory for all $0 < b < 3$, $a > 0$, $c \in \mathbb{R}$.

Note that

$$\int_t^\infty p(s)\mathrm{d}s = t^{-b}\left[\frac{a}{b} + ct^b \int_t^\infty \frac{\sin s}{s^b}\mathrm{d}s\right].$$

So for $0 < b < 3$, $a > 0$,

$$\liminf_{t\to\infty} t^3 \int_t^\infty p(s)\mathrm{d}s = \begin{cases} +\infty & \text{if } \frac{a}{b} > |c| \\ -\infty & \text{if } \frac{a}{b} < |c|. \end{cases}$$

By Hille's Theorem [4], if

$$\liminf_{t\to\infty} t^3 \int_t^\infty p(s)\mathrm{d}s > \frac{1}{4} \times \left(\frac{3}{4}\right)^3, \quad \text{that is} : \frac{a}{b} > |c|,$$

then Eq. (4.1) is oscillatory.

Therefore the oscillation conditions of Eq. (5.3) that we get improve the oscillation conditions of Hille's theorem.

Example 5.2. Consider the generalized Euler–Cauchy dynamic equation

$$((x^\Delta)^\alpha)^\Delta + \frac{\beta}{(\sigma(t))^{\alpha+1}} x^\alpha(\sigma(t)) = 0, \tag{5.3}$$

for $t \in \mathbb{T}$. Take $h(t) = t^{\frac{\alpha}{\alpha+1}}$. Then

$$\int_T^\infty \{h^{\alpha+1}(\sigma(t))p(t) - [h^\Delta(t)]^{\alpha+1} r(t)\}\Delta t \tag{5.4}$$

$$= \int_T^\infty \left\{\frac{\beta}{\sigma(t)} - \left[\left(t^{\frac{\alpha}{\alpha+1}}\right)^\Delta\right]^{\alpha+1}\right\}\Delta t. \tag{5.5}$$

If $\mathbb{T} = \mathbb{R}$, then the dynamic equation (5.3) is the half-linear Euler–Cauchy differential equation $((x^\Delta)^\alpha)^\Delta + \frac{\beta}{t^{\alpha+1}} x^\alpha(t) = 0$ and in this case $(t^{\frac{\alpha}{\alpha+1}})^\Delta = \frac{\alpha}{\alpha+1} t^{-\frac{1}{\alpha+1}}$. Therefore (5.5) can be rewritten as

$$\int_T^\infty \left\{\frac{\beta}{\sigma(t)} - \left[\left(t^{\frac{\alpha}{\alpha+1}}\right)^\Delta\right]^{\alpha+1}\right\}\Delta t = \int_T^\infty \frac{1}{t}\left[\beta - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\right]\Delta t = \infty$$

provided that $\beta > (\frac{\alpha}{\alpha+1})^{\alpha+1}$. Hence every solution of (5.3) oscillates if $\beta > (\frac{\alpha}{\alpha+1})^{\alpha+1}$, which agrees with the well-known oscillatory behavior of (5.3).

If $\mathbb{T} = \mathbb{Z}$, then (5.3) is the half-linear Euler–Cauchy difference equation

$$((x^\Delta)^\alpha)^\Delta + \frac{\beta}{(t+1)^{\alpha+1}} x(t+1) = 0,$$

and we have $\left(t^{\frac{\alpha}{\alpha+1}}\right)^\Delta = (t+1)^{\frac{\alpha}{\alpha+1}} - t^{\frac{\alpha}{\alpha+1}}$. Therefore (5.5) can be rewritten as

$$\int_T^\infty \left\{\frac{\beta}{\sigma(t)} - \left[(t^{\frac{\alpha}{\alpha+1}})^\Delta\right]^{\alpha+1}\right\}\Delta t = \int_T^\infty \left\{\frac{1}{t+1}\left[\beta - (t+1)[(t+1)^{\frac{\alpha}{\alpha+1}} - t^{\frac{\alpha}{\alpha+1}}]^{\alpha+1}\right]\right\}\Delta t.$$

Note that

$$\lim_{t\to\infty}(t+1)\left[(t+1)^{\frac{\alpha}{\alpha+1}} - t^{\frac{\alpha}{\alpha+1}}\right]^{\alpha+1} = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

So

$$\int_T^\infty \left\{\frac{1}{t+1}\left[\beta - (t+1)[(t+1)^{\frac{\alpha}{\alpha+1}} - t^{\frac{\alpha}{\alpha+1}}]^{\alpha+1}\right]\right\}\Delta t = \infty$$

provided that $\beta > (\frac{\alpha}{\alpha+1})^{\alpha+1}$. Hence every solution of (5.3) oscillates if $\beta > (\frac{\alpha}{\alpha+1})^{\alpha+1}$, which agrees with the well-known oscillatory behavior of (5.3).

If $\mathbb{T} = q_0^{\mathbb{N}} = \{1, q, q^2, \ldots\}$, $q > 1$. Then the dynamic equation (5.3) is the q-difference equation

$$((x^{\Delta})^{\alpha})^{\Delta} + \frac{\beta}{(qt)^{\alpha+1}} x(qt) = 0,$$

and in this case, (5.5) can be rewritten as

$$\int_T^{\infty} \left\{ \frac{\beta}{\sigma(t)} - \left[\left(t^{\frac{\alpha}{\alpha+1}} \right)^{\Delta} \right]^{\alpha+1} \right\} \Delta t = \int_T^{\infty} \frac{1}{qt} \left\{ \beta - q \left[\frac{q^{\frac{\alpha}{\alpha+1}} - 1}{q-1} \right]^{\alpha+1} \right\} \Delta t = \infty$$

provided that $\beta > q \left[\frac{q^{\frac{\alpha}{\alpha+1}} - 1}{q-1} \right]^{\alpha+1}$. Hence every solution of (5.3) oscillates if $\beta > q \left[\frac{q^{\frac{\alpha}{\alpha+1}} - 1}{q-1} \right]^{\alpha+1}$.

Note that $q \left[\frac{q^{\frac{\alpha}{\alpha+1}} - 1}{q-1} \right]^{\alpha+1}$ is different from $(\frac{\alpha}{\alpha+1})^{\alpha+1}$ which is the well-known critical constant from the continuous and the discrete cases.

The interested reader may give additional examples. We remark that the results in the example above may not be obtained by any existing criteria, as far as the authors are aware.

References

[1] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
[3] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[4] W.A. Coppel, Disconjugacy, in: Lecture Notes in Mathematics, vol. 220, Springer-Verlag, 1971.
[5] L. Erbe, Oscillation criteria for second order linear equations on a time scale, Canad. Appl. Math. Q. 9 (2001) 1–31.
[6] Lynn Erbe, Oscillation theorems for second order linear differential equation, Pacific J. Math. 35 (2) (1970) 337–343.
[7] Lynn Erbe, Oscillation theorems for second order nonlinear differential equations, Proc. Amer. Math. Soc. 24 (1970) 811–814.
[8] P. Řehák, Half-linear dynamic equations on time scales, Habilitation Thesis, 2005.
[9] Miloš Ráb, Kriterien für die Oszillation der Lösungen der Differentialgleichung $[p(x)y']' + q(x)y = 0$, Časopis Pěst. 84 (1959) 335–370; 85 (1960) 91.
[10] Aurel Wintner, A criterion of oscillatory stability, Quart. Appl. Math. 7 (1949) 115–117.
[11] O. Došlý, P. Řehák, Half-linear Differential Equation, in: North-Holland Mathematics Studies, vol. 202, Elsevier, Amsterdam, 2005.
[12] J. Baoguo, L. Erbe, A. Peterson, Some new comparison results for second order linear dynamic equations (submitted for publication). Q1