AN OSCILLATION RESULT FOR A NONLINEAR DYNAMIC EQUATION ON A TIME SCALE

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ABSTRACT. We obtain an oscillation criterion for solutions to the nonlinear dynamic equation

$$x^{\Delta\Delta} + q(t)x^{\Delta^\sigma} + p(t)(f \circ x^\sigma) = 0,$$

on time scales. In particular, no explicit sign assumptions are made with respect to the coefficient $p(t)$ which is different from most results for such equations. We illustrate the results by several examples, including a nonlinear Emden–Fowler dynamic equation.

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1. Introduction

Consider the second order nonlinear dynamic equation

$$x^{\Delta\Delta} + q(t)x^{\Delta^\sigma} + p(t)(f \circ x^\sigma) = 0, \tag{1.1}$$

where p and q are real–valued, right–dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$. We assume throughout that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$f'(x) > 0, \quad xf(x) > 0 \quad \text{for} \quad x \neq 0. \tag{1.2}$$

Although we assume q is a nonnegative function we do not make any explicit sign assumptions on p in contrast to most results on nonlinear oscillations.

For completeness, we recall the following concepts related to the notion of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and, since boundedness and oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The *forward* and *backward jump operators* are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be *left–dense* if $\rho(t) = t$, *right–dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left–scattered* if $\rho(t) < t$ and *right–scattered* if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous* (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

One of the main interests will be the case when f is of superlinear growth, (at least for large x), say

$$f(x) = x^{2n+1}, \quad n \geq 1. \tag{1.3}$$

In several papers ([4], [14]), (1.1) has been studied with $p > 0$ and assuming the nonlinearity has the property

$$\left| \frac{f(x)}{x} \right| \geq K \quad \text{for} \quad x \neq 0. \tag{1.4}$$

This essentially says that the equation is, in some sense, not too far from being linear. We shall see below that the oscillation of solutions of the nonlinear equation (1.1) is closely related to oscillation of a related linear equation

$$(r(t)x^\Delta)^\Delta + \lambda p_1(t)x^\sigma = 0, \tag{1.5}$$

where $\lambda > 0$, for which many oscillation criteria are known (see e.g. [1]–[5], [8], [10], and [13]). In particular, we will obtain the time scale analogues of the results due to Erbe [6], [7] for the continuous case $\mathbb{T} = \mathbb{R}$ and will extend some recent results of [12]. We shall restrict attention to solutions of (1.1) which exist on some interval of the form $[T_x, \infty)$. where $T_x \in \mathbb{T}$ may depend on the particular solution. This paper is organized as follows. In Section 2 we present some preliminary results on the chain rule, integration by parts, and an auxiliary lemma. Section 3 contains the main results on oscillation and several examples are given in Section 4.

2. **Preliminary Results**

On an arbitrary time scale $\mathbb{T}$, the usual chain rule from calculus is no longer valid (see Bohner and Peterson [3], pp 31). One form of the extended chain rule, due to S. Keller [15] and generalized to measure chains by C. Pötzsche [16], is as follows. (See also Bohner and Peterson [3], pp 32.)

Lemma 2.1. *Assume $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable* *and satisfies*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t). \tag{2.1}$$

We shall also need the following integration by parts formula (cf. [3]), which is a simple consequence of the product rule and which we formulate as follows:

Lemma 2.2. *Let $a, b \in \mathbb{T}$ and assume $f^{\Delta}, g^{\Delta} \in C_{rd}$. Then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t)g(t)\Delta t. \tag{2.2}$$

Before stating the next result, we recall that a solution of equation (1.1) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory.

We say that a function $g : \mathbb{T} \to \mathbb{R}$ satisfies condition (A) if the following condition holds:

$$\liminf_{t\to\infty} \int_T^t g(s)\Delta s \geq 0 \quad \text{and} \quad \not\equiv 0 \tag{2.3}$$

for all large T. It can be shown that (2.3) implies either $\int_a^{\infty} g(s)\Delta s = +\infty$ or that

$$\int_T^{\infty} g(s)\Delta s = \lim_{t\to\infty} \int_T^t g(s)\Delta s$$

exists and satisfies $\int_T^{\infty} g(s)\Delta s \geq 0$ for all large T.

We state the following lemma which gives another simple consequence of condition (A).

Lemma 2.3. *Suppose that g satisfies condition (A). Then there exists $T_1 \geq T_0$ so that*

$$\int_{T_1}^t g(s)\Delta s \geq 0 \quad \textit{for all} \quad t \geq T_1. \tag{2.4}$$

Proof. For any $T \geq T_0$ define

$$H(T) := \liminf_{t\to\infty} \int_T^t g(s)\Delta s.$$

Then $H(T) \geq 0$ and is $\not\equiv 0$ for all large T. Then we may suppose that $H(T) > 0$ for some $T > T_0$ fixed. If there does not exist $T_1 > T$ as claimed in the lemma, then we define

$$T_1 = T_1(T) := \sup\{t > T : \int_T^t g(s)\Delta s < 0\}.$$

If $T_1 = \infty$, then choosing $t_n \to \infty$ such that $\int_T^{t_n} g(s)\Delta s < 0$ for all n, we have that

$$H(T) \leq \liminf_{n\to\infty} \int_T^{t_n} g(s)\Delta s \leq 0,$$

which is a contradiction to $H(T) > 0$. Hence, we must have $T_1 < \infty$ which implies $\int_{T_1}^t g(s)\Delta s \geq 0$ for all $t \geq T_1$.

□

3. **Main Results**

The first result is an oscillation result for (1.1) based on a linear comparison technique. In the statement of the theorem we let the function $r(t)$ (see [3]) be given in terms of the generalized exponential function by $r(t) := e_q(t, t_0)$. (Since $q(t) \geq 0$, it follows (see [3]) that $r(t) > 0$ for all $t \geq t_0$ and we recall that $r(t)$ may be characterized as the unique solution of the IVP $r^\Delta = q(t)r$, $r(t_0) = 1$.)

Theorem 3.1. *Let $\lambda > 0$, $K > 0$, and assume that f satisfies*

$$f'(x) \geq \frac{f(x)}{x} \geq \lambda \quad \text{for} \quad |x| \geq K. \tag{3.1}$$

Suppose also that Equation (1.5) *is oscillatory, that the function $p_1(t) := r(t)p(t)$ satisfies condition (A), and that the following hold:*

$$\int_{t_0}^\infty \frac{\Delta s}{r(s)} = \infty \tag{3.2}$$

and

$$\limsup_{t\to\infty} \left(\int_{t_0}^t \sigma(s)p_1(s)\Delta s - Mr(t) \right) = \infty \tag{3.3}$$

for all $M > 0$. Then all solutions of (1.1) *are oscillatory.*

Proof. If the result is not true, then we may suppose without loss of generality that $x(t)$ is a nonoscillatory solution of (1.1) with $x(t) > 0$ for all $t \geq T$ where $T, t \in \mathbb{T}$. A similar argument will be valid if $x(t) < 0$ for all large t. After multiplying by $r(t)$, (1.1) becomes

$$(r(t)x^\Delta)^\Delta + p_1(t)(f \circ x^\sigma) = 0, \tag{3.4}$$

where $p_1(t) = r(t)p(t)$, $t \in [t_0, \infty)$. Dividing equation (3.4) by $(f \circ x^\sigma)(t)$ and integrating by parts (Lemma 2.2) for $t \geq T$ gives

$$\begin{aligned} &\frac{r(t)x^\Delta(t)}{f(x(t))} - \int_T^t r(s)x^\Delta(s) \left(\frac{1}{f(x(s))} \right)^\Delta \Delta s \\ &+ \int_T^t p_1(s)\Delta s = \frac{r(T)x^\Delta(T)}{f(x(T))} \end{aligned} \tag{3.5}$$

for $t \geq T$. We note from the Chain Rule (Lemma 2.1) and quotient rule that

$$\begin{aligned}\int_T^t r(s)x^\Delta(s)\left(\frac{1}{f(x(s))}\right)^\Delta \Delta s &= -\int_T^t r(s)x^\Delta(s)\frac{(f(x(s)))^\Delta}{f(x(s))f(x(\sigma(s)))}\Delta s \\ &= -\int_T^t r(s)x^\Delta(s)\left\{\int_0^1 f'(x(s)+h\mu(s)x^\Delta(s))dh\right\}\frac{x^\Delta(s)}{f(x(s))f(x(\sigma(s)))}\Delta s \\ &\leq 0,\end{aligned} \tag{3.6}$$

since $f'(x) > 0$. Consequently, from equation (3.5) and (3.6) we have

$$\frac{r(t)x^\Delta(t)}{(f\circ x)(t)} + \int_T^t p_1(s)\Delta s \leq \frac{r(T)x^\Delta(T)}{(f\circ x)(T)} \tag{3.7}$$

for $t \in [T, \infty)$.

Suppose first that $x^\Delta(T_1) \leq 0$ for some large $T_1 \geq T$. Then since $p_1(t)$ satisfies condition (A), it follows by taking the lim inf as $t \to \infty$ of the left side of (3.7) (with T replaced by T_1) that $x^\Delta(t) < 0$ for all large t, say $t \geq T_2 \geq T_1$. We may assume by Lemma 2.3 that

$$\int_{T_2}^t p_1(s)\Delta s \geq 0 \tag{3.8}$$

for all $t \geq T_2$.

We next integrate by parts (Lemma 2.2) the second term in equation (3.4) to get

$$\begin{aligned}&\int_{T_2}^t p_1(s)f(x(\sigma(s)))\Delta s \\ &= f(x(t))\int_{T_2}^t p_1(s)\Delta s - \int_{T_2}^t (f(x(s)))^\Delta \int_{T_2}^s p_1(\eta)\Delta\eta\Delta s.\end{aligned} \tag{3.9}$$

By the Chain Rule (Lemma 2.1) we have (since (1.2) holds and $x^\Delta(t) \leq 0$)

$$(f\circ x)^\Delta(t) = \left\{\int_0^1 f'(x(t)+h\mu(t)x^\Delta(t))dh\right\}x^\Delta(t) \leq 0, \tag{3.10}$$

for all $t \geq T_2$. Consequently, it follows that

$$\int_{T_2}^t p_1(s)f(x(\sigma(s)))\Delta s \geq 0$$

for all $t \geq T_2$, and so from equation (3.4) we obtain

$$r(t)x^\Delta(t) \leq r(T_2)x^\Delta(T_2) < 0,$$

for all $t \geq T_2$. Therefore,

$$x(t) \leq x(T_2) + r(T_2)x^\Delta(T_2)\int_{T_2}^t \frac{\Delta s}{r(s)} \to -\infty, \tag{3.11}$$

which is a contradiction since $x(t) > 0$ for all large t.

It follows that we must have $x^\Delta(t) > 0$ for all large t, say $t \geq T$. Suppose next that $x(t) \leq K$ for all large t. Multiplying the first term in (3.4) by $\frac{\sigma(t)}{f(x(\sigma(t)))}$ and integrating gives

$$\begin{aligned}
&\int_T^t \frac{\sigma(s)\left(r(s)x^\Delta(s)\right)^\Delta}{f(x(\sigma(s)))}\Delta s \\
&= \frac{tr(t)x^\Delta(t)}{f(x(t))} - \frac{Tr(T)x^\Delta(T)}{f(x(T))} - \int_T^t r(s)x^\Delta(s)\left(\frac{s}{f(x(s))}\right)^\Delta \Delta s \\
&= \frac{tr(t)x^\Delta(t)}{f(x(t))} - \frac{Tr(T)x^\Delta(T)}{f(x(T))} \\
&- \int_T^t r(s)x^\Delta(s)\left(\frac{f(x(s)) - s(f(x(s)))^\Delta}{f(x(s))f(x(\sigma(s)))}\right)\Delta s \\
&= \frac{tr(t)x^\Delta(t)}{f(x(t))} - \frac{Tr(T)x^\Delta(T)}{f(x(T))} \\
&- \int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s + \int_T^t \frac{sr(s)x^\Delta(s)}{f(x(s))f(x(\sigma(s)))}(f(x(s)))^\Delta \Delta s.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_T^t \frac{sr(s)x^\Delta(s)(f(x(s)))^\Delta}{f(x(s))f(x(\sigma(s)))}\Delta s \\
&= \int_T^t \frac{sr(s)x^\Delta(s)}{f(x(s))f(x(\sigma(s)))}\left(\int_0^1 f'(x_h(s))dh\,x^\Delta(s)\right)\Delta s \\
&> 0,
\end{aligned}$$

since $f'(x) > 0$, where

$$x_h(s) := x(s) + h\mu(s)x^\Delta(s).$$

Therefore we get

$$\begin{aligned}
&\int_T^t \frac{\sigma(s)(r(s)x^\Delta(s))^\Delta}{f(x(\sigma(s)))}\Delta s \\
&\geq \frac{tr(t)x^\Delta(t)}{f(x(t))} - \frac{Tr(T)x^\Delta(T)}{f(x(T))} - \int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}.
\end{aligned} \tag{3.12}$$

Therefore, from equation (3.4) we have after a multiplication by $\frac{\sigma(t)}{f(x(\sigma(t)))}$ and integration

$$\int_T^t \frac{\sigma(s)(r(s)x^\Delta(s))^\Delta}{f(x(\sigma(s)))}\Delta s + \int_T^t \sigma(s)p_1(s)\Delta s = 0 \tag{3.13}$$

and so by (3.12) we have after rearranging

$$\begin{aligned}
&\frac{tr(t)x^\Delta(t)}{f(x(t))} + \int_T^t \sigma(s)p_1(s)\Delta s \\
&\leq \frac{Tr(T)x^\Delta(T)}{f(x(T))} + \int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s.
\end{aligned} \tag{3.14}$$

Now since r is nondecreasing and f is increasing, we have

$$\begin{aligned}\int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s &\leq \frac{r(t)}{f(x(T))}\int_T^t x^\Delta(s)\Delta s\\ &\leq r(t)\frac{(K-x(T))}{f(x(T))}\\ &\leq r(t)M,\end{aligned}$$

where

$$M := \frac{K-x(T)}{f(x(T))} \geq 0.$$

Therefore, from (3.14) we get

$$\frac{tr(t)x^\Delta(t)}{f(x(t))} + \int_T^t \sigma(s)p_1(s)\Delta s - Mr(t) \leq \frac{Tr(T)x^\Delta(T)}{f(x(T))}. \tag{3.15}$$

From (3.3), for any $M > 0$,

$$\limsup_{t\to\infty}\left\{\int_T^t \sigma(s)p_1(s)\Delta s - Mr(t)\right\} = +\infty$$

so it follows that $\frac{tr(t)x^\Delta(t)}{f(x(t))} < 0$ for some large t, which is a contradiction, since we showed above that $x^\Delta(t) > 0$ must hold for all large t. Hence, we conclude that $x(t) > K$ for all sufficiently large t and so without loss of generality $x(t) > K$ for $t \geq T$ and so by our assumption (3.1)

$$\frac{f(x(t))}{x(t)} \geq \lambda > 0, \quad t \geq T. \tag{3.16}$$

Next let z be the solution of (1.5) with $z(T) = 0$ and $r(T)z^\Delta(T) = 1$. Since (1.5) is oscillatory, there exists $T_1 > T$ such that

$$z^\Delta(t) > 0 \quad \text{on} \quad [T, T_1) \quad \text{and} \quad z^\Delta(T_1) \leq 0. \tag{3.17}$$

If we denote $g(x) := \frac{f(x)}{x}$, then $g(x(t)) \geq \lambda$ for all $t \geq T$ so we have

$$\begin{aligned}0 &\leq \int_T^{T_1} r(t)(g(x(t))-\lambda)(z^\Delta(t))^2\Delta t\\ &= \left[r(t)z(t)z^\Delta(t)(g(x(t))-\lambda)\right]_T^{T_1} - \int_T^{T_1} z^\sigma(t)[r(t)z^\Delta(t)(g(x(t))-\lambda)]^\Delta\Delta t\\ &= r(T_1)z(T_1)z^\Delta(T_1)(g(x(T_1))-\lambda) - \int_T^{T_1} z^\sigma(t)(r(t)z^\Delta(t))^\Delta(g(x^\sigma(t))-\lambda)\Delta t\\ &- \int_T^{T_1} r(t)z^\sigma(t)z^\Delta(t)(g(x(t)))^\Delta\Delta t\\ &\leq \lambda\int_T^{T_1} p_1(t)(z^\sigma(t))^2(g(x^\sigma(t))-\lambda)\Delta t - \int_T^{T_1} r(t)z^\sigma(t)z^\Delta(t)(g(x(t)))^\Delta\Delta t.\end{aligned}$$

(3.18)

We note that

$$\begin{aligned}(g(x(t)))^{\Delta} &= \frac{x(t)(f(x(t)))^{\Delta} - f(x(t))x^{\Delta}(t)}{x(t)x^{\sigma}(t)} \\ &= \frac{x(t)\left(\int_0^1 f'(x_h(t))dh\right)x^{\Delta}(t) - f(x(t))x^{\Delta}(t)}{x(t)x^{\sigma}(t)}. \qquad (3.19)\end{aligned}$$

Now since $x^{\Delta}(t) > 0$ it follows that

$$x_h(t) \geq x(t) > 0, \quad \text{for } \ 0 \leq h \leq 1$$

and so by (1.2) we have

$$f'(x_h(t)) \geq \frac{f(x_h(t))}{x_h(t)} \geq \frac{f(x(t))}{x(t)}.$$

Thus,

$$(g(x(t)))^{\Delta} \geq \frac{x(t)\left(\int_0^1 \frac{f(x(t))}{x(t)}dh\right)x^{\Delta}(t) - f(x(t))x^{\Delta}(t)}{x(t)x^{\sigma}(t)} = 0.$$

Using this in (3.18) we get (since $\lambda > 0$, and $z^{\sigma}(t) > 0$)

$$\begin{aligned}0 &\leq \int_T^{T_1} p_1(t)(z^{\sigma}(t))^2(g(x^{\sigma}(t)) - \lambda)\Delta t \qquad (3.20)\\ &= \int_T^{T_1} \left\{\frac{z^{\sigma}(t)}{x^{\sigma}(t)}\left[p_1(t)z^{\sigma}(t)f(x^{\sigma}(t)) - \lambda p_1(t)z^{\sigma}(t)x^{\sigma}(t)\right]\right\}\Delta t.\end{aligned}$$

Notice that

$$\begin{aligned}&\left(r(t)z^{\Delta}(t)x(t) - r(t)x^{\Delta}(t)z(t)\right)^{\Delta} \\ = \;& (r(t)z^{\Delta}(t))^{\Delta}x^{\sigma}(t) + r(t)z^{\Delta}(t)x^{\Delta}(t) - (r(t)x^{\Delta}(t))^{\Delta}z^{\sigma}(t) - r(t)x^{\Delta}(t)z^{\Delta}(t) \\ = \;& p_1(t)f(x^{\sigma}(t))z^{\sigma}(t) - \lambda p_1(t)z^{\sigma}(t)x^{\sigma}(t).\end{aligned}$$

Therefore from (3.20) we get

$$\begin{aligned}0 &\leq \int_T^{T_1} \frac{z^{\sigma}(t)}{x^{\sigma}(t)}(r(t)z^{\Delta}(t)x(t) - r(t)x^{\Delta}(t)z(t))^{\Delta}\Delta t \\ &= \left[\frac{z(t)}{x(t)}(r(t)z^{\Delta}(t)x(t) - r(t)x^{\Delta}(t)z(t))\right]_T^{T_1} \\ &- \int_T^{T_1}\left(\frac{z(t)}{x(t)}\right)^{\Delta}(r(t)z^{\Delta}(t)x(t) - r(t)x^{\Delta}(t)z(t))\Delta t \\ &= \frac{z(T_1)}{x(T_1)}(r(T_1)z^{\Delta}(T_1)x(T_1) - r(T_1)x^{\Delta}(T_1)z(T_1)) \\ &- \int_T^{T_1}\frac{r(t)(z^{\Delta}(t)x(t) - x^{\Delta}(t)z(t))^2}{x(t)x^{\sigma}(t)}\Delta t \\ &< 0\end{aligned}$$

which is a contradiction. This completes the proof. □

Corollary 3.2. *Let $\lambda > 0$ and assume that equation* (1.5) *is oscillatory. Suppose in addition that* (3.2) *and* (3.3) *hold. Then all solutions of the generalized Emden–Fowler equation*

$$(r(t)x^{\Delta})^{\Delta} + p_1(t)(x^{\sigma})^{2n+1} = 0, \tag{3.21}$$

are oscillatory, where $n \geq 1$ is an integer.

Proof. Notice that with $f(x) := x^{2n+1}$, all the assumptions of Theorem 3.1 hold with $K := \lambda^{\frac{1}{2n}}$. □

Remark 3.3. Theorem 3.1 extends the results of [12] in a significant way. We do not require that the inequality

$$f'(x) \geq \frac{f(x)}{x} > 0$$

hold for all $x \neq 0$ but **only** for sufficiently large $|x| \geq K$, for some $K > 0$. In addition, we can then choose

$$\lambda = \lambda_0 := \frac{f(K)}{K} > 0$$

and then require oscillation of equation (1.5) for $\lambda = \lambda_0$. Therefore, we do not need to assume that equation (1.5) is oscillatory for **all** $\lambda > 0$. In particular, if equation (1.5) is conditionally oscillatory, then the results of Theorem 3.1 and Corollary 3.2 apply.

4. Examples

Clearly, equation (1.5) is oscillatory iff equation

$$\left(\frac{1}{\lambda} r(t)x^{\Delta}\right)^{\Delta} + p_1(t)x^{\sigma} = 0 \tag{4.1}$$

is oscillatory. It was shown in Erbe [8, Corollary 7] (see also Bohner and Peterson [3]) that

$$(r(t)x^{\Delta})^{\Delta} + p_1(t)x^{\sigma} = 0 \tag{4.2}$$

is oscillatory if there exists a sequence $\{t_k\} \subset \mathbb{T}$ with $\lim_{k\to\infty} t_k = \infty$ and $\mu(t_k) > 0$ such that

$$\limsup_{k\to\infty} \left(P_1(t_k) - \frac{r(t_k)}{\mu(t_k)}\right) = \infty, \tag{4.3}$$

where $P_1(t) := \int_{t_0}^{t} p_1(s)\Delta s$. We can therefore conclude that all solutions of (1.1) oscillate if p_1 satisfies condition (A), and (3.1), (3.2), and (3.3) hold along with

$$\limsup_{k\to\infty} \left(P_1(t_k) - \frac{r(t_k)}{\lambda\mu(t_k)}\right) = \infty. \tag{4.4}$$

We note that there is no assumption on the boundedness of r and μ. One may also apply averaging techniques or the telescoping principle to give some more sophisticated results (see Erbe, Kong, and Kong [9] and Erbe [8]). We leave this to the interested reader.

As a second example, suppose that $\mathbb{T}$ is such that there exists a sequence of points $t_k \in \mathbb{T}$ with $t_k \to \infty$ and positive numbers M, μ_0 such that $r(t_k) \leq M$ and $\mu(t_k) \geq \mu_0$. Then if $\sum_1^\infty \mu(t_k)p_1(t_k) = \infty$ it follows from results of Erbe, Kong, and Kong [9, Corollary 4.1] that all solutions of (1.5) are oscillatory for all $\lambda > 0$. Consequently, all solutions of (1.1) are oscillatory, if, in addition, p_1 satisfies condition (A) and (3.1), (3.2), and (3.3) hold.

As a third example, we consider the case when $\mathbb{T} = \mathbb{Z}$. If f has the form of (1.3) (i.e., $f(x) = x^{2n+1}$), $r(t) \equiv 1$, and $p(t) = p_1(t) = \frac{\lambda}{t\sigma(t)}$, then it is known that equation (1.5) is oscillatory if $\lambda > \frac{1}{4}$, and is nonoscillatory if $\lambda \leq \frac{1}{4}$. Since in this case (3.1) holds trivially, it follows from Theorem 3.1 that all solutions of (1.1) (which is the same as equation (3.21) in this case) are oscillatory.

Remark 4.1. From Theorem 4.64 in [3] (Leighton–Wintner Theorem) it follows that equation (1.5) is oscillatory for all $\lambda > 0$ if

$$\int_a^\infty \frac{1}{r(t)}\Delta t = \int_a^\infty p_1(t)\Delta t = +\infty. \tag{4.5}$$

Since the second condition in (4.5) implies that p_1 satisfies (A), Theorem 3.1 implies that all solutions of the Emden–Fowler equation (3.21) are oscillatory. That is, the Leighton–Wintner Theorem is valid for (3.21) and more generally for (1.1) if (1.2) and (3.3) hold. We note again that there are no explicit sign conditions on $p_1(t)$. For the special case when $\mathbb{T} = \mathbb{Z}$ and (1.1) is

$$\Delta^2 x_k + p_k(x_{k+1})^{2n+1} = 0, \tag{4.6}$$

where $n \in \mathbb{N}$, it follows that (4.6) is oscillatory if

$$\sum_{k=1}^\infty p_k = +\infty. \tag{4.7}$$

That is (4.7) implies that the linear equation

$$\Delta^2 x_k + \lambda p_k x_{k+1} = 0 \tag{4.8}$$

is oscillatory for all $\lambda > 0$ and so oscillation of (4.6) is a consequence of Theorem 3.1.

If we consider equation (4.8) with $\lambda = 1$, then Theorem 4.51 of [3] (see also [11]) implies that (4.8) is oscillatory if for any $k \geq 1$ there exists $k_1 \geq k$ such that

$$\lim_{m\to\infty} \sum_{j=k_1}^m p_j \geq 1. \tag{4.9}$$

Consequently, all solutions of (4.6) are oscillatory if (3.3) and (4.9) hold and condition (A) is satisfied by $\{p_n\}$.

Let us next consider the Euler–Cauchy dynamic equation

$$x^{\Delta\Delta} + \frac{\lambda}{t\sigma(t)}x = 0 \tag{4.10}$$

on the time scale $\mathbb{T} = r^{\mathbb{N}_0} = \{1, r, r^2, \cdots\}$, where $r > 1$. If $\lambda > \frac{1}{4}$, then it follows (see [12]) that (4.10) is oscillatory and if $0 \le \lambda \le \frac{1}{4}$, (4.10) is nonoscillatory. We conclude again that all solutions of (1.1) are oscillatory on this time scale provided (3.1) holds, since in this case it can be shown that (3.3) holds also for $p_1(t) = \frac{1}{t\sigma(t)}$.

As a final example, we consider the case when $\mathbb{T} = \mathbb{R}$ and let

$$q(t) \equiv 0, \quad p(t) = \frac{\lambda}{t^2} + \frac{\beta \sin t}{t},$$

where $\lambda, \beta > 0$. It is known (see Willett [17]) that the equation

$$y'' + \left(\frac{\lambda}{t^2} + \frac{\beta \sin t}{t}\right) y = 0$$

is oscillatory if

$$\lambda > \frac{1}{4} - \frac{\beta^2}{2}.$$

If, in addition, $\lambda > \beta$, then it follows that p satisfies condition (A) and that (3.3) holds and we also note that p changes sign for arbitrarily large t. Therefore if f satisfies (3.1), then all solutions of (1.1) are oscillatory.

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