On the Asymptotic Behavior of Solutions of Emden-Fowler Equations on Time Scales

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ABSTRACT. Consider the Emden-Fowler dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(t) = 0, \ \ \alpha > 0, \tag{0.1}$$

where $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, α is quotient of odd positive integers. We prove that if $\int_{t_0}^{\infty} t^{\alpha}|p(t)|\Delta t < \infty$, (and when $\alpha = 1$ we also assume $\lim_{t\to\infty} tp(t)\mu(t) = 0$), then (0.1) has a solution $x(t)$ with the property that

$$\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0.$$

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1. Introduction

Consider the second order Emden-Fowler dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(t) = 0, \tag{1.1}$$

where $p : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous (defined below), $\alpha > 0$, α is the quotient of odd positive integers.

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When $\mathbb{T} = \mathbb{R}$, the dynamic equation (1.1) is the second order Emden-Fowler differential equation

$$x''(t) + p(t)x^{\alpha}(t) = 0. \tag{1.2}$$

The Emden–Fowler equation (1.2) has several interesting physical applications in astrophysics (cf. Bellman [**8**] and Fowler [**12**]). Moore and Nehari [**13**] established the following: If $p(t)$ is positive and continuous and $\alpha \geq 1$, then (1.1) has solutions for which

$$\lim_{t\to\infty} \frac{x(t)}{t} = A > 0$$

if and only if

$$\int^{\infty} t^{\alpha} p(t) dt < \infty. \tag{1.3}$$

This is related to results of Atkinson [**1**] who showed that if $\alpha > 1$, $p(t) \geq 0$ and is nonincreasing, then (1.3) implies that all solutions of (1.1) are nonoscillatory. We refer to [**3**], [**6**] and [**7**] for additional results for the oscillation of (1.1). Wong [**14**, Theorem 2] establishes the sufficiency part of the above Moore–Nehari theorem without an assumption as to the sign of $p(t)$.

In this paper, by using a generalized Gronwall's inequality on time scales and an idea used by Wong [**14**], we prove that if

$$\int^{\infty} t^{\alpha} |p(t)| \Delta t < \infty,$$

(and when $\alpha = 1$ we assume $\lim_{t\to\infty} tp(t)\mu(t) = 0$), then equation (1.1) has a solution for which

$$\lim_{t\to\infty} \frac{x(t)}{t} = A > 0.$$

For completeness, (see [**9**] and [**10**] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c, d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c, d]_{\mathbb{T}}^{\kappa}$ denotes the interval $[c, d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c, d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We say $p : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous and write

$p \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ provided p is continuous at each right-dense point in $[t_0,\infty)_{\mathbb{T}}$ and at each left-dense point in $(t_0,\infty)_{\mathbb{T}}$ the left hand limit of p exists (finite). We say that $x:\mathbb{T}\to\mathbb{R}$ is (delta) differentiable at $t\in\mathbb{T}$ provided

$$x^{\Delta}(t) := \lim_{s\to t}\frac{x(t)-x(s)}{t-s},$$

exists when $\sigma(t)=t$ (here by $s\to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t)>t$

$$x^{\Delta}(t) := \frac{x(\sigma(t))-x(t)}{\mu(t)}.$$

Note that if $\mathbb{T}=\mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T}=\mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales. Section 2 is devoted to a few preliminary results, the main result is in Section 3 and we include several examples in Section 4.

2. Preliminary Lemmas

Let $\hat{\mathbb{T}} := \{t\in\mathbb{T} : \mu(t)>0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later, imposes a lower bound on the graininess function $\mu(t)$, for $t\in\hat{\mathbb{T}}$. More precisely, we introduce the following (see [**11**] and [**5**]).

Condition (C) We say that $\mathbb{T}$ satisfies condition C if there is an $M>0$ such that

$$\chi(t)\le M\mu(t), \quad t\in\mathbb{T}.$$

We note that if $\mathbb{T}$ satisfies condition (C), then the set

$$\check{\mathbb{T}} = \{t\in\mathbb{T}|\ t>0 \text{ is isolated or right-scattered or left-scattered}\}$$

is necessarily countable.

The following lemma is [**9**, Corollary 6.7].

Lemma 2.1. *Suppose that $y(t)$ and $p(t)$ are rd-continuous and $p\ge 0$. Then*

$$y(t)\le y_0+\int_{t_0}^{t} p(s)y(s)\Delta s \quad \textit{for all} \quad t\ge t_0, \tag{2.1}$$

implies

$$y(t)\le y_0 e_p(t,t_0) \quad \textit{for all} \quad t\ge t_0.$$

Lemma 2.2. *Suppose that $p\in\mathcal{R}$ and $\lim_{t\to\infty}p(t)\mu(t)=0$. Then there is a $T\in[t_0,\infty)_{\mathbb{T}}$ such that*

$$|e_p(t,T)|\le \exp\left[\int_T^t 2|p(\tau)|\Delta\tau\right]$$

for $t \in [T, \infty)_{\mathbb{T}}$. *If, in addition,* $\int_{t_0}^{\infty} |p(s)|\Delta s < \infty$, *then* $e_p(t, t_0)$ *is bounded on* $[t_0, \infty)_{\mathbb{T}}$.

PROOF. By the definition of $e_p(t, t_0)$ ([**9**], Page 57)

$$e_p(t, t_0) = \exp\left(\int_{t_0}^{t} \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \tag{2.2}$$

where

$$\xi_h(p(\tau)) = \begin{cases} \frac{1}{h} Log(1 + hp(\tau)), & h > 0 \\ p(\tau), & h = 0, \end{cases} \tag{2.3}$$

where *Log* is the principal logarithm function.

Since $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$, there is a $\delta > 0$ such that $|\ln(1+x)| \leq 2|x|$, for $|x| < \delta$. Using the hypothesis $\lim_{t\to\infty} p(t)\mu(t) = 0$, there is a $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$|p(t)\mu(t)| < \delta, \quad t \in [T, \infty)_{\mathbb{T}}.$$

Assume $\tau \in [T, \infty)_{\mathbb{T}}$. If, in addition, $\mu(\tau) > 0$, we have that

$$\begin{aligned} \left|\xi_{\mu(\tau)}(p(\tau))\right| &= \left|\frac{Log[1 + \mu(\tau)p(\tau)]}{\mu(\tau)}\right| \\ &= \frac{|\ln(1 + \mu(\tau)p(\tau))|}{\mu(\tau)} \leq 2|p(\tau)|. \end{aligned}$$

On the other hand if $\mu(\tau) = 0$, we have

$$|\xi_{\mu(\tau)}(p(\tau))| = |p(\tau)| \leq 2|p(\tau)|.$$

Hence for all $\tau \in [T, \infty)_{\mathbb{T}}$, we have

$$|\xi_{\mu(\tau)}(p(\tau))| \leq 2|p(\tau)|.$$

Therefore, by (2.2), we get that for $t \in [T, \infty)_{\mathbb{T}}$

$$\begin{aligned} |e_p(t, T)| &= |e^{\int_T^t \xi_{\mu(\tau)}(p(\tau)\Delta\tau}| \\ &\leq \left|\exp\left[\int_T^t 2|p(\tau)|\Delta\tau\right]\right| \\ &\leq \exp\left[\int_{t_0}^t 2|p(\tau)|\Delta\tau\right]. \end{aligned}$$

The last statement in this lemma follows from this last inequality and the semi group property [**9**, Theorem 2.36] $e_p(t, t_0) = e_p(t, T)e_p(T, t_0)$. □

LEMMA 2.3. *Suppose that* $[t_0, \infty)_{\mathbb{T}}$ *satisfies condition (C),* $x^{\Delta}(t)$ *is rd-continuous, and* $f : (0, \infty) \to (0, \infty)$ *is continuous and nonincreasing with* $F'(x) = f(x)$, $x \in \mathbb{R}$. *Then, we have that*

$$\int_{t_0}^{t} f(x(s))x^{\Delta}(s)\Delta s \geq F(x(t)) - F(x(t_0)), \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

PROOF. Since $[t_0,\infty)_{\mathbb{T}}$ satisfies property (C), $[t_0,\infty)_{\mathbb{T}} = \cup_{i=0}^{\infty}[t_i,t_{i+1}]_{\mathbb{T}}$, where for each $i \geq 0$ either $\sigma(t_i) = t_{i+1} > t_i$ or $[t_i,t_{i+1}]_{\mathbb{T}} = [t_i,t_{i+1}]_{\mathbb{R}}$. From the additivity of the integral it suffices to show that

$$\int_{t_i}^{t_{i+1}} f(x(s))x^{\Delta}(s)\Delta s \geq F(x(t_{i+1})) - F(x(t_i)) \tag{2.4}$$

for each $i \geq 0$. First consider the case $\sigma(t_i) = t_{i+1} > t_i$. We first consider the subcase $x(t_i) \leq x(t_{i+1})$. Using the fact that f is nonincreasing we have

$$\begin{aligned}\int_{t_i}^{t_{i+1}} f(x(s))x^{\Delta}(s)\Delta s &= \int_{t_i}^{\sigma(t_i)} f(x(s))x^{\Delta}(s)\Delta s \\ &= f(x(t_i))x^{\Delta}(t_i)\mu(t_i) \\ &= f(x(t_i))[x(t_{i+1}) - x(t_i)] \\ &\geq \int_{x(t_i)}^{x(t_{i+1})} f(x)dx \\ &= F(x(t_{i+1})) - F(x(t_i))\end{aligned}$$

and so (2.4) holds in this case. Next consider the subcase $x(t_i) > x(t_{i+1})$. In this case

$$\begin{aligned}\int_{t_i}^{t_{i+1}} f(x(s))x^{\Delta}(s)\Delta s &= \int_{t_i}^{\sigma(t_i)} f(x(s))x^{\Delta}(s)\Delta s \\ &= f(x(t_i))x^{\Delta}(t_i)\mu(t_i) \\ &= -f(x(t_i))[x(t_i) - x(t_{i+1})] \\ &\geq -\int_{x(t_{i+1})}^{x(t_i)} f(x)dx \\ &= F(x(t_{i+1})) - F(x(t_i))\end{aligned}$$

and so also in the subcase $x(t_i) > x(t_{i+1})$, it follows that (2.4) holds.

Finally, if $[t_i,t_{i+1}]_{\mathbb{T}} = [t_i,t_{i+1}]_{\mathbb{R}}$ then

$$\int_{t_i}^{t_{i+1}} f(x(t))x^{\Delta}(t)\Delta t = \int_{t_i}^{t_{i+1}} f(x(t))x'(t)dt = F(x(t_{i+1})) - F(x(t_i))$$

and so (2.4) holds in this case as well. □

Our next lemma is a sublinear analogue of the Gronwall inequality.

LEMMA 2.4. *Assume* $[t_0,\infty)_{\mathbb{T}}$ *satisfies condition (C),* $p(t) \geq 0$*,* $q(t) \geq 0$ *are rd-continuous,* $y_0 > 0$*, and* $0 < \alpha < 1$*. If* $y(t) \geq 0$*, satisfies*

$$y(t) \leq y_0 + \int_{t_0}^{t} p(s)y(s)\Delta s + \int_{t_0}^{t} q(s)y^{\alpha}(s)\Delta s$$

for $t \in [t_0,\infty)_{\mathbb{T}}$*, then*

$$y(t) \leq e_p(t,t_0)\left\{y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s\right\}^{\frac{1}{1-\alpha}}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

PROOF. Let

$$z(t) := y_0 + \int_{t_0}^{t} p(s)y(s)\Delta s + \int_{t_0}^{t} q(s)y^{\alpha}(s)\Delta s > 0$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. Then by hypothesis $y(t) \le z(t)$ on $[t_0, \infty)_{\mathbb{T}}$ and

$$\begin{aligned} z^{\Delta}(t) &= p(t)y(t) + q(t)y^{\alpha}(t) \\ &\le p(t)z(t) + q(t)z^{\alpha}(t) \\ &= p(t)[z^{\sigma}(t) - \mu(t)z^{\Delta}(t)] + q(t)z^{\alpha}(t). \end{aligned}$$

It follows that

$$z^{\Delta}(t) - \frac{p(t)}{1+\mu(t)p(t)} z^{\sigma}(t) \le \frac{q(t)}{1+\mu(t)p(t)} z^{\alpha}(t).$$

Hence

$$z^{\Delta}(t) + (\ominus p)(t) z^{\sigma}(t) \le \frac{q(t)}{1+\mu(t)p(t)} z^{\alpha}(t).$$

Multiplying by the integrating factor $e_{\ominus p}(t, t_0)$ we get

$$\begin{aligned} [e_{\ominus p}(t,t_0)z(t)]^{\Delta} &\le e_{\ominus p}(t,t_0)\frac{q(t)}{1+\mu(t)p(t)} z^{\alpha}(t) \\ &= e_{\ominus p}^{1-\alpha}(t,t_0)\frac{q(t)}{1+\mu(t)p(t)} [e_{\ominus p}(t,t_0)z(t)]^{\alpha}. \end{aligned}$$

Letting

$$v(t) := e_{\ominus p}(t,t_0)z(t) > 0,$$

we have

$$\frac{v^{\Delta}(t)}{v^{\alpha}(t)} \le e_{\ominus p}^{1-\alpha}(t,t_0)\frac{q(t)}{1+\mu(t)p(t)}.$$

Integrating from t_0 to t we obtain

$$\int_{t_0}^{t} \frac{v^{\Delta}(s)}{v^{\alpha}(s)}\Delta s \le \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s.$$

Applying Lemma 2.3 (with $f(x) = \frac{1}{x^{\alpha}}$) to the left hand side of this last inequality gives

$$\int_{t_0}^{t} \frac{v^{\Delta}(s)}{v^{\alpha}(s)}\Delta s \ge \frac{v^{1-\alpha}(t)}{1-\alpha} - \frac{v^{1-\alpha}(t_0)}{1-\alpha}.$$

It then follows that

$$\frac{v^{1-\alpha}(t)}{1-\alpha} - \frac{v^{1-\alpha}(t_0)}{1-\alpha} \le \int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s \tag{2.5}$$

and consequently

$$v^{1-\alpha}(t) \le y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s.$$

Then

$$v(t) = e_{\ominus p}(t,t_0)z(t) \le \left\{ y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s \right\}^{\frac{1}{1-\alpha}}$$

which gives us the desired result

$$y(t) \le z(t) \le e_p(t,t_0)\left\{ y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s \right\}^{\frac{1}{1-\alpha}}.$$

□

REMARK 2.5. When $p(t) \equiv 0$ in Lemma 2.4 we have that

$$y(t) \le y_0 + \int_{t_0}^{t} q(s)y^{\alpha}(s)\Delta s \quad \text{for all} \quad t \in [t_0,\infty)_{\mathbb{T}}$$

implies

$$y(t) \le \left\{ y_0^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} q(s)\Delta s \right\}^{\frac{1}{1-\alpha}}.$$

for $t \in [t_0,\infty)_{\mathbb{T}}$.

The superlinear analogue of Lemma 2.4 is the following result:

LEMMA 2.6. *If in Lemma 2.4 we replace* $0 < \alpha < 1$ *by* $\alpha > 1$ *and assume we can pick* $y_0 > 0$ *such that*

$$y_0^{1-\alpha} > (\alpha-1)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s$$

for $t \in [t_0,\infty)_{\mathbb{T}}$*, then*

$$y(t) \le \frac{e_p(t,t_0)}{\left\{ y_0^{1-\alpha} - (\alpha-1)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s \right\}^{\frac{1}{\alpha-1}}}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$.

PROOF. The proof starts out the same as in the proof of Lemma 2.4 until we get (2.5). Solving this equation for $v(t)$ we get (here we use the assumption $y_0^{1-\alpha} > (\alpha-1)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s$)

$$v(t) \le \frac{1}{\left\{ y_0^{1-\alpha} - (\alpha-1)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s \right\}^{\frac{1}{\alpha-1}}}.$$

Since $v(t) = e_{\ominus p}(t,t_0)z(t)$ and $e_{\ominus p}(t,t_0) = \frac{1}{e_p(t,t_0)}$ we have that

$$y(t) \le z(t) \quad \le \quad \frac{e_p(t,t_0)}{\left\{ y_0^{1-\alpha} - (\alpha-1)\int_{t_0}^{t} e_{\ominus p}^{1-\alpha}(s,t_0)\frac{q(s)}{1+\mu(s)p(s)}\Delta s \right\}^{\frac{1}{\alpha-1}}}$$

□

3. Asymptotic Behavior of Solutions

We now prove our main result

THEOREM 3.1. *Assume* $[t_0,\infty)_{\mathbb{T}}$ *satifies condition (C),* $\alpha > 0$ *is the quotient of odd positive integers, and*

$$\int_{t_0}^{\infty} t^{\alpha}|p(t)|\Delta t < \infty,$$

(and if $\alpha = 1$ *we assume* $\lim_{t\to\infty} tp(t)\mu(t) = 0$*), Then*

$$x^{\Delta\Delta} + p(t)x^{\alpha}(t) = 0, \tag{3.1}$$

has a solution satisfying $\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0$.

PROOF. Without loss of generality we can assume $t_0 \geq 1$. Assume $x(t)$ is a solution of (3.1) with $x(t_0) \neq 0$ and let

$$k(t_0) := |x(t_0)| + |x^{\Delta}(t_0)| > 0.$$

Now equation (3.1) is equivalent to the integral equation

$$x(t) = x(t_0) + x^{\Delta}(t_0)(t - t_0) - \int_{t_0}^{t} (t - \sigma(s))p(s)x^{\alpha}(s)\Delta s. \tag{3.2}$$

Thus for $t \in [t_0,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned} |x(t)| &\leq |x(t_0)| + |x^{\Delta}(t_0)|t + t\int_{t_0}^{t} |p(s)x^{\alpha}(s)|\Delta s \\ &\leq k(t_0)t + t\int_{t_0}^{t} |p(s)x^{\alpha}(s)|\Delta s, \quad (\text{using} \quad t \geq 1) \\ &= k(t_0)t + t\int_{t_0}^{t} s^{\alpha}|p(s)| \left(\frac{|x(s)|}{s}\right)^{\alpha} \Delta s. \end{aligned}$$

Letting $y(t) := \frac{|x(t)|}{t}$, we obtain

$$y(t) \leq k(t_0) + \int_{t_0}^{t} s^{\alpha}|p(s)|y^{\alpha}(s)\Delta s, \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{3.3}$$

Consider first the case $0 < \alpha < 1$. By Remark 2.5, we get

$$\begin{aligned} y(t) &\leq \left\{k^{1-\alpha}(t_0) + (1-\alpha)\int_{t_0}^{t} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{1-\alpha}} \\ &\leq \left\{k^{1-\alpha}(t_0) + (1-\alpha)\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{1-\alpha}} =: B. \end{aligned} \tag{3.4}$$

So we have $y(t) \leq B$, that is $|x(t)| \leq Bt$. Since

$$x^{\Delta}(t) = x^{\Delta}(t_0) - \int_{t_0}^{t} p(s)x^{\alpha}(s)\Delta s \tag{3.5}$$

and

$$\int_{t_0}^{t} |p(s)x^\alpha(s)|\,\Delta s \le B^\alpha \int_{t_0}^{\infty} |p(s)|s^\alpha \Delta s < \infty. \tag{3.6}$$

we have that $\lim_{t\to\infty} x^\Delta(t) = A$ exists. Therefore if $\epsilon > 0$ is given, then there exists a $T \in [t_0,\infty)_{\mathbb{T}}$, such that $A-\epsilon < x^\Delta(t) < A+\epsilon$, for $t \in [T,\infty)_{\mathbb{T}}$. By the time scales Mean Value Theorem [**10**, Theorem 1.14], we get that if $t \in [T,\infty)_{\mathbb{T}}$ there exist $\tau_t, \xi_t \in [T,t)_{\mathbb{T}}$ such that

$$A-\epsilon < x^\Delta(\tau_t) \le \frac{x(t)-x(T)}{t-T} \le x^\Delta(\xi_t) < A+\epsilon.$$

This implies for all $t \in [T,\infty)_{\mathbb{T}}$ that

$$(A-\epsilon)\left(1-\frac{T}{t}\right) + \frac{x(T)}{t} < \frac{x(t)}{t} < (A+\epsilon)\left(1-\frac{T}{t}\right) + \frac{x(T)}{t}.$$

Therefore we get that

$$\lim_{t\to\infty} \frac{x(t)}{t} = A. \tag{3.7}$$

From (3.5) and (3.6), we have

$$|x^\Delta(t)| \ge |x^\Delta(t_0)| - B^\alpha \int_{t_0}^{\infty} s^\alpha |p(s)|\Delta s. \tag{3.8}$$

We want to show that we can find a solution $x(t)$ so that the constant A in (3.7) is nonzero. To see this we (still assume $t_0 \ge 1$) let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ be fixed but arbitrary and let $x_{t_1}(t)$ be a family of solutions of (1.1) whose initial conditions satisfy

$$|x_{t_1}(t_1)| = C > 0, \quad |x_{t_1}^\Delta(t_1)| = D > 0, \quad t_1 \in [t_0,\infty)_{\mathbb{T}},$$

where C and D are constants (do not depend on t_1). Then by the proof of (3.8) we obtain

$$\begin{aligned} |x_{t_1}^\Delta(t)| &\ge |x_{t_1}^\Delta(t_1)| - B^\alpha(t_1) \int_{t_1}^{\infty} s^\alpha |p(s)|\Delta s \\ &= D - B^\alpha(t_1) \int_{t_1}^{\infty} s^\alpha |p(s)|\Delta s \end{aligned} \tag{3.9}$$

where

$$B(t_1) := \left\{ (C+D)^{1-\alpha} + (1-\alpha) \int_{t_1}^{\infty} s^\alpha |p(s)|\Delta s \right\}^{\frac{1}{1-\alpha}}.$$

Since

$$\lim_{t_1\to\infty} B^\alpha(t_1) \int_{t_1}^{\infty} s^\alpha |p(s)|\Delta s = 0$$

we can pick $t_1 \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large so that (using (3.9)) we have that

$$|x_{t_1}^\Delta(t)| \ge \frac{1}{2}D > 0, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

It is then easy to see that for such a t_1 we have that

$$\lim_{t\to\infty} \frac{x_{t_1}(t)}{t} = A \neq 0.$$

This completes the proof when $0 < \alpha < 1$.

Now consider the case when $\alpha = 1$. The proof is the same as the above proof up to (3.3). That is we get (3.3) with $\alpha = 1$, namely,

$$y(t) \le k(t_0) + \int_{t_0}^{t} s|p(s)|y(s)\Delta s, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

Then we use Lemma 2.1 to obtain

$$y(t) \le k(t_0)e_r(t, t_0), \quad \text{where} \quad r = r(t) := t|p(t)|. \tag{3.10}$$

Since $\int_{t_0}^{\infty} r(t)\Delta t < \infty$ and $\lim_{t\to\infty} r(t)\mu(t) = 0$ we get from Lemma 2.2 that $e_r(t, t_0)$ is bounded and hence there is a constant $B > 0$ such that $y(t) \le B$ for $t \in [t_0, \infty)_{\mathbb{T}}$. The rest of the proof is the same as in the case $0 < \alpha < 1$ after (3.4).

Next assume $\alpha > 1$. At the outset of this proof assume that $x(t)$ is a solution of (1.1), with $x^{\Delta}(t_0) = 0$ and $x_0 := x(t_0)$ is chosen so that $k(t_0) = |x(t_0)| > 0$ satisfies

$$|x_0|^{1-\alpha} = k^{1-\alpha}(t_0) > (\alpha - 1)\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$ (we can do this because we are assuming $\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s < \infty$.) It follows that

$$|x_0|^{1-\alpha} = k^{1-\alpha}(t_0) > (\alpha - 1)\int_{t_0}^{t} s^{\alpha}|p(s)|\Delta s$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Using the fact that (3.4) holds we get from Lemma 2.6 that

$$\begin{aligned} y(t) &\le \frac{1}{\left\{|x_0|^{1-\alpha} - (\alpha-1)\int_{t_0}^{t} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{\alpha-1}}} \\ &\le \frac{1}{\left\{|x_0|^{1-\alpha} - (\alpha-1)\int_{t_0}^{\infty} s^{\alpha}|p(s)|\Delta s\right\}^{\frac{1}{\alpha-1}}} =: B \end{aligned}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. It follows that $|x(t)| \le Bt$ for $t \in [t_0, \infty)_{\mathbb{T}}$ and the proof now proceeds as in the case $0 < \alpha < 1$ following (3.4). □

4. Examples

We consider in this section several examples to illustrate the results obtained. (We deal mainly with the more challenging case when $p(t)$ is not of one sign.)

EXAMPLE 4.1. Let $0 < \alpha \leq 1$, $\mathbb{T} = \mathbb{N}$ and

$$p(n) := a(-1)^n n^{-b}.$$

Then the equation

$$\Delta^2 x(n) + p(n)x^\alpha(n+1) = 0, \quad 0 < \alpha < 1 \tag{4.1}$$

has a nonoscillatory solution if $a > 0$ and $b > \alpha$ (see [**3**]). From Theorem 3.1, we conclude that if $b > \alpha + 1$, then

$$\sum_{n_0}^{\infty} n^\alpha |p(n)| < \infty$$

and therefore it follows that (4.1) has a solution satisfying

$$\lim_{n\to\infty} \frac{x(n)}{n} = A \neq 0. \tag{4.2}$$

Note also that (4.2) holds in the case $\alpha = 1$ since $b > 2$ implies that $\lim_{n\to\infty} np(n) = 0$.

EXAMPLE 4.2. Let $\alpha > 1$ and let

$$p(n) := \frac{a}{(n+1)n^b} + \frac{c(-1)^n}{n^b}, \quad n \in \mathbb{N}.$$

It was shown in [**6**] that if $a > 0$ and $0 < b \leq 1$, then all solutions of (4.1) are oscillatory. Notice however that (4.1) has a nonoscillatory solution such that (4.2) holds if $b > \alpha + 1$, since

$$\sum_{n=1}^{\infty} n^\alpha |p(n)| \leq a \sum_{n=1}^{\infty} n^{\alpha-b-1} + c \sum_{n=1}^{\infty} n^{\alpha-b} < \infty$$

REMARK 4.3. It is easy to give additional examples for the q-difference equation case using ideas for oscillation and nonoscillation in the references [**2**], [**3**], [**4**], [**5**] and [**6**]. We note also that in the case $\mathbb{T} = \mathbb{R}$, the differential equation

$$x'' + p(t)x^\alpha = 0 \tag{4.3}$$

has a solution $x(t)$ satisfying

$$\lim_{t\to\infty} \frac{x(t)}{t} = A \neq 0 \tag{4.4}$$

if $0 < \alpha \leq 1$, where

$$p(t) = \frac{\sin t}{t^b}, \quad t \in [1, \infty)$$

and where $b > \alpha + 1$.

Also, in the superlinear case $\alpha > 1$ and with

$$p(t) = \frac{a}{t^{b+1}} + \frac{c \sin t}{t^b},$$

then (4.3) has a solution saisfying (4.2) if $a > 0$, $c \neq 0$ and $b > \alpha + 1$.

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