Belohorec-type Oscillation Theorem for Second Order Sublinear Dynamic Equations on Time Scales

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Consider the Emden-Fowler sublinear dynamic equation
\[ x^\Delta\Delta(t) + p(t)x^\alpha(\sigma(t)) = 0, \] (0.1)
where \( p \in C(\mathbb{T}, \mathbb{R}) \), where \( \mathbb{T} \) is a time scale, \( 0 < \alpha < 1 \), \( \alpha \) is the quotient of odd positive integers. When \( p(t) \) is allowed to take on negative values, we obtain a Belohorec-type oscillation theorem for (0.1). As an application, we get that the sublinear difference equation
\[ \Delta^2 x(n) + p(n)x^\alpha(n + 1) = 0, \] (0.2)
is oscillatory, if
\[ \sum_{n=0}^{\infty} n^\alpha p(n) = \infty, \]
and the sublinear q-difference equation
\[ x^\Delta\Delta(t) + p(t)x^\alpha(qt) = 0. \] (0.3)
where \( t \in q\mathbb{N}_0, q > 1 \), is oscillatory, if
\[ \int_{1}^{\infty} t^\alpha p(t)\Delta t = \infty. \]

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1 Introduction

Consider the second order sublinear dynamic equation
\[ x^\Delta\Delta + p(t)x^\alpha(\sigma(t)) = 0, \] (1.1)
where \( p \in C(\mathbb{T}, \mathbb{R}) \), \( \alpha > 0 \) is the quotient of odd positive integers and where \( \mathbb{T} \) is a time scale with \( \sup \mathbb{T} = \infty \). Equation (1.1) is called superlinear if \( \alpha > 1 \) and sublinear if \( 0 < \alpha < 1 \). We call an equation oscillatory if all its continuable solutions are oscillatory.

When \( \mathbb{T} = \mathbb{R} \), the dynamic equation (1.1) is the second order nonlinear differential equation
\[ x''(t) + p(t)x^\alpha(t) = 0. \] (1.2)

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When \( T = \mathbb{N}_0 \), the dynamic equation (1.1) is the second order nonlinear difference equation
\[
\Delta^2 x(n) + p(t)x^n(n + 1) = 0. \tag{1.3}
\]

When \( p(t) \) is nonnegative, stronger oscillation results exist for the nonlinear equation (1.2) when \( \alpha \neq 1 \), notably the following:

**Theorem A** (Atkinson [1]). Let \( \alpha > 1 \). Then (1.2) is oscillatory if and only if
\[
\int_{\infty}^{\infty} tp(t) \, dt = \infty. \tag{1.4}
\]

**Theorem B** (Belohorec [5]). Let \( 0 < \alpha < 1 \). Then (1.2) is oscillatory if and only if
\[
\int_{\infty}^{\infty} t^\alpha p(t) \, dt = \infty. \tag{1.5}
\]

When \( p(t) \) is allowed to take on negative values, for \( \alpha > 1 \), Kiguradze [17] proved that (1.4) is sufficient for the differential equation (1.2) to be oscillatory and for \( 0 < \alpha < 1 \) Belohorec [6] proved that (1.5) is sufficient for the differential equation (1.2) to be oscillatory. These results have been further extended by Kwong and Wong [18].

When \( p(n) \) is nonnegative, J. W. Hooker and W. T. Patula [13, Theorem 4.1], and A. Mingarelli [19], respectively proved the following:

**Theorem C** Let \( \alpha > 1 \). Then (1.3) is oscillatory if and only if
\[
\sum_{1}^{\infty} np(n) = \infty. \tag{1.6}
\]

**Theorem D** Let \( 0 < \alpha < 1 \). Then (1.3) is oscillatory if and only if
\[
\sum_{1}^{\infty} n^\alpha p(n) = \infty. \tag{1.7}
\]

For the case \( \alpha > 1 \) and when \( p(n) \) is allowed to take on negative values, we proved in [2] that (1.6) is sufficient for the difference equation (1.3) to be oscillatory.

In this paper, we prove that when \( p(n) \) is allowed to take on negative values, for \( 0 < \alpha < 1 \), (1.7) is sufficient for the oscillation of the difference equation (1.3). In particular, we get that the sublinear difference equation
\[
\Delta^2 x(n) + \left[ \frac{1}{n^{\beta+1}} + \frac{b(-1)^n}{n^\beta} \right] x^n(n + 1) = 0,
\]
is oscillatory, if \( 0 \leq \beta \leq \alpha \) and is nonoscillatory if \( \beta > \alpha \). We note that the transformation used here is different from that in [6], which can not be applied to more general time scales. Moreover, the technique of proof in the superlinear and sublinear cases are essentially different and also differ substantially from the case \( T = \mathbb{R} \).

For completeness, (see [8] and [9] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let \( T \) be a time scale (i.e., a closed nonempty subset of \( \mathbb{R} \)) with \( \sup T = \infty \). The forward jump operator is defined by
\[
\sigma(t) = \inf\{s \in T : s > t\},
\]
and the backward jump operator is defined by
\[
\rho(t) = \sup\{s \in T : s < t\},
\]
where \( \sup \emptyset = \inf T \), where \( \emptyset \) denotes the empty set. If \( \sigma(t) > t \), we say \( t \) is right-scattered, while if \( \rho(t) < t \) we say \( t \) is left-scattered. If \( \sigma(t) = t \) we say \( t \) is right-dense, while if \( \rho(t) = t \) and \( t \not= \inf T \) we say \( t \) is left-dense. Given a time scale interval \([c, d]_T := \{t \in T : c \leq t \leq d\}\) in \( T \) the notation \([c, d]_T^n \) denotes the interval \([c, d]_T\).
in case \( \rho(d) = d \) and denotes the interval \([c, d]_T\) in case \( \rho(d) < d \). The graininess function \( \mu \) for a time scale \( T \) is defined by \( \mu(t) = \sigma(t) - t \), and for any function \( f : T \to \mathbb{R} \) the notation \( f^{\sigma}(t) \) denotes \( f(\sigma(t)) \). We say that \( x : T \to \mathbb{R} \) is differentiable at \( t \in T \) provided
\[
x^\Delta(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s},
\]
even exists when \( \sigma(t) = t \) (here by \( s \to t \) it is understood that \( s \) approaches \( t \) in the time scale) and when \( x \) is continuous at \( t \) and \( \sigma(t) > t \)
\[
x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.
\]
Note that if \( T = \mathbb{R} \), then the delta derivative is just the standard derivative, and when \( T = \mathbb{Z} \) the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales (for example the time scale \( q^{\mathbb{N}_0} := \{1, q, q^2, \cdots\} \) which is very important in quantum theory [14]).

2 Main Theorem

In the case when \( T \) is such that \( \mu(t) \) is not eventually identically zero, we define the set of all right-scattered points by \( \hat{T} := \{t \in T : \mu(t) > 0\} \) and note that \( \hat{T} \) is necessarily countable. We let \( \chi \) denote the characteristic function of \( \hat{T} \). The following condition, which will be needed later, imposes a lower bound on the graininess function \( \mu(t) \), for \( t \in \hat{T} \). More precisely, we introduce the following: (see [10]).

**Condition (C):** We say that \( T \) satisfies condition (C) if there is an \( M > 0 \) such that
\[
\chi(t) \leq M\mu(t), \quad t \in \hat{T}.
\]
We note that if \( T \) satisfies condition (C), then the subset \( \hat{T} \) of \( T \) defined by
\[
\hat{T} = \{t \in T | t > 0 \text{ is right-scattered or left-scattered}\}
\]
is also necessarily countable and, of course, \( \hat{T} \subset T \). So we can suppose that
\[
\hat{T} = \{t_i \in T | 0 < t_1 < t_2 < \cdots < t_n < \cdots\}.
\]

**Remark 2.1** To clarify the arguments below, we let \( A := \{n \in \mathbb{N} : (t_{n-1}, t_n) \subset T\} \) so that we can write
\[
T = \hat{T} \cup [\bigcup_{n \in A} (t_{n-1}, t_n)].
\]

**Theorem 2.2** Assume that \( T \) satisfies condition (C) and let \( \hat{T} \) be given by (2.2). If there exists a real number \( \beta, 0 \leq \beta \leq 1 \) such that
\[
\int_{t_1}^{\infty} \sigma^{\alpha\beta}(t)p(t)\Delta t = \infty,
\]
then (1.1) is oscillatory.

**Proof.** In this proof we use the notation \( (t^\beta)^\Delta|_{\sigma(t)} \) to mean \( (t^\beta)^\Delta \) evaluated at \( \sigma(t) \). For the sake of contradiction, assume that (1.1) is nonoscillatory. Then without loss of generality there is a solution \( x(t) \) of (1.1) and a \( T \in \mathbb{R} \) with \( x(t) > 0 \), for all \( t \in [T, \infty)_T \). Making the substitution \( x(t) = t^\beta u(t) \) in (1.1), where \( 0 \leq \beta \leq 1 \) and noticing (see Remark 2.1) that \( x^\Delta(t) = (t^\beta)^\Delta u(t) + t^\beta u^\Delta(t), x^\Delta(t) = (t^\beta)^\Delta u(t) + \beta t^\Delta u(t) + (t^\beta)^\Delta + u^\Delta(t) + \beta \sigma(t)u^\Delta(t), \) we get that
\[
(t^\beta)^\Delta u(t) + (t^\beta)^\Delta u(t) + (t^\beta)^\Delta u^\Delta(t) + \beta \sigma(t)u^\Delta(t) = 0. \quad (2.3)
\]
Multiplying both sides of (2.3) by $\frac{1}{u^\alpha(\sigma(s)))}$, integrating from $T$ to $t$, and using an integration by parts formula we get

$$
\int_T^t (s^\beta)\Delta u(\sigma(s)) \Delta s + \int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))\Delta}{u^\alpha(\sigma(s))}\right) \Delta s
+ \int_T^t \left(\frac{s^\beta u^\Delta(s)}{u^\alpha(\sigma(s))}\right) \Delta s + \int_T^t \left[\frac{s^\beta}{u^\alpha(s)}\right] \Delta s
= \int_T^t u^\Delta(s) \Delta s + \int_T^t (\sigma(s))^{\alpha\beta} p(s) \Delta s = 0. \tag{2.4}
$$

Next, using the quotient rule and then Pötzsche’s chain rule [8, Theorem 1.90] gives

$$
\int_T^t \left[\frac{s^\beta}{u^\alpha(s)}\right] \Delta u(\Delta s)
= \int_T^t \frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))} \Delta s - \int_T^t \frac{s^\beta [u^\alpha(s)]^2}{u^\alpha(s) u^\alpha(\sigma(s))} \Delta s
= \int_T^t \frac{(s^\beta)\Delta u(\alpha)}{u^\alpha(\sigma(s))} \Delta s - \alpha \int_T^t \left[\frac{s^\beta}{u^\alpha(\sigma(s))}\right] \Delta s
\leq \int_T^t \frac{(s^\beta)\Delta}{u^\alpha(\sigma(s))} u^\Delta(s) \Delta s,
$$

where we used the fact that $u_h(s) := u(s) + h u(T) u^\Delta(s) = (1-h) u(s) + h u^\alpha(s) \geq 0$. Using this last inequality in (2.4) we get

$$
\int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))}\right) \Delta s + \int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))\Delta}{u^\alpha(\sigma(s))}\right) \Delta s
+ \int_T^t \frac{t^b u^\Delta(t)}{u^\alpha(T)} - T^b u^\Delta(T) + \int_T^t (\sigma(s))^{\alpha\beta} p(s) \Delta s \leq 0. \tag{2.5}
$$

Note that

$$
\int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))\Delta}{u^\alpha(\sigma(s))}\right) \Delta s
= \left(\frac{(t^\beta)\Delta u(T)}{u^\alpha(T)}\right) - \left(\frac{(t^\beta)\Delta u(T)}{u^\alpha(T)}\right) - \int_T^t \frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))} \Delta s
+ \int_T^t \frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))} \Delta s
$$

Let us define $A := \frac{T^b u^\Delta(T)}{u^\alpha(T)}$, $B := \frac{(t^b)\Delta u(T)}{u^\alpha(T)}$. Then we get from (2.5) and (2.6) that

$$
\left(\frac{(t^\beta)\Delta u(T)}{u^\alpha(T)}\right) - \left(\frac{(t^\beta)\Delta u(T)}{u^\alpha(T)}\right) - B + \int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))}\right) \Delta s
+ \int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))}\right) \Delta s
= \left(\frac{t^b u^\Delta(t)}{u^\alpha(t)}\right) - A + \int_T^t (\sigma(s))^{\alpha\beta} p(s) \Delta s \leq 0. \tag{2.7}
$$

Note that $(t^\beta)\Delta = \beta \int_0^t [(1-h)t + h \sigma(t)]^{\beta-1} dh > 0$, for $t > 0$. So the first term of (2.7) is positive. From (2.7), we get that

$$
-B + \int_T^t \left(\frac{(s^\beta)\Delta u(\sigma(s))}{u^\alpha(\sigma(s))}\right) \Delta s + \left(\frac{t^b u^\Delta(t)}{u^\alpha(t)}\right) - A + \int_T^t (\sigma(s))^{\alpha\beta} p(s) \Delta s \leq 0. \tag{2.8}
$$
Since $0 < \beta \leq 1$, one can use the Pötzsch-Potzsche chain rule to show that $(t^\beta)^{\Delta}$ is nonincreasing. Using the second mean value theorem [9, Theorem 5.45], we get that for each $t \in [T, \infty)$:

$$\int_T^t \left( \frac{u^\alpha(s)\Delta(s)^\beta u(\sigma(s))}{u^\alpha(s)u^\alpha(\sigma(s))} \right) \Delta s = \Lambda(t)(s^\beta)^{\Delta}T, \quad (2.9)$$

where $m_t < \Lambda(t) < M_t$, and where $m_t$ and $M_t$ denote the infimum and supremum, respectively, of the function $\int_T^t \frac{(u^\alpha(s)\Delta u(\sigma(s)))}{u^\alpha(s)u^\alpha(\sigma(s))} \Delta s$.

Next, we will show that

$$\int_T^t \left( \frac{u^\alpha(s)\Delta u(\sigma(s))}{u^\alpha(s)u^\alpha(\sigma(s))} \right) \Delta s \geq \frac{\alpha}{1 - \alpha} [-u^{1-\alpha}(T)]. \quad (2.10)$$

Assume first that $t = t_{i-1} < t_i = \sigma(t)$. We get that

$$L_i(u) := \int_{t_{i-1}}^{t_i} \left( \frac{u^\alpha(s)^{\Delta} u(\sigma(s))}{u^\alpha(s)u^\alpha(\sigma(s))} \right) \Delta s = \left[ \frac{1}{u^\alpha(t_{i-1})} - \frac{1}{u^\alpha(t_i)} \right] u(t_i). \quad (2.11)$$

Setting $v(t_i) := \frac{1}{u^\alpha(t_i)}$, we have that

$$L_i(u) := \frac{v(t_{i-1}) - v(t_i)}{v^\frac{1}{\alpha}(t_i)}. \quad (2.12)$$

We consider the two possible cases $v(t_{i-1}) \geq v(t_i)$ and $v(t_{i-1}) < v(t_i)$. First if $v(t_{i-1}) \geq v(t_i)$ we have that

$$\frac{v(t_{i-1}) - v(t_i)}{v^\frac{1}{\alpha}(t_i)} \geq \int_{v(t_i)}^{v(t_{i-1})} \frac{1}{s^\frac{1}{\alpha}} ds \quad (2.13)$$

$$= \frac{\alpha}{1 - \alpha} [v^{1-\frac{1}{\alpha}}(t_i) - v^{1-\frac{1}{\alpha}}(t_{i-1})].$$

On the other hand if $v(t_{i-1}) < v(t_i)$, then

$$\frac{v(t_i) - v(t_{i-1})}{v^\frac{1}{\alpha}(t_i)} \leq \int_{v(t_{i-1})}^{v(t_i)} \frac{1}{s^\frac{1}{\alpha}} ds \quad (2.13)$$

$$= \frac{\alpha}{1 - \alpha} [v^{1-\frac{1}{\alpha}}(t_{i-1}) - v^{1-\frac{1}{\alpha}}(t_i)].$$

which implies that

$$\frac{v(t_{i-1}) - v(t_i)}{v^\frac{1}{\alpha}(t_i)} \geq \frac{\alpha}{1 - \alpha} [v^{1-\frac{1}{\alpha}}(t_i) - v^{1-\frac{1}{\alpha}}(t_{i-1})].$$

Hence, whenever $t = t_{i-1} < t_i = \sigma(t)$, we have that

$$L_i(u) \geq \frac{\alpha}{1 - \alpha} [v^{1-\frac{1}{\alpha}}(t_i) - v^{1-\frac{1}{\alpha}}(t_{i-1})]. \quad (2.14)$$

If the real interval $[t_{i-1}, t_i] \subset T$, it is easy to see that

$$L_i(u) := \int_{t_{i-1}}^{t_i} \left( \frac{u^\alpha(s)^{\Delta} u(\sigma(s))}{u^\alpha(s)u^\alpha(\sigma(s))} \right) \Delta s = \int_{t_{i-1}}^{t_i} \frac{\alpha u'(s)}{u^\alpha(s)} ds \quad (2.15)$$

$$= \frac{\alpha}{1 - \alpha} [u^{1-\alpha}(t_i) - u^{1-\alpha}(t_{i-1})]$$

$$= \frac{\alpha}{1 - \alpha} [v^{1-\frac{1}{\alpha}}(t_i) - v^{1-\frac{1}{\alpha}}(t_{i-1})].$$

Note that since $T$ satisfies condition (C), we have from (2.14), (2.15) and the additivity of the integral that

$$\int_T^t \left( \frac{u^\alpha(s)^{\Delta} u(\sigma(s))}{u^\alpha(s)u^\alpha(\sigma(s))} \right) \Delta s \geq \frac{\alpha}{1 - \alpha} [v^{1-\frac{1}{\alpha}}(t) - v^{1-\frac{1}{\alpha}}(T)] \quad (2.16)$$
If there exists a real number $\beta$, then
\[ \frac{\alpha}{1-\alpha} \left[ u_{1-\alpha}(t) - u_{1-\alpha}(T) \right] \geq \frac{\alpha}{1-\alpha} \left[ -u_{1-\alpha}(T) \right]. \]

Therefore, we get that (2.10) holds. From (2.8), (2.9), (2.10), we get that (1.1)
\[ \sigma \rho \]
where $r$ function is nonincreasing. We note, as was established in [2], that this need not hold (i.e., one can find an example [2, Example 2.3] where $r$ is not monotone).

In order to extend Theorem 2.2 in this manner, as in [2], we shall make an additional assumption. As in [2] we introduce the function $r(t)$ defined by
\[ r(t) = \begin{cases} t, & \text{if } t \text{ is right-dense and left-scattered} \\ \rho(t), & \text{otherwise}. \end{cases} \quad (2.19) \]

It is clear that $r(\sigma(t)) = t$ for any time scale $T$ satisfying condition (C). The additional assumption which will be needed is a monotonicity assumption on the expression $r(\beta(t))^\Delta$. That is, we shall assume that $(r(\beta(t))^\Delta$ is nonincreasing. Clearly, if $T = \mathbb{R}$, $T = \mathbb{Z}_0$ or $T = q^\mathbb{Z}$, it is easy to see that $(r(\beta(t))^\Delta = (\rho(\beta(t))^\Delta$ is nonincreasing. We note, as was established in [2], that this need not hold (i.e., one can find an example [2, Example 2.3] where $(\rho(\beta(t))^\Delta$ is not monotone).

**Theorem 2.3** Assume that $T$ satisfies condition (C) and let \[ \hat{T} = \{ t_i \in T | 0 < t_1 < t_2 < \cdots < t_n < \cdots \}. \]
If there exists a real number $\beta$, $0 \leq \beta \leq 1$ such that the delta derivative $(r(\beta(t))^\Delta$ is nonincreasing, where the function $r(t)$ is defined in (2.19), and
\[ \int_{t_1}^{\infty} t^{\alpha} \rho|p(t)| \Delta t = \infty, \]
then (1.1) is oscillatory.
Proof. Assume that (1.1) is nonoscillatory. Then without loss of generality there is a solution \( x(t) \) of (1.1) and a \( T \in \mathbb{T} \) with \( x(t) > 0 \), for all \( t \in [T, \infty) \). Making the substitution \( x(t) = r^\beta(t)u(t) \) and noticing that

\[
\begin{align*}
\Delta x(t) &= (r^\beta(t))\Delta u(\sigma(t)) + r^\beta(t)u^\Delta(t) \\
\Delta^2 x(t) &= (r^\beta(t))\Delta^2 u(\sigma(t)) + (r^\beta(t))\Delta[\sigma(t)]u(\sigma(t))^\Delta \\
&\quad+ (r^\beta(t))\Delta u^\Delta(t) + r^\beta(\sigma(t))u^\Delta(t) \\
&= (r^\beta(t))\Delta^2 u(\sigma(t)) + (r^\beta(t))\Delta[\sigma(t)]u(\sigma(t))^\Delta \\
&\quad+ (r^\beta(t))\Delta u^\Delta(t) + t^\beta u^\Delta(t)
\end{align*}
\]

we get from (1.1) that

\[
(r^\beta(t))\Delta^2 u(\sigma(t)) + (r^\beta(t))\Delta[\sigma(t)]u(\sigma(t))^\Delta + (r^\beta(t))\Delta u^\Delta(t) + t^\beta u^\Delta(t) = 0.
\]

Multiplying by \( \frac{1}{u^\alpha(\sigma(t))} \) and integrating from \( T \) to \( t \) we get

\[
\int_T^t \frac{(r^\beta(s))\Delta^2 u(\sigma(s))}{u^\alpha(\sigma(s))} \Delta s + \int_T^t \frac{(r^\beta(s))\Delta[\sigma(s)]u(\sigma(s))^\Delta}{u^\alpha(\sigma(s))} \Delta s = 0.
\]

(2.20)

But, using \( r(\sigma(t)) = t \), and then integrating by parts we get that the fourth term in (2.20) is given by

\[
\int_T^t \frac{s^\beta u^\Delta(s)}{u^\alpha(s)} \Delta s = \int_T^t \frac{r^\beta(s)u^\Delta(\sigma(s))}{u^\alpha(\sigma(s))} \Delta s
\]

\[
= \left[ \frac{r^\beta(s)u^\Delta(s)}{u^\alpha(s)} \right]_T^t - \int_T^t \frac{[r^\beta(s)]^\beta u^\Delta(s)}{u^\alpha(s)} \Delta s + \int_T^t \frac{r^\beta(s)[u^\alpha(s)]^\beta u^\Delta(s)}{u^\alpha(s)u^\alpha(\sigma(s))} \Delta s.
\]

Using the Pötzsche chain rule we get the third term on the last line satisfies

\[
\int_T^t \frac{r^\beta(s)[u^\alpha(s)]^\beta u^\Delta(s)}{u^\alpha(s)u^\alpha(\sigma(s))} \Delta s = \alpha \int_T^t \frac{r^\beta(s)}{u^\alpha(s)u^\alpha(\sigma(s))} \Delta s
\]

\[
= \left[ \frac{r^\beta(s)}{u^\alpha(s)u^\alpha(\sigma(s))} \right]_T^t + \int_T^t \frac{r^\beta(s)[u^\alpha(s)]^\beta u^\Delta(s)}{u^\alpha(s)u^\alpha(\sigma(s))} \Delta s \geq 0,
\]

which implies that the fourth term in (2.20) satisfies the inequality

\[
\int_T^t \frac{s^\beta u^\Delta(s)}{u^\alpha(s)} \Delta s \geq \left[ \frac{r^\beta(s)u^\Delta(s)}{u^\alpha(s)} \right]_T^t - \int_T^t \frac{(r^\beta(s))^\beta u^\Delta(s)}{u^\alpha(\sigma(s))} \Delta s.
\]

(2.21)

Using this last inequality we get from (2.20) that

\[
\int_T^t \frac{(r^\beta(s))\Delta^2 u(\sigma(s))}{u^\alpha(\sigma(s))} \Delta s + \int_T^t \frac{(r^\beta(s))\Delta[\sigma(s)]u(\sigma(s))^\Delta}{u^\alpha(\sigma(s))} \Delta s
\]

\[
\quad+ \frac{r^\beta(t)u^\Delta(t)}{u^\alpha(t)} - \frac{r^\beta(T)u^\Delta(T)}{u^\alpha(T)} + \int_T^t s^\alpha p(s) \Delta s \leq 0.
\]
Note that
\[
\int_T^t \frac{(r^\beta(t)\Delta u(s))}{u^\alpha(s)} \Delta s = \frac{(r^\beta(t)\Delta u(s))}{u^\alpha(t)} - \frac{(r^\beta(t)\Delta u(T))}{u^\alpha(T)} - \int_T^t \frac{(r^\beta(s)\Delta u(s))}{u^\alpha(s)} \Delta s + \int_T^t \frac{(u^\alpha(s)\Delta (r^\beta(s)\Delta u(s)))}{u^\alpha(s)u^\alpha(s)} \Delta s.
\]

Let \( A := \frac{(r^\beta(T)\Delta u(T))}{u^\alpha(T)} \), \( B := \frac{(r^\beta(T)\Delta u(T))}{u^\alpha(T)} \). From (2.22), (2.23), we get that
\[
\frac{r^\beta(t)u(t)}{u^\alpha(t)} - B + \int_T^t \frac{(u^\alpha(s)\Delta (r^\beta(s)\Delta u(s)))}{u^\alpha(s)u^\alpha(s)} \Delta s - A + \int_T^t s^\alpha\beta p(s) \Delta s \leq 0.
\]

Note that \( r^\beta(t) \geq 0 \) and \( u^\alpha(t) = 0 \), so \( (r^\beta(t)) \Delta = \beta \int_0^1 [(1 - h)r(t) + ht]^{\beta-1} dh r^\Delta(t) \geq 0 \), for \( t > 0 \). So
\[
\frac{(r^\beta(t)\Delta u(s))}{u^\alpha(s)} \geq 0 \text{ and from (2.24), we get that}
\]
\[
\frac{r^\beta(t)u(t)}{u^\alpha(t)} - B + \int_T^t \frac{(u^\alpha(s)\Delta (r^\beta(s)\Delta u(s)))}{u^\alpha(s)u^\alpha(s)} \Delta s - A + \int_T^t s^\alpha\beta p(s) \Delta s \leq 0.
\]

Since \( (r^\beta(s)\Delta \) is nonincreasing, by the second mean value theorem [9, Theorem 5.45]), we get that for each \( t \in [T, \infty) \)
\[
\int_T^t \frac{(u^\alpha(s)\Delta (r^\beta(s)\Delta u(s)))}{u^\alpha(s)u^\alpha(s)} \Delta s = (r^\beta(s))\Delta |T\Lambda(t)
\]
where \( m_u \leq \Lambda(t) \leq M_u \), and where \( m_u \) and \( M_u \) denote the infimum and supremum, respectively, of the function \( \int_T^t \frac{(u^\alpha(s)\Delta (r^\beta(s)\Delta u(s)))}{u^\alpha(s)u^\alpha(s)} \Delta s \).

From (2.10), we get that
\[
\int_T^t \frac{(u^\alpha(s)\Delta u(s))}{u^\alpha(s)u^\alpha(s)} \Delta s \geq \frac{\alpha}{1 - \alpha} [-u^{1-\alpha}(T)].
\]

Therefore from (2.25), (2.26), we get that
\[
\frac{r^\beta(t)u(t)}{u^\alpha(t)} - B - (r^\beta(s))\Delta |T\frac{\alpha u^{1-\alpha}(T)}{1 - \alpha} - A + \int_T^t s^\alpha\beta p(s) \Delta s \leq 0.
\]

Since \( \int_T^\infty \sigma^\alpha\beta(p(t)) \Delta t = \infty \), from (2.28), there exists \( T_1 \geq T \) such that for \( t \geq T_1 \), we have
\[
\frac{r^\beta(t)(u(t))\Delta}{u^\alpha(t)} \leq -1.
\]

Integrating from \( T_1 \) to \( t \), using an inequality of [3] and [9, Theorem 5.68] and noticing that \( r(t) \leq t \), we get that, as \( t \to \infty \)
\[
\frac{u^{1-\alpha}(t) - u^{1-\alpha}(T_1)}{1 - \alpha} \leq \int_{T_1}^t \frac{u^\Delta(s)}{u^\alpha(s)} \Delta s \leq -\int_{T_1}^t \frac{1}{r^\beta(s)} \Delta s \leq -\int_{T_1}^t \frac{1}{s^\beta} \Delta s \to -\infty.
\]
Therefore \( u^{1-\alpha}(t) < 0 \), for large \( t \), which is a contradiction. Thus equation (1.1) is oscillatory.

When \( \mathbb{T} = \mathbb{Z} \) and \( \mathbb{T} = q^\mathbb{N}_0, q > 1 \), we get the following corollaries. Corollary 2.4 shows that with no sign assumption on \( p(n) \), the condition

\[
\sum_{n} n^\alpha p(n) = \infty
\]

is sufficient for the oscillation of the difference equation (1.3).

**Corollary 2.4** Assume \( \mathbb{T} = \mathbb{Z} \). If

\[
\sum_{n} n^\alpha p(n) = \infty
\]

then (1.3) is oscillatory.

**Corollary 2.5** Assume \( \mathbb{T} = q^\mathbb{N}_0 \). If

\[
\int_{1}^{\infty} t^\alpha p(t) \Delta t = \infty
\]

then the \( q \)-difference equation \( x^\Delta \Delta(t) + p(t)x^\alpha(t) = 0 \), is oscillatory.

**Remark 2.6** By using the idea in Theorem 2.2, we can give a simple proof of Belohorec’s Theorem (see [6]). Consider the differential equation

\[
x^{\nu} + p(t)x^{\alpha}(t) = 0, \quad 0 < \alpha < 1,
\]

(2.29)

where \( \int_{1}^{\infty} t^\beta p(t) dt = \infty \), for some \( 0 \leq \beta \leq 1 \). Suppose \( x(t) > 0 \) for \( t \geq T > 1 \) is a nonoscillatory solution. Let \( x(t) = t^\beta u(t) \). Then from (2.29), we get

\[
\beta(\beta - 1) t^{\beta - 2} u(t) + 2 \beta t^{\beta - 1} u'(t) + t^\beta u''(t) + t^{\alpha \beta} p(t)u^\alpha(t) = 0.
\]

Multiplying \( \frac{1}{u^\alpha(t)} \) and integrating from \( T \) to \( t \), we get

\[
\beta(\beta - 1) \int_{T}^{t} s^{\beta - 2} \frac{u(s)}{u^\alpha(s)} ds + 2 \beta \int_{T}^{t} s^{\beta - 1} \frac{u'(s)}{u^\alpha(s)} ds + \int_{T}^{t} s^\beta \frac{u''(s)}{u^\alpha(s)} ds + \int_{T}^{t} s^{\alpha \beta} p(s) ds = 0.
\]

(2.30)

Note that

\[
\int_{T}^{t} s^{\beta} \frac{u''(s)}{u^\alpha(s)} ds = t^\beta \frac{u''(t)}{u^\alpha(t)} - \frac{T^\beta u''(T)}{u^\alpha(T)} - \int_{T}^{t} \frac{\beta s^{\beta - 1} u'(s)}{u^\alpha(s)} ds + \int_{T}^{t} \frac{\alpha s^{\beta} [u'(s)]^2}{u^{\alpha + 1}(s)} ds \geq t^\beta \frac{u''(t)}{u^\alpha(t)} - \frac{T^\beta u''(T)}{u^\alpha(T)} - \int_{T}^{t} \frac{\beta s^{\beta - 1} u'(s)}{u^\alpha(s)} ds.
\]

So by (2.30), we get that

\[
\beta(\beta - 1) \int_{T}^{t} s^{\beta - 2} \frac{u(s)}{u^\alpha(s)} ds + \beta \int_{T}^{t} s^{\beta - 1} \frac{u'(s)}{u^\alpha(s)} ds + \int_{T}^{t} s^{\alpha \beta} p(s) ds \leq 0.
\]

(2.31)

Applying integration by parts to the second term in (2.31), we get that

\[
\beta \int_{T}^{t} s^{\beta - 1} \frac{u'(s)}{u^\alpha(s)} ds = \frac{\beta t^{\beta - 1} u(t)}{u^\alpha(t)} - \frac{\beta T^{\beta - 1} u(T)}{u^\alpha(T)} - \beta(\beta - 1) \int_{T}^{t} s^{\beta - 2} \frac{u(s)}{u^\alpha(s)} ds + \alpha \beta \int_{T}^{t} \frac{1}{s^{1 - \beta}} \frac{u'(s)}{u^\alpha(s)} ds.
\]

(2.32)
But, since $0 \leq \beta \leq 1$, \( \frac{1}{1-\beta} \) is nonincreasing, we have by the (continuous) second mean value theorem [4, Exercise 30 N, page 236] there is a point $\xi \in (T, t)$, which could depend on $t$, such that

\[
\int_T^t \frac{1}{s^{1-\beta}} \frac{u'(s)}{u^\alpha(s)} ds = \frac{1}{T^{1-\beta}} \int_T^\xi \frac{u'(s)}{u^\alpha(s)} ds
\]

\[
= \frac{1}{T^{1-\beta}} \left[ \frac{u^{1-\alpha}(s)}{1-\alpha} \right]_T^\xi \geq -\frac{u^{1-\alpha}(T)}{(1-\alpha)T^{1-\beta}}.
\]

Then from (2.31), (2.32), and (2.33) there is a constant $K$ such that

\[
\frac{\beta t^{\beta-1}u(t)}{u^\alpha(t)} + \frac{t^{\beta}u'(t)}{u^\alpha(t)} \leq K - \int_t^T \frac{1}{s^{1-\beta}} \left[ \frac{u^{1-\alpha}(s)}{1-\alpha} \right] ds.
\]

Since $\int_\infty^\infty t^{\alpha\beta}p(t) dt = \infty$, it follows from (2.34), that there exists $T_1 \in \mathbb{T}$ such that for $t \in [T_1, \infty)$,

\[
\frac{u'(t)}{u^\alpha(t)} \leq -\frac{1}{t^\beta}.
\]

Integrating from $T_1$ to $t$, we get that as $t \to \infty$

\[
\frac{1}{1-\alpha} [u^{1-\alpha}(t) - u^{1-\alpha}(T_1)] \leq -\int_{T_1}^t \frac{1}{s^\beta} ds \to -\infty.
\]

Therefore $u^{1-\alpha}(t) < 0$, for large $t$, which is a contradiction. Thus, equation (2.29) is oscillatory.

### 3 Example

Consider the sublinear difference equation

\[
\Delta^2 x(n) + \left[ \frac{1}{n^{\beta+1}} + b \frac{(-1)^n}{n^\beta} \right] x^\alpha(n+1) = 0
\]

(3.1)

where $\beta \geq 0$, and $b$ is any real number. Since

\[
\sum_{n=1}^\infty n^\alpha \left[ \frac{1}{n^{\beta+1}} + b \frac{(-1)^n}{n^\beta} \right] = \infty,
\]

(3.2)

for $0 \leq \beta \leq \alpha$, by Corollary 2.4, (3.1) is oscillatory.

Note that we can also use Theorem 2.2 to obtain the fact that (3.1) is oscillatory, since (taking $\alpha = 1$)

\[
\sum_{n=1}^\infty (n+1)^{\alpha\beta} \left[ \frac{1}{n^{\beta+1}} + b \frac{(-1)^n}{n^\beta} \right] = \sum_{n=1}^\infty \left( 1 + \frac{1}{n} \right)^{\beta} \left[ \frac{1}{n} + b(-1)^n \right]
\]

\[
= \sum_{n=1}^\infty \left( 1 + \frac{\beta}{n} + o \left( \frac{1}{n} \right) \right) \left[ \frac{1}{n} + b(-1)^n \right]
\]

\[
= \sum_{n=1}^\infty \left( \frac{1}{n} + b(-1)^n + \beta \frac{(-1)^n}{n} + O \left( \frac{1}{n^2} \right) \right)
\]

\[
= \infty.
\]
In the following, using [11, Theorem 2.1], we will prove that when $\beta > \alpha$, (3.1) has a nonoscillatory solution, which shows sharpness of the oscillation result. By [7] and [11], we have

$$
\sum_{n}^{\infty} \frac{1}{i^{\beta+1}i^{\beta}} = \frac{1}{\beta n^{\beta}} + o \left( \frac{1}{n^{\beta}} \right).
$$

$$
\sum_{n}^{\infty} \frac{(-1)^i}{i^{\beta}} = \frac{(-1)^n}{2n^{\beta}} + o \left( \frac{1}{n^{\beta}} \right).
$$

So

$$
\left| \sum_{n}^{\infty} \left( \frac{1}{i^{\beta+1}i^{\beta}} + \frac{(-1)^i}{i^{\beta}} \right) \right| \leq \frac{A}{n^{\beta}},
$$

where $A$ is constant. Let $F(n) = \frac{A+n\epsilon}{n^{\beta}}$, then we have

$$
P(n) \leq F(n+1).
$$

At the end of Example 1 in [11], we proved that

$$
n^{\alpha}|(n+1)^{-\beta} - n^{-\beta}| \sim \frac{\beta}{n^{1-\alpha+\beta}}.
$$

Using this we obtain

$$
\int_{1}^{\infty} t^{\alpha}|F\Delta(t)|\Delta < \infty.
$$

By [11, Theorem 2.1], we get that (3.1) has a nonoscillatory solution.

References