Oscillation of Sublinear Emden-Fowler Dynamic Equations on Time Scales

Jia Baoguo*, Lynn Erbe, and Allan Peterson
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130, U.S.A.
lerbe2@math.unl.edu, apeterson1@math.unl.edu,
Jia Baoguo
School of Mathematics and Computer Science
Zhongshan University
Guangzhou, China, 510275
mcsjbg@mail.sysu.edu.cn

This paper is dedicated to Professor Peter E. Kloeden

Abstract. Consider the Emden-Fowler sublinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(\sigma(t)) = 0, \tag{0.1}$$

where $p \in C(\mathbb{T}, R)$, where $\mathbb{T}$ is a time scale, $0 < \alpha < 1$, α is the quotient of odd positive integers. We obtain a Kamenev-type oscillation theorem for (0.1). As applications, we get that the sublinear difference equation

$$\Delta^2 x(n) + b(-1)^n n^c x^{\alpha}(n+1) = 0, \tag{0.2}$$

where $0 < \alpha < 1$, $b > 0$, $c > 1$, is oscillatory.
Keywords and Phrases: oscillation; Emden-Fowler equation; sublinear
2000 AMS Subject Classification: 34K11, 39A10, 39A99.

1. Introduction

Consider the second order sublinear dynamic equation

$$x^{\Delta\Delta} + p(t)x^{\alpha}(\sigma(t)) = 0, \tag{1.1}$$

where $p \in C(\mathbb{T}, R)$, $0 < \alpha < 1$, and where α is the quotient of odd positive integers. When $\mathbb{T} = \mathbb{R}$ and $\alpha = 1$, the dynamic equation (1.1) is the second

*Supported by the Guangdong Provincial Natural Science Foundation of China .

order sublinear differential equation

$$x''(t) + p(t)x^{\alpha}(t) = 0. \tag{1.2}$$

When $\alpha = 1$, the differential equation (1.2) is the second order linear differential equation

$$x'' + p(t)x = 0. \tag{1.3}$$

Wintner [**11**] proved if

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left[\int_{t_0}^{s} p(\tau)d\tau \right] ds = \infty, \tag{1.4}$$

all solutions of (1.3) are oscillatory. Hartman [**10**] has proved (1.4) cannot be replaced by

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left[\int_{t_0}^{s} p(\tau)d\tau \right] ds = \infty. \tag{1.5}$$

However, in the nonlinear cases this is not true. Kamenev [**8**] proved that (1.5) ensures that all regular solutions of (1.2) (i.e., all solutions infinitely continuable to the right) are oscillatory, for $0 < \alpha < 1$. Many additional references to earlier work for both the superlinear ($\alpha > 1$) and the sublinear ($0 < \alpha < 1$) cases may be found in Wong [**6**] and the references therein.

In this paper, we extend Kamenev's oscillation theorem to dynamic equations on time scales and as an application, we show that the sublinear difference equation

$$\Delta^2 x(n) + b(-1)^n n^c x^{\alpha}(n+1) = 0 \tag{1.6}$$

is oscillatory, for $0 < \alpha < 1$, $b > 0$, $c > 1$. This equation is a discrete analog of the equation (1.2) with $p(t) = t^{\lambda} \sin t$.

For completeness, (see [**4**] and [**5**] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\sup \varnothing = \inf \mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c, d]_{\mathbb{T}} := \{t \in \mathbb{T} : c \le t \le d\}$ in $\mathbb{T}$ the notation $[c, d]^{\kappa}{}_{\mathbb{T}}$ denotes the interval $[c, d]_{\mathbb{T}}$ in case $\rho(d) = d$ and denotes the interval $[c, d)_{\mathbb{T}}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by

$\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^\Delta(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales.

2. Main Theorem

Following Philos [**7**], we consider a non-negative kernel function $h(t,s)$ defined on $D = \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. We shall assume that $h(t,s)$ satisfies the following conditions:

(H_1) $h(t,t) \equiv 0$ for $t \geq t_0$,

(H_2) $h^{\Delta_s}(t,s) \leq 0$ for $t \geq s \geq t_0$,

where $h^{\Delta_s}(t,s)$ denotes the partial delta derivative of h with respect to s.

(H_3) $h^{\Delta_s^2}(t,s) \geq 0$ for $t \geq s \geq t_0$,

where $h^{\Delta_s^2}(t,s)$ denotes the second order partial delta derivative of h with respect to s.

(H_4) $-h^{-1}(t,t_0)h^{\Delta_s}(t,s)|_{s=t_0} \leq M_0$ for large t.

We will also need the following second mean value theorem (see [**5**, page 143]).

Lemma 2.1. *Let f be a bounded function that is integrable on $[a,b]_\mathbb{T}$. Let m_F and M_F be the infimum and supremum, respectively, of the function $F(t) = \int_a^t f(s)\Delta s$ on $[a,b]_\mathbb{T}$. Suppose that g is nonincreasing with $g(t) \geq 0$ on $[a,b]_\mathbb{T}$. Then there is some number Λ with $m_F \leq \Lambda \leq M_F$ such that*

$$\int_a^b f(t)g(t)\Delta t = g(a)\Lambda.$$

Let $\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following (see [**2**] and [**3**]).

Condition (C) We say that $\mathbb{T}$ satisfies condition C if there is an $M > 0$ such that

$$\chi(t) \leq M\mu(t), \quad t \in \mathbb{T}.$$

We note that if $\mathbb{T}$ satisfies condition (C), then the set

$$\check{\mathbb{T}} = \{t \in \mathbb{T} |\ t > 0 \text{ is isolated or right-scattered or left-scattered}\}$$

is necessarily countable.

THEOREM 2.2. *Assume that $\mathbb{T}$ satisfies condition (C) and suppose there exists a non-negative kernel function $h(t,s)$ on D satisfying $(H_1)--(H_4)$, such that*

$$\limsup_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} h(t,\sigma(s))p(s)\Delta s = \infty. \tag{2.1}$$

Then all regular solutions (i.e., all solutions infinitely continuable to the right) of (1.1) are oscillatory.

REMARK 2.3. In the case $\mathbb{T} = \mathbb{R}$, an additional requirement was imposed on $h(t,s)$, namely, $\frac{\partial h}{\partial s}(t,s)|_{s=t} \equiv 0$. Therefore when $\mathbb{T} = \mathbb{R}$, the above theorem improves [**6**, Theorem 1].

REMARK 2.4. Let $h(t,s) = (t-s)^\gamma, \gamma > 0$. It is easy to see that $h(t,s)$ satisfies $(H_1)--(H_4)$. So we get the following Kamenev-type criterion on time scales (Note that in [**6**] it is assumed that $\gamma > 1$.)

We define for $t > t_0$

$$G(t,t_0) := \frac{1}{(t-t_0)^\gamma} \int_{t_0}^{t} [t-\sigma(s)]^\gamma p(s)\Delta s.$$

COROLLARY 2.5. *Assume that $\mathbb{T}$ satisfies condition (C). If there exists real number $\gamma > 0$ such that*

$$\limsup_{t\to\infty} G(t,t_0) = \infty, \tag{2.2}$$

then all regular solutions of (1.1) are oscillatory.

REMARK 2.6. Using integration by parts, we have

$$\begin{aligned} & \frac{1}{t}\int_{t_0}^{t} \left[\int_{t_0}^{s} p(\tau)\Delta\tau\right] \Delta s \\ = & \frac{1}{t}\left[t\int_{t_0}^{t} p(s)\Delta s - \int_{t_0}^{t} (\sigma(s))p(s)\Delta s\right] \\ = & \frac{1}{t}\int_{t_0}^{t} [t-\sigma(s)]p(s)\Delta s. \end{aligned}$$

So when $\mathbb{T} = \mathbb{R}$, $\gamma = 1$, Corollary 2.5 is Kamenev's theorem [**8**].

PROOF. Assume that (1.1) is nonoscillatory. Then without loss of generality there is a solution $x(t)$ of (1.1) and a $T \in \mathbb{T}$ with $x(t) > 0$, for all $t \in [T, \infty)_{\mathbb{T}}$. Make the Riccati substitution $w(t) = \frac{x^{\Delta}(t)}{x^{\alpha}(t)}$, $t \geq t_0$. Differentiating w and using the Pötzsche chain rule [**4**, Theorem 1.90] we get that

$$w^{\Delta}(t) = -p(t) - w^2(t)\frac{x^{\alpha}(t)}{x^{\alpha}(\sigma(t))}\left[\int_0^1 \alpha(x_h(t))^{\alpha-1}dh\right],$$

where $x_h(t) = x(t) + h\mu(t)x^{\Delta}(t) = (1-h)x(t) + hx(\sigma(t)) > 0$. So we get that

$$w^{\Delta}(t) \leq -p(t). \tag{2.3}$$

Multiplying (2.3) by $h(t, \sigma(s))$ and integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} h(t,\sigma(s))w^{\Delta}(s)\Delta s \leq -\int_{t_0}^{t} h(t,\sigma(s))p(s)\Delta s. \tag{2.4}$$

Now integrating by parts and using the second mean value theorem (Lemma 2.1) and $(H_1) - -(H_3)$, we get that

$$\begin{aligned}\int_{t_0}^{t} h(t,\sigma(s))w^{\Delta}(s)\Delta s &= -h(t,t_0)w(t_0) - \int_{t_0}^{t} h^{\Delta_s}(t,s)w(s)\Delta s \\ &= -h(t,t_0)w(t_0) - h^{\Delta_s}(t,s)|_{s=t_0}\Lambda,\end{aligned} \tag{2.5}$$

where $m_x \leq \Lambda \leq M_x$, and where m_x and M_x denote the infimum and supremum, respectively, of the function $\int_{t_0}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\Delta s$.

In the following, we will obtain an estimate for m_x, i.e., a lower bound for the function $\int_{t_0}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\Delta s$.

Assume first that $t = t_1 < t_2 = \sigma(t)$. Then

$$\int_{t}^{\sigma(t)} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\Delta s = \frac{x^{\Delta}(t)\mu(t)}{x^{\alpha}(t)} = \frac{x(\sigma(t)) - x(t)}{x^{\alpha}(t)}. \tag{2.6}$$

We consider the two possible cases $x(t) \leq x(\sigma(t))$ and $x(t) > x(\sigma(t))$. First if $x(t) \leq x(\sigma(t))$ we have that

$$\frac{x(\sigma(t)) - x(t)}{x^{\alpha}(t)} \geq \int_{x(t)}^{x(\sigma(t))} \frac{1}{s^{\alpha}}ds = \frac{1}{1-\alpha}[x^{1-\alpha}(\sigma(t)) - x^{1-\alpha}(t)]. \tag{2.7}$$

On the other hand if $x(t) > x(\sigma(t))$, then

$$\frac{x(t) - x(\sigma(t))}{x^{\alpha}(t)} \leq \int_{x(\sigma(t))}^{x(t)} \frac{1}{s^{\alpha}}ds = \frac{1}{1-\alpha}[x^{1-\alpha}(t) - x^{1-\alpha}(\sigma(t))],$$

which implies that

$$\frac{x(\sigma(t)) - x(t)}{x^{\alpha}(t)} \geq \frac{1}{1-\alpha}[x^{1-\alpha}(\sigma(t)) - x^{1-\alpha}(t)]. \tag{2.8}$$

Hence, whenever $t_1 = t < \sigma(t) = t_2$, we have that from (2.6) and (2.7) in the one case and (2.4) and (2.6) in the other case that

$$\int_{t_1}^{t_2} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\Delta s \geq \frac{1}{1-\alpha}[x^{1-\alpha}(\sigma(t)) - x^{1-\alpha}(t)]. \tag{2.9}$$

If the real interval $[t_1, t_2] \subset \mathbb{T}$, then

$$\int_{t_1}^{t_2} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\Delta s = \frac{1}{1-\alpha}[x^{1-\alpha}(t_2) - x^{1-\alpha}(t_1)] \tag{2.10}$$

and so (2.9) holds.

Note that since $\mathbb{T}$ satisfies condition (C), we have from (2.9), (2.10) and the additivity of the integral that

$$\int_{t_0}^{t} \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\Delta s \geq \frac{1}{1-\alpha}[x^{1-\alpha}(t) - x^{1-\alpha}(t_0)] \geq -\frac{x^{1-\alpha}(t_0)}{1-\alpha}. \tag{2.11}$$

So

$$\Lambda \geq m_x \geq -\frac{x^{1-\alpha}(t_0)}{1-\alpha}. \tag{2.12}$$

From (2.4), (2.5) and (2.12), we have that

$$-h(t,t_0)w(t_0) + h^{\Delta_s}(t,s)|_{s=t_0} \cdot \frac{x^{1-\alpha}(t_0)}{1-\alpha} \leq -\int_{t_0}^{t} h(t,\sigma(s))p(s)\Delta s. \tag{2.13}$$

Dividing by $h(t, t_0)$ and using (H_4), we arrive at

$$w(t_0) + M_0 \cdot \frac{x^{1-\alpha}(t_0)}{1-\alpha} \geq \frac{1}{h(t,t_0)}\int_{t_0}^{t} h(t,\sigma(s))p(s)\Delta s. \tag{2.14}$$

We can now use (2.1) to deduce from (2.14) a desired contradiction upon taking lim sup as $t \to \infty$. Thus equation (1.1) is oscillatory. □

3. Example

EXAMPLE 3.1. Consider the difference equation

$$\Delta^2 x(n) + p(n)x^{\alpha}(n+1) = 0 \tag{3.1}$$

where $p(n) = b(-1)^n n^c$, $b > 0$, $c > 1$. We need the following two lemmas. The first lemma may be regarded as a discrete version of L'Hopital's rule and can be found in [**4**, page 48].

LEMMA 3.2. *(Stolz–Cesáro Theorem) Let* $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be two sequences of real number. If* b_n *is positive, strictly increasing and unbounded and the following limit exists:*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

We will use Lemma 3.2 to prove the following result.

LEMMA 3.3. *For each real number $d > 0$, we have*

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{m} i^d}{m^{d+1}} = \frac{1}{d+1}, \tag{3.2}$$

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{m} i^d - \frac{m^{d+1}}{d+1}}{m^d} = \frac{1}{2}. \tag{3.3}$$

For each real number $c > 1$, we have

$$\lim_{m\to\infty} \frac{\sum_{i=1}^{m} i^c - \frac{m^{c+1}}{c+1} - \frac{m^c}{2}}{m^{c-1}} = \frac{c}{12}. \tag{3.4}$$

PROOF. Here we only prove (3.4) as the proofs of (3.2) and (3.3) are similar.

By Taylor's formula, we have

$$(1+\frac{1}{m})^a = 1 + \frac{a}{m} + \frac{a(a-1)}{2m^2} + \frac{a(a-1)(a-2)}{6m^3} + o\left(\frac{1}{m^3}\right), \tag{3.5}$$

for any real number a. For $c > 1$, by (3.5) and the Stolz–Cesáro Theorem (Lemma 3.2), it is easy to see that

$$\begin{aligned} & \lim_{m\to\infty} \frac{\sum_{i=1}^{m} i^c - \frac{m^{c+1}}{c+1} - \frac{m^c}{2}}{m^{c-1}} \\ = & \lim_{m\to\infty} \frac{(m+1)^c - \frac{(m+1)^{c+1}}{c+1} - \frac{(m+1)^c}{2} + \frac{m^{c+1}}{c+1} + \frac{m^c}{2}}{(m+1)^{c-1} - m^{c-1}} \\ = & \lim_{m\to\infty} \frac{\frac{m}{2}(1+\frac{1}{m})^c - \frac{m^2}{c+1}(1+\frac{1}{m})^{c+1} + \frac{m^2}{c+1} + \frac{m}{2}}{(1+\frac{1}{m})^{c-1} - 1}. \end{aligned} \tag{3.6}$$

By (3.5), we have

$$\left(1+\frac{1}{m}\right)^{c-1} = 1 + \frac{c-1}{m} + o\left(\frac{1}{m}\right), \tag{3.7}$$

$$\left(1+\frac{1}{m}\right)^{c} = 1 + \frac{c}{m} + \frac{c(c-1)}{2m^2} + o\left(\frac{1}{m^2}\right), \tag{3.8}$$

$$\left(1+\frac{1}{m}\right)^{c+1} = 1 + \frac{c+1}{m} + \frac{c(c+1)}{2m^2} + \frac{c(c+1)(c-1)}{6m^3} + o\left(\frac{1}{m^3}\right). \tag{3.9}$$

Using (3.7)-(3.9) in (3.6), it follows that (3.4) holds. □

So given $0 < \epsilon < 1$, for large m, we have the inequality

$$\frac{m^{c+1}}{c+1} + \frac{m^c}{2} + \frac{c(1-\epsilon)}{12} m^{c-1} < \sum_{i=1}^{m} i^c < \frac{m^{c+1}}{c+1} + \frac{m^c}{2} + \frac{c(1+\epsilon)}{12} m^{c-1}. \tag{3.10}$$

Therefore for $t = m$, by integrating by parts we have

$$\begin{aligned}
I(t) &=: \frac{1}{t}\int_1^t \left[\int_1^s p(\tau)\Delta\tau\right]\Delta s \\
&= \frac{1}{t}\left[t\int_1^t p(s)\Delta s - \int_1^t (\sigma(s))p(s)\Delta s\right] \\
&= \int_1^t p(s)\Delta s - \frac{1}{t}\int_1^t (s+1)p(s)\Delta s \\
&= b\left[\sum_{n=1}^{m-1}(-1)^n n^c - \frac{1}{m}\sum_{n=1}^{m-1}(-1)^n n^c(n+1)\right] \\
&= b\left[\sum_{n=1}^{m-2}(-1)^n n^c - \frac{1}{m}\sum_{n=1}^{m-2}(-1)^n n^c(n+1)\right],
\end{aligned}$$

since the terms corresponding to $n = m - 1$ cancel. Letting $m = 2k$, by (3.10) we have by rearranging

$$\begin{aligned}
&\frac{I(2k)}{b} = -\sum_{i=1}^{2k-2} i^c + 2^{c+1}\sum_{i=1}^{k-1} i^c + \frac{1}{2k}\sum_{i=1}^{2k-2} i^{c+1} \\
-\ & \frac{2^{c+2}}{2k}\sum_{i=1}^{k-1} i^{c+1} + \frac{1}{2k}\sum_{i=1}^{2k-2} i^c - \frac{2^{c+1}}{2k}\sum_{i=1}^{k-1} i^c \\
\geq\ & -\frac{(2k-2)^{c+1}}{c+1} - \frac{(2k-2)^c}{2} - \frac{c(1+\epsilon)}{12}\cdot(2k-2)^{c-1} \\
+\ & \frac{(2k-2)^{c+1}}{c+1} + \frac{(2k-2)^{c+1}}{2(k-1)} + \frac{c(1-\epsilon)}{12(k-1)^2}\cdot(2k-2)^{c+1} \\
+\ & \frac{(2k-2)^{c+2}}{2k(c+2)} + \frac{(2k-2)^{c+1}}{4k} + \frac{(c+1)(1-\epsilon)}{24k}\cdot(2k-2)^c \\
-\ & \frac{(2k-2)^{c+2}}{2k(c+2)} - \frac{(2k-2)^{c+1}}{2k} - \frac{(c+1)(1+\epsilon)}{6k}\cdot(2k-2)^c \\
+\ & \frac{(2k-2)^{c+1}}{2k(c+1)} + \frac{(2k-2)^c}{4k} + \frac{c(1-\epsilon)}{24k}\cdot(2k-2)^{c-1} \\
-\ & \frac{(2k-2)^{c+1}}{2k(c+1)} - \frac{(2k-2)^{c+1}}{2k(2k-2)} - \frac{c(1+\epsilon)}{6k}\cdot(2k-2)^{c-1}.
\end{aligned}$$

Observing that the first terms in the last six lines add to zero, and regrouping the last twelve terms in three groups of four respectively we can write

$$\begin{aligned}\frac{I(2k)}{b} &\geq (2k-2)^{c-1}\left[-(k-1)-\frac{c(1+\epsilon)}{12}+2(k-1)+\frac{c(1-\epsilon)}{3}\right]\\ &+ (2k-2)^{c-1}\left[\frac{(k-1)^2}{k}+(c+1)(1-\epsilon)\frac{k-1}{12k}\right.\\ &- \left.\frac{2(k-1)^2}{k}-(c+1)(1+\epsilon)\frac{k-1}{3k}\right]\\ &+ (2k-2)^{c-1}\left[\frac{k-1}{2k}+\frac{c(1-\epsilon)}{24k}-\frac{k-1}{k}-\frac{c(1+\epsilon)}{6k}\right].\end{aligned}$$

Factoring out $(2k-2)^{c-1}$ and simplifying we get

$$\frac{I(2k)}{b} \geq (2k-2)^{c-1}\left[\left(\frac{1}{4}-\frac{5(1+2c)\epsilon}{12}\right)+O\left(\frac{1}{k}\right)\right]. \tag{3.11}$$

Take $0<\epsilon<\frac{3}{5(2c+1)}$. From (3.11), for $c>1$, we obtain that

$$\limsup_{t\to\infty} I(t)=\infty.$$

By Corollary 2.5, equation (1.1) is oscillatory for $b>0$, $c>1$.

Remark 3.4. In fact, by (3.5) and (3.10), we can also prove that

$$\lim_{t\to\infty} I(2k+1)=0, \quad \text{for} \quad b>0, \;\; c>1.$$

Example 3.5. Consider the q-difference equation

$$x^{\Delta\Delta}(t)+p(t)x^{\alpha}(qt)=0 \tag{3.12}$$

where $p(t)=b(-1)^n t^c$, $t=q^n\in\mathbb{T}=q_0^{\mathbb{N}}$, $q>1$, $b>0$, $c>-1$, $0<\alpha<1$.

For $t=q^m$, we have

$$\begin{aligned}I(t) &=: \frac{1}{t}\int_1^t\left[\int_1^s p(\tau)\Delta\tau\right]\Delta s\\ &= \int_1^t p(s)\Delta s-\frac{1}{t}\int_1^t \sigma(s)p(s)\Delta s\\ &= b\left[\sum_{n=1}^{m-1}(-1)^n q^{cn}(q-1)q^n-\frac{1}{q^m}\sum_{n=1}^{m-1}(-1)^n q^{(c+1)n+1}(q-1)q^n\right].\end{aligned}$$

Take $m=2k$. We get that

$$\begin{aligned}\frac{I(q^{2k})}{b(q-1)q^{c+1}} &= -\frac{q^{(2k-1)(c+1)}+1}{q^{c+1}+1}+\frac{q^2(q^{(2k-1)(c+2)}+1)}{q^{2k}(q^{c+2}+1)}\\ &= \frac{q^{(2k-1)(c+1)}(q-1)+q^{-2k+1}(q^{c+1}+1)-q^{c+2}-1}{(1+q^{c+1})(1+q^{c+2})}.\end{aligned}$$

So

$$\limsup_{t\to\infty} I(t) = \infty.$$

By Corollary 2.5, equation (3.12) is oscillatory for $b > 0$, $c > -1$.

References

[1] Martin Bohner, Lynn Erbe, and Allan Peterson, Oscillation for nonlinear second order dynamic equations on a time scale, J. Math. Anal. Appl., 301 (2005) 491–507.

[2] Lynn Erbe, Oscillation criteria for second order linear equations on a time scale, Canad. Appl. Math. Quart., 9 (2001) 346–375.

[3] Jia Baoguo, Lynn Erbe and Allan Peterson, A Wong-type oscillation theorem for second order linear dynamic equations on time scales, J. Difference Equs. Appl., to appear.

[4] M. Bohner and A. Peterson, Dynamic Equation on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[5] M. Bohner and A. Peterson, editors, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[6] J. S. W. Wong, Oscillation criteria for second order nonlinear differential equations involving general means, J. Math. Anal. Appl., 247 (2000), 489-505.

[7] Ch. G. Philos, Oscillation theorems for linear differential equations of second order, Arch. Appl. 53(1989), 482-492.

[8] I. V. Kamenev, Some specially nonlinear oscillation theorems, Mat. Zametki, 10 (1971) 129–134.

[9] Walter Kelley and Allan Peterson, The Theory of Differential Equations Classical and Qualitative, Pearson Education, Inc., Upper Saddle River, New Jersey, 2004.

[10] P. Hartman, On nonoscillatory linear differential equations of second order, Amer. J. Math., 74 (1952) 489–500.

[11] A. Wintner, A criterion of oscillatory stability, Quart. Appl. Math., 7 (1940) 115–117.