Correction of Theorem 4.13

**Theorem 4.13** (Discrete Floquet’s Theorem) If \( \Phi(t) \) is a fundamental matrix for the Floquet system (4.16), then \( \Phi(t+p) \) is also a fundamental matrix and
\[
\Phi(t+p) = \Phi(t)C,
\]
where
\[
C = \Phi^{-1}(0)\Phi(p). \tag{4.21}
\]
Furthermore, there is a nonsingular matrix function \( P(t) \) and a nonsingular constant matrix \( B \) such that
\[
\Phi(t) = P(t)B^t, \tag{4.22}
\]
where \( P(t) \) is periodic with period \( p \).

**Proof.** Assume \( \Phi(t) \) is a fundamental matrix for the Floquet system (4.16). If \( \Psi(t) \equiv \Phi(t+p) \), then \( \Psi(t) \) is nonsingular for all \( t \), and
\[
\Psi(t+1) = \Phi(t+p+1) = A(t+p)\Phi(t+p) = A(t)\Psi(t),
\]
so \( \Phi(t+p) \) is a fundamental matrix for Eq. (4.16). By Theorem 4.10, there is a nonsingular matrix \( C \) such that
\[
\Phi(t+p) = \Phi(t)C,
\]
for all integers \( t \). Letting \( t = 0 \) and solving for \( C \), we get that Eq (4.21) holds.

By Lemma 4.1, there is a nonsingular matrix \( B \) so that \( B^p = C \). Let
\[
P(t) = \Phi(t)B^{-t}. \tag{4.23}
\]
Note that \( P(t) \) is nonsingular for all \( t \) and since
\[
P(t+p) = \Phi(t+p)B^{-(t+p)} = \Phi(t)CB^{-p}B^{-t} = \Phi(t)B^{-t} = P(t),
\]
\( P(t) \) has period \( p \). Solving Eq. (4.23) for \( \Phi(t) \), we get Eq. (4.22).

**Definition 4.2.** Let \( \Phi(t) \) and \( C \) be as in Floquet’s theorem (Theorem 4.13), then the eigenvalues \( \mu \) of the matrix \( C \) are called the “Floquet multipliers” of the Floquet system (4.16).

Since fundamental matrices of a linear system are not unique, we must show that Floquet multipliers are well defined. Let \( \Phi(t) \) and \( \Psi(t) \) be fundamental matrices for the Floquet system (4.16) and let
\[
C_1 = \Phi^{-1}(0)\Phi(p) \quad \text{and} \quad C_2 = \Psi^{-1}(0)\Psi(p).
\]
It remains to show that $C_1$ and $C_2$ have the same eigenvalues. By Theorem 4.10 there is a matrix $F$ so that
\[ \Psi(t) = \Phi(t)F, \]
for all integers $t$. Hence,
\[ C_2 = \Psi^{-1}(0)\Psi(p) = [\Phi(0)F]^{-1}[\Phi(p)F] = F^{-1}\Phi^{-1}(0)\Phi(p)F = F^{-1}C_1F. \]

Since
\[ \det(C_2 - \lambda I) = \det(F^{-1}C_1F - I) = \det F^{-1}(C_1 - \lambda I)F = \det(C_1 - \lambda I), \]
$C_1$ and $C_2$ have the same eigenvalues. Hence Floquet multipliers are well defined.

**Remark** The Floquet multipliers of the Floquet system (4.16) are the eigenvalues of the matrix
\[ D = A(p - 1)A(p - 2)\ldots A(0). \]

To see this let $\Phi(t)$ be the fundamental matrix of the Floquet system (4.16) satisfying $\Phi(0) = I$. Then the Floquet multipliers are the eigenvalues of
\[ D = \Phi^{-1}(0)\Phi(p) = \Phi(p). \]

Iterating the equation
\[ \Phi(t + 1) = A(t)\Phi(t), \]
we get that
\[ D = \Phi(p) = [A(p - 1)A(p - 2)\ldots A(0)]\Phi(0) = A(p - 1)A(p - 2)\ldots A(0), \]
which is the desired result.

Here are some simple examples of Floquet multipliers.

**Example 4.14.** For the scalar equation
\[ u(t + 1) = (-1)^tu(t), \]
the coefficient function $a(t) = (-1)^t$ has minimum period 2, and $d = a(1)a(0) = -1$, so $\mu = -1$ is the Floquet multiplier.

**Example 4.15.** Find the Floquet multipliers for the Floquet system
\[ u(t + 1) = \begin{bmatrix} 0 & 1 \\ (-1)^t & 0 \end{bmatrix} u(t). \]
The coefficient matrix $A(t)$ is periodic with minimum period 2, so

$$D = A(1)A(0)$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Consequently, $\mu_1 = 1$, $\mu_2 = -1$ are the Floquet multipliers.