Solving Initial Value Problems in Nabla Fractional Calculus

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Sample Initial Value Problems

Example (Continuous Example)

$$\begin{cases} y''(t) = t^2 \\ y(0) = 3 \\ y'(0) = 5 \end{cases}$$

$y(t) = \frac{t^4}{12} + 5t + 3$

Example (Discrete Fractional Example)

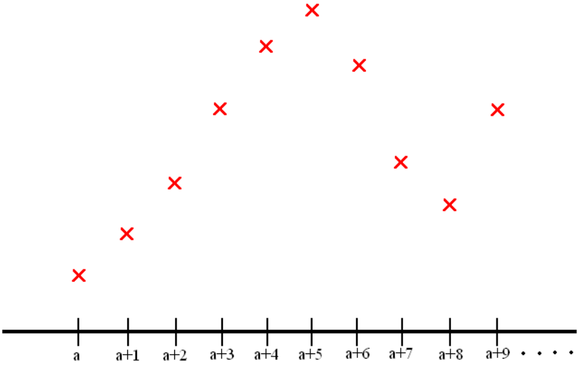
$$\begin{cases} \nabla_0^{1.4} f(t) = t^{\overline{2}}, \quad t \in \mathbb{N}_{a+2} \\ f(2) = 1 \\ \nabla f(2) = 2 \end{cases}$$

$f(t) \approx -7.44 t^{\overline{.4}} + 1.62 t^{\overline{-.6}} + 1.61 t^{\overline{3.4}}$

Outline

Outline

Domains of Functions in the Discrete Case



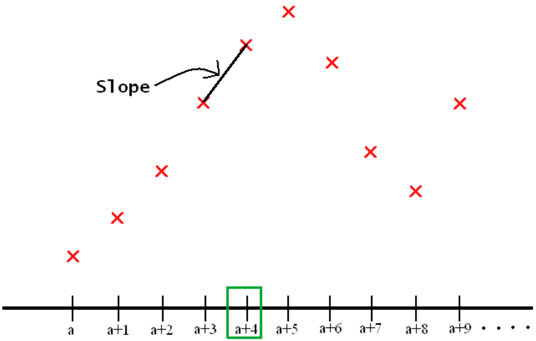
Definition (Domain of $\mathbb{N}_a$)

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\}.$$

Nabla Difference Operator

Definition (Nabla Difference Operator)

$\nabla f(t) := f(t) - f(t-1), t \in \mathbb{N}_{a+1}$

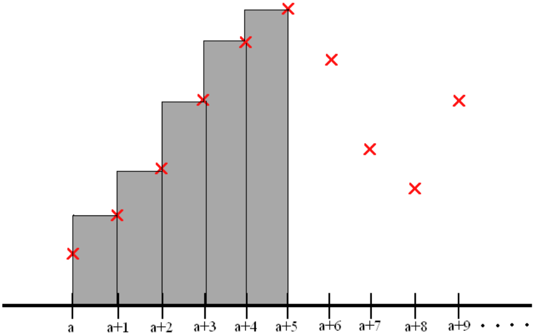


$$\nabla f(a+4) = f(a+4) - f(a+3)$$

Nabla Definite Integrals

Definition

$\nabla t = \sum_{t=c+1}^{d} f(t)$, for $c < d$.



$\int_a^{a+5} f(t)$

Outline

Example of a Nabla Difference

Example

Consider

$$\nabla t^2 = t^2 - (t-1)^2 = t^2 - (t^2 - 2t + 1) = 2t - 1.$$

Rising Factorial Function

Definition (Rising Factorial Function)

For $k, n \in \mathbb{N}$, the rising factorial function is

$$k^{\overline{n}} := k(k+1)\cdots(k+n-1) = \frac{(k+n-1)!}{(k-1)!}.$$

Example

$$3^{\overline{2}} = 3 \cdot (3+1) = 3 \cdot 4 = 12$$

Difference of a Rising Factorial Function

Example

Consider

$$t^{\overline{2}} = t \cdot (t+1).$$

$$\nabla t^{\overline{2}} = t \cdot (t+1) - (t-1) \cdot t = t^2 + t - t^2 + t = 2t.$$

Integral Sums for Integers

Theorem (Repeated Integrals)

$$\nabla_a^{-n} f(t) := \frac{1}{(n-1)!} \int_a^t (t-(s-1))^{\overline{n-1}} f(s) \nabla s$$

Gamma Function

The gamma function is an extension of the factorial functions for non-integer values.

Definition (Gamma Function)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Properties

- $\Gamma(n) = (n-1)!, \text{ for } n \in \mathbb{N}.$
- $x \cdot \Gamma(x) = \Gamma(x+1).$

Gamma Function

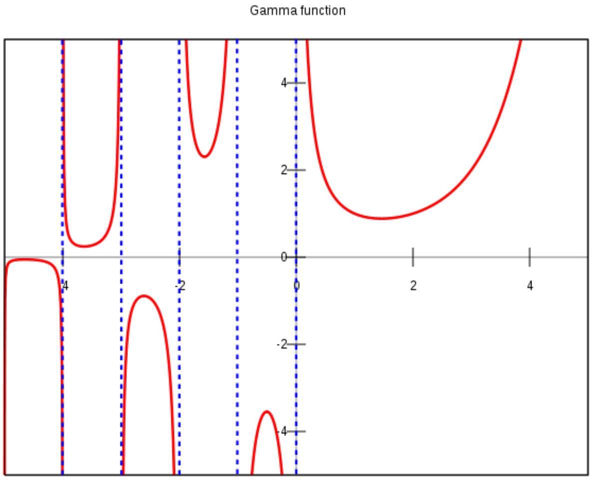


Figure: http://en.wikipedia.org/wiki/Gamma_function

Extending the Rising Factorial Function

Definition (Rising Function)

For $k, \nu \in \mathbb{R}$, the rising function is

$$k^{\overline{\nu}} := \frac{\Gamma(k+\nu)}{\Gamma(k)}.$$

Example

$$3^{\overline{2}} = 3 \cdot (3+1) = \frac{\Gamma(3+2)}{\Gamma(3)} = \frac{\Gamma(5)}{\Gamma(3)} = \frac{4!}{2!} = 12$$

$$3^{\overline{2.05}} = \frac{\Gamma(3+2.05)}{\Gamma(3)} = \frac{\Gamma(5.05)}{\Gamma(3)} \approx 12.94$$

Extending the Repeated Integrals Formula

Definition (Nabla Fractional Sum)

Let $\nu > 0$, then the v^{th}-order fractional sum is

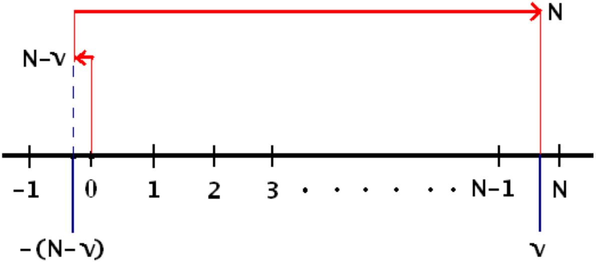
$$\nabla_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_a^t (t-(s-1))^{\overline{\nu-1}} f(s) \nabla s.$$

Defining the Fractional Difference

Definition (Nabla Fractional Difference)

Let $\nu > 0$, and choose $N \in \mathbb{N}$ such that $N - 1 < \nu \leq N$. Then the ν^{th}-order fractional difference is

$$\nabla_a^{\nu} f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t), t \in \mathbb{N}_{a+N}.$$



Fractional Difference Notation

Example

Consider the 1.9^{th}-order difference.

$$\nabla_a^{1.9} f(t) = \nabla^2 \left(\frac{1}{\Gamma(1.9)} \int_a^t (t-(s-1))^{\overline{1.9-1}} f(s) \nabla s \right).$$

Example

Consider the 2^{nd}-order difference. $\nabla^2 f(t) = f(t) - 2f(t-1) + f(t-2)$.

- Non-whole order differences must denote a base, i.e. $\nabla_a^{1.9} f(t)$.
- Whole order differences do not depend on a base, thus the subscript is omitted, i.e. $\nabla^2 f(t)$.

Alternative Definition of the Fractional Difference

Theorem (Alternative Definition of a Fractional Difference)

The following statements are equivalent for $\nu > 0$ and $N \in \mathbb{N}$ chosen such that $N - 1 < \nu < N$.

$$\nabla_a^{\nu} f(t) = \nabla^N \nabla_a^{-(N-\nu)} f(t),$$

$$\nabla_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t-(s-1))^{\overline{-\nu-1}} f(s),$$

for $\nu \notin \mathbb{N}_0$.

Unified Definition of the Fractional Sums and Differences

1. The ν^{th}-order fractional sum of f is given by

$$\nabla_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-(s-1))^{\overline{\nu-1}} f(s), \text{ for } t \in \mathbb{N}_a.$$

2. The ν^{th}-order fractional difference of f is given by

$$\nabla_a^{\nu} f(t) := \begin{cases} \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t-(s-1))^{\overline{-\nu-1}} f(s), & \nu \notin \mathbb{N}_0 \\ \nabla^N f(t), & \nu = N \in \mathbb{N}_0 \end{cases}$$

for $t \in \mathbb{N}_{a+N}$.

Continuity of the Fractional Difference

Theorem (Continuity of the Nabla Fractional Difference)

Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. Then the fractional difference $\nabla_a^\nu f$ is continuous with respect to $\nu \geq 0$.

Consider the sequence $\{\nabla_a^{1.9} f(a+3), \nabla_a^{1.99} f(a+3), \nabla_a^{1.999} f(a+3), \cdots\}$. This theorem implies that as the difference approaches 2, it depends less and less on its base and behaves more and more like the whole order $\nabla^2 f(a+3)$

Outline

Fractional Taylor Monomials

Definition (Fractional Order Taylor Monomials)

For $\nu \in \mathbb{R} \backslash \{-1, -2, ...\}$, define the Taylor monomial as

$$h_\nu^a(t) = h_\nu(t, a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

Taylor Monomial Example

Example

Consider

$$h_2(t,0) = \frac{t^{\overline{2}}}{\Gamma(3)}$$

$$\nabla h_2(t,0) = \frac{t^{\overline{2}} - (t-1)^{\overline{2}}}{2} = t = \frac{t^{\overline{1}}}{\Gamma(2)} = h_1(t,0)$$

$$\nabla^2 h_2(t,0) = t - (t-1) = 1 = h_0(t,0).$$

Generalized Power Rule

Theorem (Generalized Power Rule)

$$\nabla_a^\nu h_\mu(t,a) := h_{\mu-\nu}(t,a)$$

or

$$\nabla_a^\nu (t-a)^{\overline{\mu}} := \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\overline{\mu-\nu}}$$

Power Rule Example

Example

$$\nabla^{.95} t^{\overline{2}} = \frac{\Gamma(2+1)}{\Gamma(2-.95+1)} t^{\overline{(2-.95)}}$$

$$\approx 1.957 t^{\overline{1.05}}$$

Taylor Monomial Shifting

Lemma (One Step Taylor Monomial Shifting)

$$h_{\nu-N}(t, a+1) = h_{\nu-N}(t, a) - h_{\nu-N-1}(t, a).$$

Taylor Monomial Shifting

Theorem (General Taylor Monomial Shifting)

$$h_{\nu-N}(t, a+m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} h_{\nu-N-k}(t, a).$$

Outline

Composition Rules

Theorem (Fractional Nabla Sums and Differences Composition Rules)

Let $\mu, \nu > 0$ *be given, and choose* $N \in \mathbb{N}$ *such that* $N - 1 < \nu \leq N$, *then we have*

$$\nabla_a^{-\nu}\nabla_a^{-\mu}f(t) = \nabla_a^{-\nu-\mu}f(t), \text{ for } t \in \mathbb{N}_a,$$

$$\nabla_a^{\nu}\nabla_a^{-\mu}f(t) = \nabla_a^{\nu-\mu}f(t), \text{ for } t \in \mathbb{N}_{a+N}.$$

Example

$$\nabla_a^{-0.2}(\nabla_a^{-0.5}f(t)) = \nabla_a^{-0.7}f(t)$$

Composition Rules

Theorem (Fractional Nabla Sums and Whole Order Differences Composition Rules)

Let $\nu > 0$ and $k \in \mathbb{N}_0$ be given, and choose $N \in \mathbb{N}$ such that $N - 1 < \nu \leq N$, then we have

$$\nabla_{a+k}^{-\nu} \nabla^k f(t) = \nabla_{a+k}^{k-\nu} f(t) - \sum_{j=0}^{k-1} \nabla^j f(a+k) \frac{(t-a-k)^{\overline{\nu-k+j}}}{\Gamma(\nu-k+j+1)},$$

for $t \in \mathbb{N}_{a+k}$.

$$\nabla_{a+k}^{\nu} \nabla^k f(t) = \nabla_{a+k}^{k+\nu} f(t) - \sum_{j=0}^{k-1} \nabla^j f(a+k) \frac{(t-a-k)^{\overline{-\nu-k+j}}}{\Gamma(-v-k+j+1)},$$

for $t \in \mathbb{N}_{a+k+N}$.

Outline

Motivation

Example

$$\begin{cases} y' = t^2 \\ y(0) = A \end{cases}$$

$$\begin{aligned}
\mathcal{L}\{y'\}(s) &= \mathcal{L}\{t^2\}(s) \\
s\mathcal{L}\{y\}(s) - y(0) &= \frac{2!}{s^{2+1}} \\
\mathcal{L}\{y\}(s) &= \frac{2}{s^4} + \frac{A}{s} \\
y(t) &= \mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s}\right\} \\
y(t) &= \frac{t^3}{3} + A
\end{aligned}$$

Laplace Transform

Definition (Laplace Transform for Nabla Differences)

$$\mathcal{L}_a\{f\}(s) := \sum_{k=1}^{\infty}(1-s)^{k-1}f(a+k).$$

Properties

- Linear
- Exists for all functions of exponential order
- $\mathcal{L}\{g(t)\}(s) = \mathcal{L}\{t^2\}(s) \Rightarrow g(t) = t^2$

Laplace Transform of Taylor Monomials

Theorem (Laplace Transform of a Taylor Monomial)

For $\nu \in \mathbb{R} \backslash \{-1, -2, ...\}$,

$$\mathcal{L}\{h_\nu(\cdot, a)\}(s) = \frac{1}{s^{\nu+1}}.$$

Transformation of a ν^{th} Order Fractional Difference

Theorem (Transformation of a ν^{th} Order Fractional Difference)

Pick $N \in \mathbb{Z}^+$ such that $N-1 < \nu \leq N$. Then

$$\mathcal{L}_{a+N}\{\nabla_a^\nu f\}(s) = s^\nu \mathcal{L}_{a+N}\{f\}(s) + \sum_{k=0}^{N-1} \left(s^\nu \left(\frac{1}{1-s}\right)^{N-k} f(a+k+1) - s^N \left(\frac{1}{1-s}\right)^{N-k} \nabla_a^{-(N-\nu)} f(a+k+1) - \nabla_a^{\nu-k-1} f(a+N) s^k \right).$$

Transformation of N^{th} Order Nabla Difference

Applying the definition of a ν-th order nabla difference gives

$$\mathcal{L}_{a+N}\{\nabla_a^\nu f\}(s) = \mathcal{L}_{a+N}\{\nabla^N \nabla_a^{-(N-\nu)} f\}(s).$$

Theorem (Transformation of N^{th}-Order Nabla Difference)

For $f : \mathbb{N}_a \to \mathbb{R}$,

$$\mathcal{L}_{a+N}\{\nabla^N f\}(s) = s^N \mathcal{L}_{a+N}\{f\}(s) - \sum_{k=0}^{N-1} s^k \nabla^{N-k} f(a+N).$$

Laplace Shifting Theorem

Theorem (Laplace Shifting Theorem)

$$\mathcal{L}_{a+N}\{f\}(s) = \left(\frac{1}{1-s}\right)^N \mathcal{L}_a\{f\}(s) - \sum_{k=0}^{N-1}\left(\left(\frac{1}{1-s}\right)^{N-k} f(a+k+1)\right)$$

Transformation of Fractional Sums

Theorem (Transformation of Fractional Sums)

For $\nu \in \mathbb{R}^+$,

$$\mathcal{L}_a\{\nabla_a^{-\nu} f\}(s) = \frac{1}{s^\nu}\mathcal{L}_a\{f\}(s).$$

Convolution

Definition (Convolution)

The convolution of f and g is

$$(f * g)(t) := \int_a^t f(t-(s-1)+a)g(s)\nabla s, t \in \mathbb{N}_{a+1}.$$

Theorem (Convolution Theorem)

$$\mathcal{L}_a\{f * g\}(s) = \mathcal{L}_a\{f\}(s) \cdot \mathcal{L}_a\{g\}(s).$$

Fractional Sums

Theorem

For $\nu \in \mathbb{R}\backslash\{0, -1, -2, ...\}$,

$$\nabla_a^{-\nu} f(t) = (h_{\nu-1}(\cdot, a) * f(\cdot))(t).$$

Mittag-Leffler Function

Definition (Mittag-Leffler Function)

For $|p| < 1$, $\alpha > 0, \beta \in \mathbb{R}$, and $t \in \mathbb{N}_a$,

$$E_{p,\alpha,\beta}(t,a) := \sum_{k=0}^{\infty} p^k h_{\alpha k+\beta}(t,a).$$

Theorem

For $|p| < 1, \alpha > 0, \beta \in \mathbb{R}, |1-s| < 1$, *and* $|s^\alpha| > |p|$,

$$\mathcal{L}_a\{E_{p,\alpha,\beta}(\cdot,a)\}(s) = \frac{s^{\alpha-\beta-1}}{s^\alpha - p}.$$

Outline

General Solution for $1 < \nu \leq 2$

Theorem

Let $1 < \nu \leq 2$ *and* $|c| < 1$. *Then the fractional initial value problem*

$$\begin{cases} \nabla_a^\nu f(t) + cf(t) = g(t), & t \in \mathbb{N}_{a+2} \\ f(a+2) = A_0, & A_0 \in \mathbb{R} \\ \nabla f(a+2) = A_1, & A_1 \in \mathbb{R} \end{cases}$$

has the solution

$$\begin{aligned} f(t) = [E_{-c,\nu,\nu-1}(\cdot, a) * g(\cdot)] - [&g(a+1) + g(a+2) \\ &+ (\nu - 2c - 2)A_0 - (\nu - c - 1)A_1]E_{-c,\nu,\nu-1}(t,a) \\ &+ [g(a+2) + (\nu - c - 1)A_0 - \nu A_1]E_{-c,\nu,\nu-2}(t,a). \end{aligned}$$

General Solution for $1 < \nu \leq 2$

Proof.

We begin by taking the Laplace transform based at a+2 of both sides of the equation.

$$\mathcal{L}_{a+2}\{\nabla_a^\nu f\}(s) + c\mathcal{L}_{a+2}\{f\}(s) = \mathcal{L}_{a+2}\{g\}(s).$$

We apply the Laplace transform of a ν^{th} order nabla difference to the first term and the shifting theorem to the remaining terms to give

$$\frac{s^\nu + c}{(1-s)^2}\mathcal{L}_a\{f\}(s) - \frac{(s^2+c)}{(1-s)^2}f(a+1) - \frac{c}{1-s}f(a+2) -$$
$$\nabla_a^{-(1-\nu)}f(a+2) - \frac{s}{1-s}\nabla_a^{-(2-\nu)}f(a+2)$$
$$= \frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s) - \frac{1}{(1-s)^2}g(a+1) - \frac{1}{1-s}g(a+2).$$

Proof (cont'd).

Applying initial conditions gives

$$\frac{s^{\nu}+c}{(1-s)^2}\mathcal{L}_a\{f\}(s)-\frac{(s^2+c)}{(1-s)^2}[A_0-A_1]-\frac{c}{1-s}A_0-$$
$$[(1-\nu)[A_0-A_1]+A_0]-\frac{s}{1-s}[(2-\nu)[A_0-A_1]+A_0]$$
$$=\frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s)-\frac{1}{(1-s)^2}g(a+1)-\frac{1}{1-s}g(a+2).$$

Combine terms with respect to the $A_i's$

$$(s^{\nu}+c)\mathcal{L}_a\{f\}(s)+[\nu(1-s)+(s-2)(c+1)]A_0$$
$$+[c+1+\nu(s-1)]A_1=\frac{1}{(1-s)^2}\mathcal{L}_a\{g\}(s)-\frac{1}{(1-s)^2}g(a+1)$$
$$-\frac{1}{1-s}g(a+2).$$

Proof (cont'd).

Solve for the Laplace transform of f.

$$\begin{aligned}\mathcal{L}_a\{f\}(s) = \frac{1}{s^\nu + c}\mathcal{L}_a\{g\}(s) + \frac{1}{s^\nu + c}[g(a+1) + g(a+2)\\ + (\nu - 2c - 2)A_0 + (c + 1 - \nu)A_1]\\ + \frac{s}{s^\nu + c}[g(a+2) + (\nu - c - 1)A_0 - \nu A_1].\end{aligned}$$

Finally, take the inverse Laplace transform to obtain the desired result.

$$\begin{aligned}f(t) = [E_{-c,\nu,\nu-1}(\cdot, a) * g(\cdot)] - [g(a+1) + g(a+2)\\ + (\nu - 2c - 2)A_0 - (\nu - c - 1)A_1]E_{-c,\nu,\nu-1}(t, a)\\ + [g(a+2) + (\nu - c - 1)A_0 - \nu A_1]E_{-c,\nu,\nu-2}(t, a).\end{aligned}$$

□

Special Case for $1 < \nu \leq 2$

Consider the case when c=0. We then obtain the following IVP

Corollary

Let $1 < \nu \leq 2$. Then the fractional initial value problem

$$\begin{cases} \nabla_a^\nu f(t) = g(t), & t \in \mathbb{N}_{a+2} \\ f(a+2) = A_0, & A_0 \in \mathbb{R} \\ \nabla f(a+2) = A_1, & A_1 \in \mathbb{R} \end{cases}$$

has the solution

$$\begin{aligned} f(t) = &\left[(2-\nu)A_0 + (\nu-1)A_1 - g(a+1) - g(a+2)\right] h_{\nu-1}^a(t) \\ &+ \left[(\nu-1)A_0 - \nu A_1 + g(a+2)\right] h_{\nu-2}^a(t) + \nabla_a^{-\nu} g(t). \end{aligned} \tag{1}$$

Special Case for $1 < \nu \leq 2$

Proof.

From the definition of the Mittag-Leffler function, we observe the following

$$E_{0,\nu,\nu-1}(t,a) = \sum_{k=0}^{\infty} 0^k h^a_{\nu k+\nu-1}(t)$$

Note that all terms except the $k = 0$ term are zero. Therefore, by convention

$$E_{0,\nu,\nu-1}(t,a) = h^a_{\nu-1}(t) \qquad \textit{and} \qquad E_{0,\nu,\nu-2}(t,a) = h^a_{\nu-2}(t)$$

Also recall that $[h_{\nu-1}(\cdot,a) * g(\cdot)](t) = \nabla_a^{-\nu} g(t)$.
This yields the given result. □

Note: The solution can be obtained directly without the use of the Mittag-Leffler function.

Example

Consider the following 1.4^{th}-order initial value problem.

Example

$$\begin{cases} \nabla_0^{1.4} f(t) = t^{\overline{2}}, \quad t \in \mathbb{N}_{a+2} \\ f(2) = 1 \\ \nabla f(2) = 2 \end{cases}$$

Notice that this is an example of (1) with the following

$$a = 0, \qquad \nu = 1.4, \qquad N = 2$$
$$A_0 = 1, \qquad A_1 = 2, \qquad g(t) = t^{\overline{2}}$$

Example (cont'd)

We simply apply the solution of the form (1). This yields

$$f(t) = [(2 - 1.4)(1) + (1.4 - 1)(2) - (1)^{\overline{2}} - (2)^{\overline{2}}]h_{1.4-1}(t,0) + [(1.4 - 1)(1) - (1.4)(2) + (2)^{\overline{2}}]h_{1.4-2}(t,0) + \nabla_0^{-1.4} t^{\overline{2}}$$

.

Example (cont'd)

We simply apply the solution of the form (1). This yields

$$\begin{aligned} f(t) &= [(2-1.4)(1) + (1.4-1)(2) - (1)^{\overline{2}} - (2)^{\overline{2}}]h_{1.4-1}(t,0) \\ &\quad + [(1.4-1)(1) - (1.4)(2) + (2)^{\overline{2}}]h_{1.4-2}(t,0) + \nabla_0^{-1.4} t^{\overline{2}} \\ &= -6.6h_{.4}(t,0) + 3.6h_{-.6}(t,0) + \nabla_0^{-1.4} t^{\overline{2}} \end{aligned}$$

.

Example (cont'd)

We simply apply the solution of the form (1). This yields

$$\begin{aligned} f(t) &= [(2-1.4)(1) + (1.4-1)(2) - (1)^{\overline{2}} - (2)^{\overline{2}}]h_{1.4-1}(t,0) \\ &\quad + [(1.4-1)(1) - (1.4)(2) + (2)^{\overline{2}}]h_{1.4-2}(t,0) + \nabla_0^{-1.4} t^{\overline{2}} \\ &= -6.6h_{.4}(t,0) + 3.6h_{-.6}(t,0) + \nabla_0^{-1.4} t^{\overline{2}} \\ &= -6.6h_{.4}(t,0) + 3.6h_{-.6}(t,0) + \frac{\Gamma(3)}{\Gamma(2.4)} t^{\overline{3.4}} \end{aligned}$$

.

Example (cont'd)

We simply apply the solution of the form (1). This yields

$$\begin{aligned}
f(t) &= [(2-1.4)(1)+(1.4-1)(2)-(1)^{\overline{2}}-(2)^{\overline{2}}]h_{1.4-1}(t,0) \\
&\quad + [(1.4-1)(1)-(1.4)(2)+(2)^{\overline{2}}]h_{1.4-2}(t,0)+\nabla_0^{-1.4}t^{\overline{2}} \\
&= -6.6h_{.4}(t,0)+3.6h_{-.6}(t,0)+\nabla_0^{-1.4}t^{\overline{2}} \\
&= -6.6h_{.4}(t,0)+3.6h_{-.6}(t,0)+\frac{\Gamma(3)}{\Gamma(2.4)}t^{\overline{3.4}} \\
&= -6.6\frac{t^{\overline{.4}}}{\Gamma(1.4)}+3.6\frac{t^{\overline{-.6}}}{\Gamma(.4)}+\frac{\Gamma(3)}{\Gamma(2.4)}t^{\overline{3.4}}
\end{aligned}$$

.

Example (cont'd)

We simply apply the solution of the form (1). This yields

$$
\begin{aligned}
f(t) &= [(2-1.4)(1) + (1.4-1)(2) - (1)^{\overline{2}} - (2)^{\overline{2}}]h_{1.4-1}(t,0) \\
&\qquad + [(1.4-1)(1) - (1.4)(2) + (2)^{\overline{2}}]h_{1.4-2}(t,0) + \nabla_0^{-1.4} t^{\overline{2}} \\
&= -6.6 h_{.4}(t,0) + 3.6 h_{-.6}(t,0) + \nabla_0^{-1.4} t^{\overline{2}} \\
&= -6.6 h_{.4}(t,0) + 3.6 h_{-.6}(t,0) + \frac{\Gamma(3)}{\Gamma(2.4)} t^{\overline{3.4}} \\
&= -6.6 \frac{t^{\overline{.4}}}{\Gamma(1.4)} + 3.6 \frac{t^{\overline{-.6}}}{\Gamma(.4)} + \frac{\Gamma(3)}{\Gamma(2.4)} t^{\overline{3.4}} \\
&\approx -7.44 t^{\overline{.4}} + 1.62 t^{\overline{-.6}} + 1.61 t^{\overline{3.4}}.
\end{aligned}
$$

General Solution for $\nu > 0$

Consider the following initial value problem, with $\nu > 0$.

$$\begin{cases} \nabla_a^\nu f(t) = g(t), & t \in \mathbb{N}_{a+N} \\ \nabla^i f(a+N) = A_i, & i = 0, 1, ..., N-1 \end{cases}$$

To solve this IVP, we split it into two problems

- The non-homogeneous problem with homogeneous initial conditions
- The homogeneous problem with non-homogeneous initial conditions

The Non-homogeneous Problem with Homogeneous Initial Conditions

Theorem

Let $\nu > 0$. Then the fractional initial value problem

$$\begin{cases} \nabla_a^\nu f(t) = g(t), & t \in \mathbb{N}_{a+N} \\ \nabla^i f(a+N) = 0, & i = 0, 1, ..., N-1 \end{cases}$$

has the solution

$$f(t) = \nabla_a^{-\nu} g(t) - \sum_{k=0}^{N-1} \sum_{i=0}^{k} g(a+k+1) \binom{k}{i} (-1)^i h_{\nu-1-i}^a(t).$$

The Homogeneous Problem with Non-homogeneous Initial Conditions

Conjecture

Let $\nu > 0$. *Then the fractional initial value problem*

$$\begin{cases} \nabla_a^\nu f(t) = 0, & t \in \mathbb{N}_{a+N} \\ \nabla^i f(a+N) = A_i, & i = 0, 1, ..., N-1 \end{cases}$$

has the solution

$$f(t) = \sum_{i=0}^{N-1} h^a_{\nu-1-i}(t) \sum_{j=0}^{N-1} \sum_{r=0}^{N-j-1} (-1)^r \binom{N-j-1}{r} A_r \sum_{k=0}^{i} \frac{(i-\nu+1-k)^{\overline{N-1-j}}}{\Gamma(N-j)} \binom{N}{k} (-1)^k.$$

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark

Pick N such that $N-1<\nu\leq N$. Then

$$\nabla_{a+N}^{-\nu}\nabla_a^{\nu}f(t)=0.$$

Apply the definition of a ν-th order difference.

$$\nabla_{a+N}^{-\nu}\nabla^{N}(\nabla_a^{-(N-\nu)}f(t))=0.$$

Next apply the composition rule for a ν-th order nabla sum based at $a+N$ of a whole order difference.

$$\nabla_{a+N}^{N-\nu}\nabla_a^{-(N-\nu)}f(t)-\sum_{j=0}^{N-1}\nabla^{j}\nabla_a^{-(N-\nu)}f(a+N)h_{\nu-N+j}^{a+N}(t)=0.$$

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont'd)

Apply the definition of a ν-th order sum and split the integral into two parts

$$\nabla_{a+N}^{N-\nu}\left[\frac{1}{\Gamma(N-\nu)}\left(\int_{a+N}^{t}(t-(s-1))^{\overline{N-\nu-1}}f(s)\nabla s\right.\right.$$
$$\left.\left.+\int_{a}^{a+N}(t-(s-1))^{\overline{N-\nu-1}}f(s)\nabla s\right)\right]$$
$$-\sum_{j=0}^{N-1}\nabla_a^{j-N+\nu}f(a+N)h_{\nu-N+j}^{a+N}(t)=0.$$

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont'd)

Apply the definition of a ν-th order difference

$$\nabla_{a+N}^{N-\nu}[\nabla_{a+N}^{-(N-\nu)} f(t)] + \nabla_{a+N}^{N-\nu}[\sum_{s=a+1}^{a+N} \frac{(t-(s-1))^{\overline{N-\nu-1}}}{\Gamma(N-\nu)} f(s)]$$

$$-\sum_{j=0}^{N-1} \nabla_a^{j-N+\nu} f(a+N) h_{\nu-N+j}^{a+N}(t) = 0.$$

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont'd)

Apply the composition rule for a difference composed with a sum.

$$f(t) + \nabla_{a+N}^{N-\nu}\left[\sum_{s=a+1}^{a+N} \frac{(t-(s-1))^{\overline{N-\nu-1}}}{\Gamma(N-\nu)} f(s)\right]$$

$$-\sum_{j=0}^{N-1} \nabla_a^{j-N+\nu} f(a+N) \nabla_a^{j-N+\nu} f(a+N) h_{\nu-N+j}^{a+N}(t) = 0.$$

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont'd)

Solving for f gives

$$\sum_{j=0}^{N-1} f(a+j+1)\left[\frac{(s+1-j)^{\overline{N-\nu-1}}}{\Gamma(N-\nu)}\sum_{s=j}^{N-1}\sum_{k=0}^{s}\binom{s}{k}(-1)^k h^a_{\nu-N-1-k}(t)\right.$$
$$+\sum_{r=0}^{N-1}\frac{(N-\nu-r)^{\overline{N-1-j}}}{\Gamma(N-j)}\sum_{k=0}^{r}\binom{N}{k}(-1)^k h^a_{\nu-N+r-k}(t)$$
$$\left.+\sum_{r=0}^{N-1}\frac{(N-\nu-r)^{\overline{N-1-j}}}{\Gamma(N-j)}\sum_{k=r+1}^{N}\binom{N}{k}(-1)^k h^a_{\nu-N+r-k}(t)\right]=f(t).$$

The center summation can be expressed as the desired solution.

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont'd)

$$f(t) = \sum_{j=0}^{N-1} f(a+j+1) \sum_{r=0}^{N-1} \frac{(N-\nu-r)^{\overline{N-1-j}}}{\Gamma(N-j)} \sum_{k=0}^{r} \binom{N}{k} (-1)^k h_{\nu-N+r-k}(t).$$

Write f(a+j+1) in terms of initial conditions and swap the order of summation to obtain the desired result

The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont'd)

It remains to be shown that the remaining terms go to zero.

$$\sum_{j=0}^{N-1} f(a+j+1)\left[\frac{(s+1-j)^{\overline{N-\nu-1}}}{\Gamma(N-\nu)}\sum_{s=j}^{N-1}\sum_{k=0}^{s}\binom{s}{k}(-1)^k h^a_{\nu-N-1-k}(t) + \sum_{r=0}^{N-1}\frac{(N-\nu-r)^{\overline{N-1-j}}}{\Gamma(N-j)}\sum_{k=r+1}^{N}\binom{N}{k}(-1)^k h^a_{\nu-N+r-k}(t)\right] = 0.$$

- The Taylor monomials in the above summation are expected to have zero coefficients
- Once proven, we will have the solution to the general non-homogeneous problem

Future Work

- Prove the general solution to the IVP for $\nu > 0$
- Extend the results to solutions involving the Mittag-Leffler function
- Obtain a general solution to the IVP that is not restricted for $|c| < 1$
- Obtain composition rules for sums and differences based at different points
- Recreate the results for general discrete time scales

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