Solving Initial Value Problems in Nabla Fractional Calculus

Kevin Ahrendt, Lucas Castle, Katy Yochman

University of Nebraska-Lincoln, Lamar University, Rose-Hulman Institute of Technology

Summer Research Program 2011
Mentors: Dr. Peterson and Dr. Holm

July 28, 2011
Sample Initial Value Problems

Example (Continuous Example)

\[
\begin{cases}
y''(t) = t^2 \\
y(0) = 3 \\
y'(0) = 5
\end{cases}
\]

\[y(t) = \frac{t^4}{12} + 5t + 3\]

Example (Discrete Fractional Example)

\[
\begin{cases}
\nabla_0^{1.4} f(t) = t^2, \quad t \in \mathbb{N}_{a+2} \\
f(2) = 1 \\
\nabla f(2) = 2
\end{cases}
\]

\[f(t) \approx -7.44t^{-4} + 1.62t^{-0.6} + 1.61t^{3.4}\]
Outline

1. Introduction to the Nabla Discrete Calculus
2. Fractional Sums and Differences
3. Taylor Monomials
4. Composition Rules
5. Laplace Transforms
6. Solving Initial Value Problems
Outline

1. Introduction to the Nabla Discrete Calculus
2. Fractional Sums and Differences
3. Taylor Monomials
4. Composition Rules
5. Laplace Transforms
6. Solving Initial Value Problems
Domains of Functions in the Discrete Case

Definition (Domain of $\mathbb{N}_a$)

\[ \mathbb{N}_a := \{a, a+1, a+2, \ldots\}. \]
Nabla Difference Operator

Definition (Nabla Difference Operator)

\[ \nabla f(t) := f(t) - f(t - 1), \quad t \in \mathbb{N}_{a+1} \]

\[ \nabla f(a + 4) = f(a + 4) - f(a + 3) \]
Nabla Definite Integrals

Definition

\[ \nabla t = \sum_{t=c+1}^{d} f(t), \text{ for } c < d. \]
Outline

1. Introduction to the Nabla Discrete Calculus

2. Fractional Sums and Differences

3. Taylor Monomials

4. Composition Rules

5. Laplace Transforms

6. Solving Initial Value Problems
Example of a Nabla Difference

Example

Consider

\[ \nabla t^2 = t^2 - (t - 1)^2 = t^2 - (t^2 - 2t + 1) = 2t - 1. \]
Rising Factorial Function

Definition (Rising Factorial Function)
For $k, n \in \mathbb{N}$, the rising factorial function is

$$k^\overline{n} := k(k + 1) \cdots (k + n - 1) = \frac{(k + n - 1)!}{(k - 1)!}.$$ 

Example

$$3^\overline{2} = 3 \cdot (3 + 1) = 3 \cdot 4 = 12$$
Difference of a Rising Factorial Function

Example

Consider

\[ t^2 = t \cdot (t + 1). \]

\[ \nabla t^2 = t \cdot (t + 1) - (t - 1) \cdot t = t^2 + t - t^2 + t = 2t. \]
Integral Sums for Integers

**Theorem (Repeated Integrals)**

\[
\nabla_a^{-n} f(t) := \frac{1}{(n-1)!} \int_a^t (t - (s-1))^{n-1} f(s) \nabla s
\]
Gamma Function

The gamma function is an extension of the factorial functions for non-integer values.

**Definition (Gamma Function)**

\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.
\]

**Properties**

- \( \Gamma(n) = (n - 1)! \), for \( n \in \mathbb{N} \).
- \( x \cdot \Gamma(x) = \Gamma(x + 1) \).
Gamma Function

Figure: http://en.wikipedia.org/wiki/Gamma_function
Extending the Rising Factorial Function

**Definition (Rising Function)**

For \( k, \nu \in \mathbb{R} \), the rising function is

\[
k^\nu := \frac{\Gamma(k + \nu)}{\Gamma(k)}.
\]

**Example**

\[
3^2 = 3 \cdot (3 + 1) = \frac{\Gamma(3 + 2)}{\Gamma(3)} = \frac{\Gamma(5)}{\Gamma(3)} = \frac{4!}{2!} = 12
\]

\[
3^{2.05} = \frac{\Gamma(3 + 2.05)}{\Gamma(3)} = \frac{\Gamma(5.05)}{\Gamma(3)} \approx 12.94
\]
Extending the Repeated Integrals Formula

Definition (Nabla Fractional Sum)

Let $\nu > 0$, then the $\nu^{th}$-order fractional sum is

$$\nabla_a^{−\nu} f(t) := \frac{1}{\Gamma(\nu)} \int_a^t (t - (s - 1))^{\nu - 1} f(s) \nabla s.$$
Defining the Fractional Difference

**Definition (Nabla Fractional Difference)**

Let $\nu > 0$, and choose $N \in \mathbb{N}$ such that $N - 1 < \nu \leq N$. Then the $\nu^{th}$-order fractional difference is

$$\nabla_{a}^{\nu} f(t) := \nabla_{a}^{N} \nabla_{a}^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N}.$$
Fractional Difference Notation

**Example**

Consider the $1.9^{th}$-order difference.

$$\nabla_{1.9} f(t) = \nabla^2 \left( \frac{1}{\Gamma(1.9)} \int_a^t (t - (s - 1))^{1.9-1} f(s) \nabla s \right).$$

**Example**

Consider the $2^{nd}$-order difference. $\nabla^2 f(t) = f(t) - 2f(t - 1) + f(t - 2)$.

- Non-whole order differences must denote a base, i.e. $\nabla_{1.9} f(t)$.
- Whole order differences do not depend on a base, thus the subscript is omitted, i.e. $\nabla^2 f(t)$. 
Alternative Definition of the Fractional Difference

Theorem (Alternative Definition of a Fractional Difference)

The following statements are equivalent for $\nu > 0$ and $N \in \mathbb{N}$ chosen such that $N - 1 < \nu < N$.

$$\nabla_a^\nu f(t) = \nabla_a^N \nabla_a^{-(N-\nu)} f(t),$$

$$\nabla_a^\nu f(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t-(s-1))^{-\nu-1} f(s),$$

for $\nu \notin \mathbb{N}_0$. 
Unified Definition of the Fractional Sums and Differences

1. The $\nu^{th}$-order fractional sum of $f$ is given by

$$\nabla_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t - (s - 1))^{\nu - 1} f(s), \text{ for } t \in \mathbb{N}_a.$$  

2. The $\nu^{th}$-order fractional difference of $f$ is given by

$$\nabla_a^{\nu} f(t) := \begin{cases} 
\frac{1}{\Gamma(-\nu)} \sum_{s=a+1}^{t} (t - (s - 1))^{-\nu - 1} f(s), & \nu \notin \mathbb{N}_0 \\
\nabla_{a+N} f(t), & \nu = N \in \mathbb{N}_0 
\end{cases}, \text{ for } t \in \mathbb{N}_{a+N}.$$
Continuity of the Fractional Difference

**Theorem (Continuity of the Nabla Fractional Difference)**

Let \( f : \mathbb{N}_a \rightarrow \mathbb{R} \) be given. Then the fractional difference \( \nabla^\nu_a f \) is continuous with respect to \( \nu \geq 0 \).

Consider the sequence \( \{ \nabla^{1.9}_a f(a + 3), \nabla^{1.99}_a f(a + 3), \nabla^{1.999}_a f(a + 3), \ldots \} \). This theorem implies that as the difference approaches 2, it depends less and less on its base and behaves more and more like the whole order \( \nabla^2 f(a + 3) \).
Outline

1. Introduction to the Nabla Discrete Calculus
2. Fractional Sums and Differences
3. Taylor Monomials
4. Composition Rules
5. Laplace Transforms
6. Solving Initial Value Problems
Fractional Taylor Monomials

Definition (Fractional Order Taylor Monomials)

For $\nu \in \mathbb{R} \setminus \{-1, -2, ...\}$, define the Taylor monomial as

$$h^a_\nu(t) = h_\nu(t, a) := \frac{(t - a)^\nu}{\Gamma(\nu + 1)}.$$
Taylor Monomial Example

Consider

\[ h_2(t, 0) = \frac{t^2}{\Gamma(3)} \]

\[ \nabla h_2(t, 0) = \frac{t^2 - (t - 1)^2}{2} = t = \frac{t^1}{\Gamma(2)} = h_1(t, 0) \]

\[ \nabla^2 h_2(t, 0) = t - (t - 1) = 1 = h_0(t, 0). \]
Generalized Power Rule

**Theorem (Generalized Power Rule)**

\[ \nabla^\nu_a h_\mu(t, a) := h_{\mu-\nu}(t, a) \]

or

\[ \nabla^\nu_a (t - a)^\mu := \frac{\Gamma(\mu + 1)}{\Gamma(\mu - \nu + 1)} (t - a)^{\mu-\nu} \]
Power Rule Example

Example

\[ \nabla^{.95} t^2 = \frac{\Gamma(2 + 1)}{\Gamma(2 - .95 + 1)} t^{2 - .95} \]

\[ \approx 1.957 t^{1.05} \]
Taylor Monomial Shifting

Lemma (One Step Taylor Monomial Shifting)

\[ h_{\nu-N}(t, a + 1) = h_{\nu-N}(t, a) - h_{\nu-N-1}(t, a). \]
Taylor Monomial Shifting

Theorem (General Taylor Monomial Shifting)

\[ h_{\nu-N}(t, a + m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} h_{\nu-N-k}(t, a). \]
Outline

1. Introduction to the Nabla Discrete Calculus
2. Fractional Sums and Differences
3. Taylor Monomials
4. Composition Rules
5. Laplace Transforms
6. Solving Initial Value Problems
Theorem (Fractional Nabla Sums and Differences Composition Rules)

Let $\mu, \nu > 0$ be given, and choose $N \in \mathbb{N}$ such that $N - 1 < \nu \leq N$, then we have

$$\nabla_a^{-\nu} \nabla_a^{-\mu} f(t) = \nabla_a^{-\nu-\mu} f(t), \text{ for } t \in \mathbb{N}_a,$$

$$\nabla_a^{\nu} \nabla_a^{-\mu} f(t) = \nabla_a^{\nu-\mu} f(t), \text{ for } t \in \mathbb{N}_{a+N}.$$

Example

$$\nabla_a^{-0.2} (\nabla_a^{-0.5} f(t)) = \nabla_a^{-0.7} f(t)$$
Composition Rules

Theorem (Fractional Nabla Sums and Whole Order Differences Composition Rules)

Let \( \nu > 0 \) and \( k \in \mathbb{N}_0 \) be given, and choose \( N \in \mathbb{N} \) such that \( N - 1 < \nu \leq N \), then we have

\[
\nabla_{a+k}^{-\nu} \nabla^k f(t) = \nabla_{a+k}^{k-\nu} f(t) - \sum_{j=0}^{k-1} \nabla^j f(a + k) \frac{(t - a - k)^{\nu-k+j}}{\Gamma(\nu - k + j + 1)},
\]

for \( t \in \mathbb{N}_{a+k} \).

\[
\nabla_{a+k}^{\nu} \nabla^k f(t) = \nabla_{a+k}^{k+\nu} f(t) - \sum_{j=0}^{k-1} \nabla^j f(a + k) \frac{(t - a - k)^{-\nu-k+j}}{\Gamma(-\nu - k + j + 1)},
\]

for \( t \in \mathbb{N}_{a+k+N} \).
Outline

1. Introduction to the Nabla Discrete Calculus
2. Fractional Sums and Differences
3. Taylor Monomials
4. Composition Rules
5. Laplace Transforms
6. Solving Initial Value Problems
Motivation

Example

\[
\begin{align*}
\begin{cases}
y' &= t^2 \\
y(0) &= A
\end{cases}
\end{align*}
\]

\[
\mathcal{L}\{y\}'(s) = \mathcal{L}\{t^2\}(s)
\]

\[
s\mathcal{L}\{y\}(s) - y(0) = \frac{2!}{s^{2+1}}
\]

\[
\mathcal{L}\{y\}(s) = \frac{2}{s^4} + \frac{A}{s}
\]

\[
y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s}\right\}
\]

\[
y(t) = \frac{t^3}{3} + A
\]
Laplace Transform

Definition (Laplace Transform for Nabla Differences)

\[ \mathcal{L}_a\{f\}(s) := \sum_{k=1}^{\infty} (1 - s)^{k-1} f(a + k). \]

Properties

- Linear
- Exists for all functions of exponential order
- \( \mathcal{L}\{g(t)\}(s) = \mathcal{L}\{t^2\}(s) \Rightarrow g(t) = t^2 \)
Theorem (Laplace Transform of a Taylor Monomial)

For $\nu \in \mathbb{R}\backslash\{-1, -2, \ldots\}$,

$$\mathcal{L}\{h_\nu(\cdot, a)\}(s) = \frac{1}{s^{\nu+1}}.$$
Theorem (Transformation of a $\nu^{th}$ Order Fractional Difference)

Pick $N \in \mathbb{Z}^+$ such that $N - 1 < \nu \leq N$. Then

$$
\mathcal{L}_{a+N}\{\nabla_\nu^a f\}(s) = s^\nu \mathcal{L}_{a+N}\{f\}(s) + \sum_{k=0}^{N-1} \left( s^\nu \left( \frac{1}{1-s} \right)^{N-k} f(a + k + 1) 
- s^N \left( \frac{1}{1-s} \right)^{N-k} \nabla_a^{(N-\nu)} f(a + k + 1)
- \nabla_a^{\nu-k-1} f(a + N)s^k \right).
$$
Transformation of $N^{th}$ Order Nabla Difference

Applying the definition of a $\nu$-th order nabla difference gives

$$\mathcal{L}_{a+N}\{\nabla^{\nu} f\}(s) = \mathcal{L}_{a+N}\{\nabla^N \nabla_a^{-(N-\nu)} f\}(s).$$

**Theorem (Transformation of $N^{th}$-Order Nabla Difference)**

For $f : \mathbb{N}_a \rightarrow \mathbb{R}$,

$$\mathcal{L}_{a+N}\{\nabla^N f\}(s) = s^N \mathcal{L}_{a+N}\{f\}(s) - \sum_{k=0}^{N-1} s^k \nabla^{N-k} f(a + N).$$
Laplace Shifting Theorem

**Theorem (Laplace Shifting Theorem)**

\[
\mathcal{L}_{a+N}\{f\}(s) = \left(\frac{1}{1-s}\right)^N \mathcal{L}_a\{f\}(s) - \sum_{k=0}^{N-1} \left(\left(\frac{1}{1-s}\right)^{N-k} f(a+k+1)\right)
\]
Transformation of Fractional Sums

**Theorem (Transformation of Fractional Sums)**

For $\nu \in \mathbb{R}^+$, \[
\mathcal{L}_a\{\nabla^{-\nu}_a f\}(s) = \frac{1}{s^\nu} \mathcal{L}_a\{f\}(s).
\]
Convolution

**Definition (Convolution)**

The convolution of $f$ and $g$ is

$$(f * g)(t) := \int_{a}^{t} f(t - (s - 1) + a)g(s)\nabla s, t \in \mathbb{N}_{a+1}.$$ 

**Theorem (Convolution Theorem)**

$$\mathcal{L}_a\{f * g\}(s) = \mathcal{L}_a\{f\}(s) \cdot \mathcal{L}_a\{g\}(s).$$
Theorem

For $\nu \in \mathbb{R}\{0, -1, -2, ...\}$,

$$\nabla_a^{-\nu} f(t) = (h_{\nu-1}(\cdot, a) * f(\cdot))(t).$$
Laplace Transforms

Mittag-Leffler Function

Definition (Mittag-Leffler Function)
For \(|p| < 1, \alpha > 0, \beta \in \mathbb{R}, \text{ and } t \in \mathbb{N}_a,\)

\[ E_{p,\alpha,\beta}(t, a) := \sum_{k=0}^{\infty} p^k h_{\alpha k + \beta}(t, a). \]

Theorem
For \(|p| < 1, \alpha > 0, \beta \in \mathbb{R}, |1 - s| < 1, \text{ and } |s^\alpha| > |p|,\)

\[ \mathcal{L}_a\{E_{p,\alpha,\beta}(\cdot, a)\}(s) = \frac{s^{\alpha-\beta-1}}{s^\alpha - p}. \]
Outline

1. Introduction to the Nabla Discrete Calculus
2. Fractional Sums and Differences
3. Taylor Monomials
4. Composition Rules
5. Laplace Transforms
6. Solving Initial Value Problems
General Solution for $1 < \nu \leq 2$

**Theorem**

Let $1 < \nu \leq 2$ and $|c| < 1$. Then the fractional initial value problem

\[
\begin{align*}
\begin{cases}
\nabla_\nu^\nu f(t) + cf(t) &= g(t), \quad t \in \mathbb{N}_{a+2} \\
 f(a + 2) &= A_0, \quad A_0 \in \mathbb{R} \\
 \nabla f(a + 2) &= A_1, \quad A_1 \in \mathbb{R}
\end{cases}
\end{align*}
\]

has the solution

\[
f(t) = [E_{-c,\nu,\nu-1}(\cdot, a) * g(\cdot)] - [g(a + 1) + g(a + 2) \\
+ (\nu - 2c - 2)A_0 - (\nu - c - 1)A_1]E_{-c,\nu,\nu-1}(t, a) \\
+ [g(a + 2) + (\nu - c - 1)A_0 - \nu A_1]E_{-c,\nu,\nu-2}(t, a).
\]
General Solution for $1 < \nu \leq 2$

Proof. 

We begin by taking the Laplace transform based at $a+2$ of both sides of the equation.

$$\mathcal{L}_{a+2}\{\nabla_a^{\nu} f\}(s) + c\mathcal{L}_{a+2}\{f\}(s) = \mathcal{L}_{a+2}\{g\}(s).$$

We apply the Laplace transform of a $\nu^{th}$ order nabla difference to the first term and the shifting theorem to the remaining terms to give

$$\frac{s^\nu + c}{(1 - s)^2} \mathcal{L}_a\{f\}(s) - \frac{(s^2 + c)}{(1 - s)^2} f(a + 1) - \frac{c}{1 - s} f(a + 2) -$$

$$\nabla^{- (1 - \nu)}_a f(a + 2) - \frac{s}{1 - s} \nabla^{- (2 - \nu)}_a f(a + 2)$$

$$= \frac{1}{(1 - s)^2} \mathcal{L}_a\{g\}(s) - \frac{1}{(1 - s)^2} g(a + 1) - \frac{1}{1 - s} g(a + 2).$$
Proof (cont’d).

Applying initial conditions gives

\[
\frac{s^\nu + c}{(1 - s)^2} \mathcal{L}_a\{f\}(s) - \frac{(s^2 + c)}{(1 - s)^2} [A_0 - A_1] - \frac{c}{1 - s} A_0 -
\]

\[
[(1 - \nu)[A_0 - A_1] + A_0] - \frac{s}{1 - s} [(2 - \nu)[A_0 - A_1] + A_0]
\]

\[
= \frac{1}{(1 - s)^2} \mathcal{L}_a\{g\}(s) - \frac{1}{(1 - s)^2} g(a + 1) - \frac{1}{1 - s} g(a + 2).
\]

Combine terms with respect to the \(A'_i\)’s

\[
(s^\nu + c) \mathcal{L}_a\{f\}(s) + [\nu(1 - s) + (s - 2)(c + 1)]A_0
\]

\[
+ [c + 1 + \nu(s - 1)]A_1 = \frac{1}{(1 - s)^2} \mathcal{L}_a\{g\}(s) - \frac{1}{(1 - s)^2} g(a + 1)
\]

\[- \frac{1}{1 - s} g(a + 2).
\]
Proof (cont’d).

Solve for the Laplace transform of $f$.

$$\mathcal{L}_a\{f\}(s) = \frac{1}{s^\nu + c} \mathcal{L}_a\{g\}(s) + \frac{1}{s^\nu + c} [g(a + 1) + g(a + 2)$$
$$\quad + (\nu - 2c - 2)A_0 + (c + 1 - \nu)A_1]$$
$$\quad + \frac{s}{s^\nu + c} [g(a + 2) + (\nu - c - 1)A_0 - \nu A_1].$$

Finally, take the inverse Laplace transform to obtain the desired result.

$$f(t) = [E_{-c,\nu,\nu-1}(\cdot, a) \ast g(\cdot)] - [g(a + 1) + g(a + 2)$$
$$\quad + (\nu - 2c - 2)A_0 - (\nu - c - 1)A_1]E_{-c,\nu,\nu-1}(t, a)$$
$$\quad + [g(a + 2) + (\nu - c - 1)A_0 - \nu A_1]E_{-c,\nu,\nu-2}(t, a).$$
Special Case for $1 < \nu \leq 2$

Consider the case when $c=0$. We then obtain the following IVP

**Corollary**

Let $1 < \nu \leq 2$. Then the fractional initial value problem

$$\begin{cases} \nabla_{a}^{\nu} f(t) = g(t), & t \in \mathbb{N}_{a+2} \\ f(a+2) = A_0, & A_0 \in \mathbb{R} \\ \nabla f(a+2) = A_1, & A_1 \in \mathbb{R} \end{cases}$$

has the solution

$$f(t) = [(2 - \nu)A_0 + (\nu - 1)A_1 - g(a+1) - g(a+2)] h_{\nu-1}^{a}(t)$$
$$+ [(\nu - 1)A_0 - \nu A_1 + g(a+2)] h_{\nu-2}^{a}(t) + \nabla_{a}^{\nu} g(t). \quad (1)$$
Special Case for $1 < \nu \leq 2$

**Proof.**

From the definition of the Mittag-Leffler function, we observe the following

\[
E_{0,\nu,\nu-1}(t, a) = \sum_{k=0}^{\infty} 0^k h_{\nu k + \nu - 1}(t)
\]

Note that all terms except the $k = 0$ term are zero. Therefore, by convention

\[
E_{0,\nu,\nu-1}(t, a) = h_{\nu - 1}(t) \quad \text{and} \quad E_{0,\nu,\nu-2}(t, a) = h_{\nu - 2}(t)
\]

Also recall that \([h_{\nu - 1}(\cdot, a) \ast g(\cdot)](t) = \nabla_a^{-\nu} g(t)\).

This yields the given result.

Note: The solution can be obtained directly without the use of the Mittag-Leffler function.
Example

Consider the following $1.4^{th}$-order initial value problem.

$$\begin{cases}
\nabla_{0}^{1.4} f(t) = t^2, \quad t \in \mathbb{N}_{a+2} \\
f(2) = 1 \\
\nabla f(2) = 2
\end{cases}$$

Notice that this is an example of (1) with the following

- $a = 0$, $\nu = 1.4$, $N = 2$
- $A_0 = 1$, $A_1 = 2$, $g(t) = t^2$
Example (cont’d)

We simply apply the solution of the form (1). This yields

\[ f(t) = [(2 - 1.4)(1) + (1.4 - 1)(2) - (1)^2 - (2)^2]h_{1.4-1}(t, 0) \\
+ [(1.4 - 1)(1) - (1.4)(2) + (2)^2]h_{1.4-2}(t, 0) + \nabla_0^{-1.4} t^2 \]
Example (cont’d)

We simply apply the solution of the form (1). This yields

\[
f(t) = [(2 - 1.4)(1) + (1.4 - 1)(2) - (1)^2 - (2)^2]h_{1.4-1}(t, 0) \\
+ [(1.4 - 1)(1) - (1.4)(2) + (2)^2]h_{1.4-2}(t, 0) + \nabla_{0}^{-1.4}t^2 \\
= -6.6h.4(t, 0) + 3.6h-.6(t, 0) + \nabla_{0}^{-1.4}t^2
\]
Example (cont’d)

We simply apply the solution of the form (1). This yields

\[
\begin{align*}
  f(t) &= [(2 - 1.4)(1) + (1.4 - 1)(2) - (1)\bar{2} - (2)\bar{2}]h_{1.4-1}(t, 0) \\
  & \quad + [(1.4 - 1)(1) - (1.4)(2) + (2)\bar{2}]h_{1.4-2}(t, 0) + \nabla_0^{-1.4}t^2 \\
  &= -6.6h_{.4}(t, 0) + 3.6h_{-.6}(t, 0) + \nabla_0^{-1.4}t^2 \\
  &= -6.6h_{.4}(t, 0) + 3.6h_{-.6}(t, 0) + \frac{\Gamma(3)}{\Gamma(2.4)}t^{3.4}
\end{align*}
\]
Example (cont’d)

We simply apply the solution of the form (1). This yields

\[
f(t) = [(2 - 1.4)(1) + (1.4 - 1)(2) - (1)^2 - (2)^2]h_{1.4-1}(t, 0) \\
+ [(1.4 - 1)(1) - (1.4)(2) + (2)^2]h_{1.4-2}(t, 0) + \nabla_0^{-1.4}t^2 \\
= -6.6h_{.4}(t, 0) + 3.6h_{-.6}(t, 0) + \nabla_0^{-1.4}t^2 \\
= -6.6h_{.4}(t, 0) + 3.6h_{-.6}(t, 0) + \frac{\Gamma(3)}{\Gamma(2.4)}t^{3.4} \\
= -6.6\frac{t^{.4}}{\Gamma(1.4)} + 3.6\frac{t^{-.6}}{\Gamma(.4)} + \frac{\Gamma(3)}{\Gamma(2.4)}t^{3.4}.
\]
Example (cont’d)

We simply apply the solution of the form (1). This yields

\[
f(t) = [(2 - 1.4)(1) + (1.4 - 1)(2) - (1)^2 - (2)^2]h_{1.4-1}(t, 0)
+ [(1.4 - 1)(1) - (1.4)(2) + (2)^2]h_{1.4-2}(t, 0) + \nabla_0^{-1.4} t^2
\]

\[
= -6.6 h_{1.4}(t, 0) + 3.6 h_{-0.6}(t, 0) + \nabla_0^{-1.4} t^2
\]

\[
= -6.6 h_{1.4}(t, 0) + 3.6 h_{-0.6}(t, 0) + \frac{\Gamma(3)}{\Gamma(2.4)} t^{3.4}
\]

\[
= -6.6 \frac{t^{4}}{\Gamma(1.4)} + 3.6 \frac{t^{-6}}{\Gamma(0.4)} + \frac{\Gamma(3)}{\Gamma(2.4)} t^{3.4}
\]

\[
\approx -7.44 t^{4} + 1.62 t^{-0.6} + 1.61 t^{3.4}.
\]
General Solution for $\nu > 0$

Consider the following initial value problem, with $\nu > 0$.

$$
\begin{align*}
\nabla^\nu_a f(t) &= g(t), \quad t \in \mathbb{N}_{a+N} \\
\nabla^i f(a+N) &= A_i, \quad i = 0, 1, \ldots, N-1
\end{align*}
$$

To solve this IVP, we split it into two problems

- The non-homogeneous problem with homogeneous initial conditions
- The homogeneous problem with non-homogeneous initial conditions
The Non-homogeneous Problem with Homogeneous Initial Conditions

**Theorem**

Let $\nu > 0$. Then the fractional initial value problem

\[
\begin{align*}
\nabla_a^\nu f(t) &= g(t), \quad t \in \mathbb{N}_{a+N} \\
\nabla^i f(a + N) &= 0, \quad i = 0, 1, \ldots, N - 1
\end{align*}
\]

has the solution

\[
f(t) = \nabla_a^{-\nu} g(t) - \sum_{k=0}^{N-1} \sum_{i=0}^{k} g(a + k + 1) \binom{k}{i} (-1)^i h_{\nu-1-i}^a(t).
\]
The Homogeneous Problem with Non-homogeneous Initial Conditions

Conjecture

Let $\nu > 0$. Then the fractional initial value problem

\[
\begin{aligned}
\nabla_{a}^{\nu} f(t) &= 0, \quad t \in \mathbb{N}_{a+N} \\
\nabla^{i} f(a + N) &= A_{i}, \quad i = 0, 1, \ldots, N - 1
\end{aligned}
\]

has the solution

\[
f(t) = \sum_{i=0}^{N-1} h_{\nu-1-i}^{a}(t) \sum_{j=0}^{N-1} \sum_{r=0}^{N-j-1} (-1)^{r} \binom{N-j-1}{r} A_{r} \sum_{k=0}^{i} \frac{(i - \nu + 1 - k)^{N-1-j}}{\Gamma(N-j)} \binom{N}{k} (-1)^{k}.
\]
The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark

Pick \( N \) such that \( N - 1 < \nu \leq N \). Then

\[
\nabla_a^{N-\nu} \nabla_a^{\nu} f(t) = 0.
\]

Apply the definition of a \( \nu \)-th order difference.

\[
\nabla_a^{N-\nu} \nabla^N (\nabla_a^{(N-\nu)} f(t)) = 0.
\]

Next apply the composition rule for a \( \nu \)-th order nabla sum based at \( a + N \) of a whole order difference.

\[
\nabla_a^{N-\nu} \nabla_a^{(N-\nu)} f(t) - \sum_{j=0}^{N-1} \nabla^j \nabla_a^{(N-\nu)} f(a + N) h_a^{a+N}_{\nu-N+j}(t) = 0.
\]
The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont’d)

Apply the definition of a $\nu$-th order sum and split the integral into two parts

\[
\nabla^{N-\nu}_{a+N} \left[ \frac{1}{\Gamma(N-\nu)} \left( \int_{a+N}^{t} (t-(s-1))^{N-\nu-1} f(s) \nabla s \right) \right] \\
+ \int_{a}^{a+N} (t-(s-1))^{N-\nu-1} f(s) \nabla s \\
- \sum_{j=0}^{N-1} \nabla^{j-N+\nu} f(a+N) h_{\nu-N+j}^{a+N}(t) = 0.
\]
The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont’d)

Apply the definition of a $\nu$-th order difference

$$\nabla_{a+N}^{N-\nu} [ \nabla_{a+N}^{-(N-\nu)} f(t)] + \nabla_{a+N}^{N-\nu} \left[ \sum_{s=a+1}^{a+N} \frac{(t - (s - 1))^{N-\nu-1}}{\Gamma(N-\nu)} f(s) \right]$$

$$- \sum_{j=0}^{N-1} \nabla_{a}^{-(N+\nu)} f(a + N) h_{\nu-N+j}^{a+N}(t) = 0.$$
The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont’d)

Apply the composition rule for a difference composed with a sum.

\[ f(t) + \nabla_{a+N}^{N-\nu} \left[ \sum_{s=a+1}^{a+N} \frac{(t - (s - 1))^{N-\nu-1}}{\Gamma(N-\nu)} f(s) \right] \]

\[ - \sum_{j=0}^{N-1} \nabla_{a}^{j-N+\nu} f(a + N) \nabla_{a}^{j-N+\nu} f(a + N) h_{\nu-N+j}^{a+N}(t) = 0. \]
The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont’d)

Solving for \( f \) gives

\[
\sum_{j=0}^{N-1} f(a+j+1) \left[ \frac{(s+1-j)^{N-\nu-1}}{\Gamma(N-\nu)} \sum_{s=j}^{N-1} \sum_{k=0}^{s} \binom{s}{k} (-1)^k h_{\nu-N-1-k}^a(t) \right.
\]

\[
+ \sum_{r=0}^{N-1} \frac{(N-\nu-r)^{N-1-j}}{\Gamma(N-j)} \sum_{k=0}^{r} \binom{N}{k} (-1)^k h_{\nu-N+r-k}^a(t)
\]

\[
+ \sum_{r=0}^{N-1} \frac{(N-\nu-r)^{N-1-j}}{\Gamma(N-j)} \sum_{k=r+1}^{N} \binom{N}{k} (-1)^k h_{\nu-N+r-k}^a(t) \right] = f(t).
\]

The center summation can be expressed as the desired solution.
The Homogeneous Problem with Non-homogeneous Initial Conditions

Remark (Cont’d)

\[ f(t) = \sum_{j=0}^{N-1} f(a+j+1) \sum_{r=0}^{N-1} \frac{(N - \nu - r)^{N-1-j}}{\Gamma(N-j)} \sum_{k=0}^{r} \left( \frac{N}{k} \right) (-1)^k h_{\nu-N+r-k}(t). \]

Write \( f(a+j+1) \) in terms of initial conditions and swap the order of summation to obtain the desired result.
Remark (Cont’d)

It remains to be shown that the remaining terms go to zero.

\[
\sum_{j=0}^{N-1} f(a+j+1) \left[ \frac{(s + 1 - j)^{N-\nu-1}}{\Gamma(N-\nu)} \sum_s \sum_k (-1)^k h_{\nu-N-1-k}^a(t) \right] = 0.
\]

- The Taylor monomials in the above summation are expected to have zero coefficients.
- Once proven, we will have the solution to the general non-homogeneous problem.
Future Work

- Prove the general solution to the IVP for $\nu > 0$
- Extend the results to solutions involving the Mittag-Leffler function
- Obtain a general solution to the IVP that is not restricted for $|c| < 1$
- Obtain composition rules for sums and differences based at different points
- Recreate the results for general discrete time scales
Acknowledgements

Thank You

- Dr. Allan Peterson
- Dr. Michael Holm
- UNL Summer Research Program 2011

M. Holm, Sum and difference compositions in discrete fractional calculus, *CUBO Mathematical Journal*, 13 (3),(2011)
