1. Given the vectors $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$, find

(a) (3 points) $\vec{u} \cdot \vec{v}$

(b) (3 points) $||\vec{u}||$

(c) (4 points) $\text{proj}_L \vec{u}$, where $L$ is the line in the direction of $\vec{v}$.

(d) (5 points) A vector orthogonal to both $\vec{u}$ and $\vec{v}$. 
2. Suppose that \( B = \{ \vec{b}_1, \vec{b}_2 \} \) and \( C = \{ \vec{u}_1, \vec{u}_2 \} \) are two bases for a subspace \( W \) and that \( \vec{u}_1 = 3\vec{b}_1 + 5\vec{b}_2 \), and \( \vec{u}_2 = 2\vec{b}_1 + 4\vec{b}_2 \).

(a) (4 points) Find \( P_{C \leftarrow B} \)

(b) (4 points) Find \( P_{B \leftarrow C} \)

(c) (3 points) If \( [\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \), determine \( [\vec{x}]_C \)

(d) (3 points) If \( [\vec{y}]_C = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \), determine \( [\vec{y}]_B \)

(e) (4 points) Suppose that in addition to the above information, it is known that \( P_B = \begin{bmatrix} 2 & -1 \\ -2 & 0 \end{bmatrix} \).

Use this fact to compute \( P_C \)
3. Given the matrix \( A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \), find

(a) (4 points) The eigenvalues of \( A \).

(b) (4 points) One eigenvector of \( A \).

(c) (4 points) An invertible matrix \( P \) and a matrix \( C \) of the form \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \) such that \( A = PCP^{-1} \).

4. (8 points) Find a least squares solution to \( A\bar{x} = \bar{b} \) for \( A = \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \) and \( \bar{b} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \).
5. Below are four vectors we will use in this problem. Suppose (a) \( B = \{ \vec{u}_1, \vec{u}_2 \} \) is an orthogonal basis for the vector space \( W \), (b) \( \vec{y} \) is in \( W \), and (c) \( \vec{z} \) is some other vector.

\[
\vec{u}_1 = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T \quad \vec{u}_2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T \quad \vec{y} = \begin{bmatrix} -7 & 7 & 5 \end{bmatrix}^T \quad \vec{z} = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}^T
\]

(a) (5 points) Use the fact that \( B \) is an orthogonal basis for \( W \) to write \( \vec{y} \) as a linear combination of \( \vec{u}_1 \) and \( \vec{u}_2 \). (Note: Your method of solution must utilize the orthogonality of \( B \).)

(b) (5 points) Find the best approximation to \( \vec{z} \) in the subspace spanned by \( W \).

(c) (4 points) Is \( \vec{z} \) in \( W \)? Why or why not?

6. (5 points) Find the area of the parallelogram in \( \mathbb{R}^2 \) determined by the vectors \[ \begin{bmatrix} 1 \\ 6 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -1 \end{bmatrix} \].

7. (5 points) Suppose that \( A \) is a \( 4 \times 3 \) matrix and that \( \text{Nul} \ A = \{ \vec{0} \} \). Find rank \( A \) and explain your answer.
8. (10 points) Use the Gram-Schmidt orthogonalization process to find an orthogonal basis for $W$, given that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for $W$, where

$$\vec{u}_1 = \begin{bmatrix} 1 & 1 & 0 & 2 \end{bmatrix}^T, \quad \vec{u}_2 = \begin{bmatrix} 3 & 2 & 0 & 1 \end{bmatrix}^T, \quad \vec{u}_3 = \begin{bmatrix} 2 & 0 & 1 & 1 \end{bmatrix}^T$$

9. (5 points) Determine if the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

span $\mathbb{R}^5$. 
10. For the matrix \[ A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \], find

(a) (4 points) The eigenvalues of \( A \).

(b) (4 points) An eigenvector for the smallest eigenvalue you found in part (a).

(c) (4 points) Suppose each eigenspace of \( A \) is one dimensional. Would \( A \) be diagonalizable? Explain.

11. (5 points) Suppose that \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) is a linear transformation, where \( T \left( \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \) and \( T \left( \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Find a vector \( \vec{x} \) in \( \mathbb{R}^3 \) such that \( T(\vec{x}) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \).
12. For the matrix \( A = \begin{bmatrix}
1 & 2 & 0 & -1 & -1 \\
1 & 2 & 1 & 0 & 1 \\
2 & 4 & 1 & -1 & 0 \\
3 & 6 & 0 & -3 & -2
\end{bmatrix} \), find the following:

(a) \((6 \text{ points})\) A basis for \( \text{Nul} \ A \).

(b) \((4 \text{ points})\) A basis for \( \text{Col} \ A \).

(c) \((4 \text{ points})\) A basis for \( \text{Row} \ A \).

13. \((5 \text{ points})\) If \( B \) is a \( 5 \times 8 \) matrix, what is the smallest possible dimension of \( \text{Nul} \ B \)? Explain.
14. (2 points each) True or false. Read each question carefully and circle the correct answer.

(a) T  F Any linear combination of vectors can always be written in the form $A\vec{x}$ for a suitable matrix $A$ and vector $\vec{x}$.

(b) T  F If $A$ is an invertible $n \times n$ matrix, then the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^n$.

(c) T  F If the columns of an $m \times n$ matrix $A$ are linearly independent, then these columns span $\mathbb{R}^m$.

(d) T  F The dimension of the vector space $\mathbb{P}_4$ is 4.

(e) T  F Every stochastic matrix has a steady-state vector.

(f) T  F An eigenspace of an $n \times n$ matrix $A$ is the null space of a certain matrix.

15. (5 points) Suppose that $B$ is a non-invertible $6 \times 6$ matrix. What can you say about the rank of $B$?

16. (5 points) Suppose that $A$ is an $7 \times 10$ matrix and $\dim \text{Row } A = 5$. Find $\dim \text{Nul } A$.

**Five Point Bonus:** Suppose $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is a linear transformation. Find the standard matrix of $T$, that is, the matrix $A$ such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$, given the following information:

\[
T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 5 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}
\]