1. (6 points) Let $H$ be the following set:

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a, b, c, d \text{ are real and} \begin{cases} 2a + 3b - c = 0 \\ a + 3c + 2d = 0 \end{cases} \right\}$$

Is $H$ a subspace? Why or why not?

2. (6 points) Find a basis for the null space of the matrix $\begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix}$.

3. (5 points) What is the rank of a matrix?
4. The matrices $A$ and $B$ below are known to be row equivalent.

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 & -4 & 3 & 2 \\ 3 & 9 & 1 & 0 & -1 & 2 & -1 \\ -1 & -3 & 4 & -2 & -8 & -2 & 7 \\ 0 & 0 & 3 & -1 & -5 & 2 & 5 \\ 2 & 6 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Use this information to solve the following problems.

(a) (3 points) Find $\dim(\text{Nul } A)$.

(b) (3 points) Find a basis for $\text{Col } B$.

(c) (3 points) Find a basis for $\text{Row } A$.

Write $A = [\vec{v}_1 \vec{v}_2 \vec{v}_3 \vec{v}_4 \vec{v}_5 \vec{v}_6 \vec{v}_7]$.

(d) (5 points) Express $\vec{v}_7$ as a linear combination of $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ or explain why this is not possible.

(e) (5 points) Are the vectors $\vec{v}_2$, $\vec{v}_5$, and $\vec{v}_6$ linearly independent? Explain.
5. (6 points) Consider a homogeneous system of 15 linear equations in 20 unknowns. Let \( A \) be the coefficient matrix of the system. Suppose the solution set is a 5 dimensional vector space. Do the columns of \( A \) span \( \mathbb{R}^{15} \)? Justify your answer.

6. For the basis \( B = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\} \) of \( \mathbb{R}^2 \) find the following:

   (a) (3 points) The change of basis matrix \( P_B \) from \( B \)-coordinates to the standard coordinates in \( \mathbb{R}^2 \).

   (b) (5 points) The \( B \)-coordinates of \( \vec{x} = \begin{bmatrix} 13 \\ 1 \end{bmatrix} \)

   (c) (4 points) The standard coordinates of \( [\vec{x}]_B = \begin{bmatrix} 3 \\ 11 \end{bmatrix} \)

7. (4 points) True or false: A change of coordinates matrix is always invertible. Justify your answer.
8. The Mothball County Futility Society meets weekly to debate, well, not very important things. The current discussion involves the best flavor of breath mint—is it spearmint or peppermint? After each round of lively debate, 1% of those supporting spearmint switch to peppermint, while 2% percent of peppermint proponents switch to the spearmint side.

(a) (4 points) Find a stochastic matrix for this Markov chain. Label the rows and columns.

(b) (6 points) Find the steady state vector.

9. (10 points) A $6 \times 6$ matrix $B$ has four eigenvalues: $\lambda = 1, 3, 4, 7$. What can you say about each of the following? Be as precise as possible.

(a) The dimension of the eigenspace corresponding to $\lambda = 4$.

(b) The number of linearly independent eigenvectors $B$ has.

10. (4 points) True or false: If a $6 \times 6$ matrix $A$ has less than six distinct eigenvalues, then $A$ cannot be diagonalizable. Justify your answer.
11. For the matrix $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, find

(a) (3 points) The eigenvalues of $A$.

(b) (8 points) One eigenvector for each eigenvalue.

(c) (5 points) Is $A$ diagonalizable? Justify your answer.

12. (5 points) The 12 $\times$ 12 matrix $C$ has precisely one eigenvalue, $\lambda = -3$ with the obvious multiplicity. What is the absolutely smallest possible dimension of the eigenspace corresponding to the eigenvalue $-3$? Circle your answer, and justify your choice.

(a) 0
(b) 1
(c) 2
(d) 12