Entropy-Viscosity for Navier-Stokes: An Entropy-Based LES Model

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Motivation

Eddy viscosity models for Navier Stokes:

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\triangle\boldsymbol{u} + \nabla\cdot(\nu_E S(\boldsymbol{u}))),$$
$$\nabla\cdot\boldsymbol{u} = 0,$$
$$S(u) := \tfrac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T),$$
$$\nu_E = \nu_E(\boldsymbol{u}, \nabla\boldsymbol{u}, \ldots) = \text{“Eddy viscosity”}$$

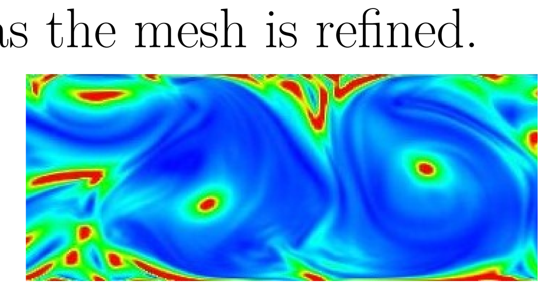
Modeling Objectives:

- ν_E should model the effect of the small (unresolved) eddies (“$\nu_E = 0$” is usual NSE).
- Idea: ν_E should only be active where numerical errors are large.
- What is the “best” choice for ν_E?

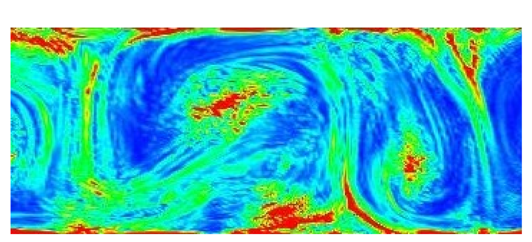
Introduction

When dealing very turbulent (i.e., high Reynolds number) problems, one quickly realizes that computer resources are limited. This implies that, in general, numerical simulations of realistic problems are essentially always under-resolved. Being under-resolved in a space-time region means that the numerical solutions experience large gradients that cannot be correctly represented on the mesh. As time progresses, the large unresolved gradients grow and propagate due to the nonlinear advection.

The usual LES approach to this dilemma is to add an artificial “eddy viscosity” ν_E which is locally proportional to, e.g., the (symmetric) gradient, as in the Smagorinsky model. However, this tends to overly damp important turbulent structures, even as the mesh is refined.



(a) Smagorinsky viscosity (normalized)



(b) Entropy-viscosity (normalized)

It is therefore desirable to find a viscosity which damps spurious oscillations, but which does not have significant influence in the smooth, resolved regions of the flow. In particular, it should be supported only in regions experiencing “numerical shocks,” and should decay to zeros as the mesh is refined.

Methods

A viscosity based on the PDE residual is one possible candidate. However, as the Burger's equation example (below) shows, this is not a robust quantity. Instead, we monitor the local entropy production (see [2, 3]), based on Scheffer's notion of local energy for suitable weak solutions [4]. We use this to build an artificial viscosity, the entropy-viscosity.

The entropy-viscosity method is self-adaptive and degenerates naturally to DNS when all the subgrid scales are represented. No filtering or deconvolution is required.

Results

Examining the energy spectra of unresolved flows, one can see that incorporating the entropy-viscosity leads to a significant gains in terms of preservation of the inertial range. In particular, flows regularized with entropy-viscosity appear to capture the Kolmogorov $-5/3$'s slope rather well in the inertial range. The effect is also apparent in the vortex tubes for an under-resolved 64^3 simulation. On the other hand, there is little qualitative effect on the vortex tubes for a resolved 256^3 simulation.

Building the Entropy-Viscosity from the Entropy Residual

- The numerical entropy residual (cf. Scheffer's notion of local energy for suitable weak solutions [4]):
$$D_h(\boldsymbol{x},t) := \partial_t(\tfrac{1}{2}|\boldsymbol{u}_h|^2) + \nabla\cdot((\tfrac{1}{2}|\boldsymbol{u}_h|^2 + p_h)\boldsymbol{u}_h) - R_e^{-1}\Delta(\tfrac{1}{2}|\boldsymbol{u}_h|^2) + R_e^{-1}(\triangle\boldsymbol{u}_h)^2 - \boldsymbol{f}\cdot\boldsymbol{u}_h.$$
- The entropy-viscosity is defined via $D_h(\boldsymbol{x},t)$, but limited by first-order viscosity:
$$\nu_E(\boldsymbol{x},t) := \min\left(\tfrac{1}{2}|\boldsymbol{u}_h(\boldsymbol{x},t)|h(\boldsymbol{x})\ ,\ ch^2(\boldsymbol{x})\frac{|D_h(\boldsymbol{x},t))|}{\|(\boldsymbol{u}_h-\bar{\boldsymbol{u}}_h)^2\|_{L^\infty(\Omega)}}\right)$$

Example: Burgers Equation

Riemann Initial Value Problem:

$$\begin{cases} \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0, \quad (x,t)\in\mathbb{R}\times\mathbb{R}_+, \\ u(x,0) = u_0(x) = 1 - H(x) = \begin{cases} 1, & x\le 0, \\ 0, & x > 0. \end{cases} \end{cases}$$

“Physical” Solution:

$$u(x,t) = 1 - H\left(x - \tfrac{1}{2}t\right)$$

PDE Residual:

$$\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = \tfrac{1}{2}H' - \tfrac{1}{2}H' = 0$$

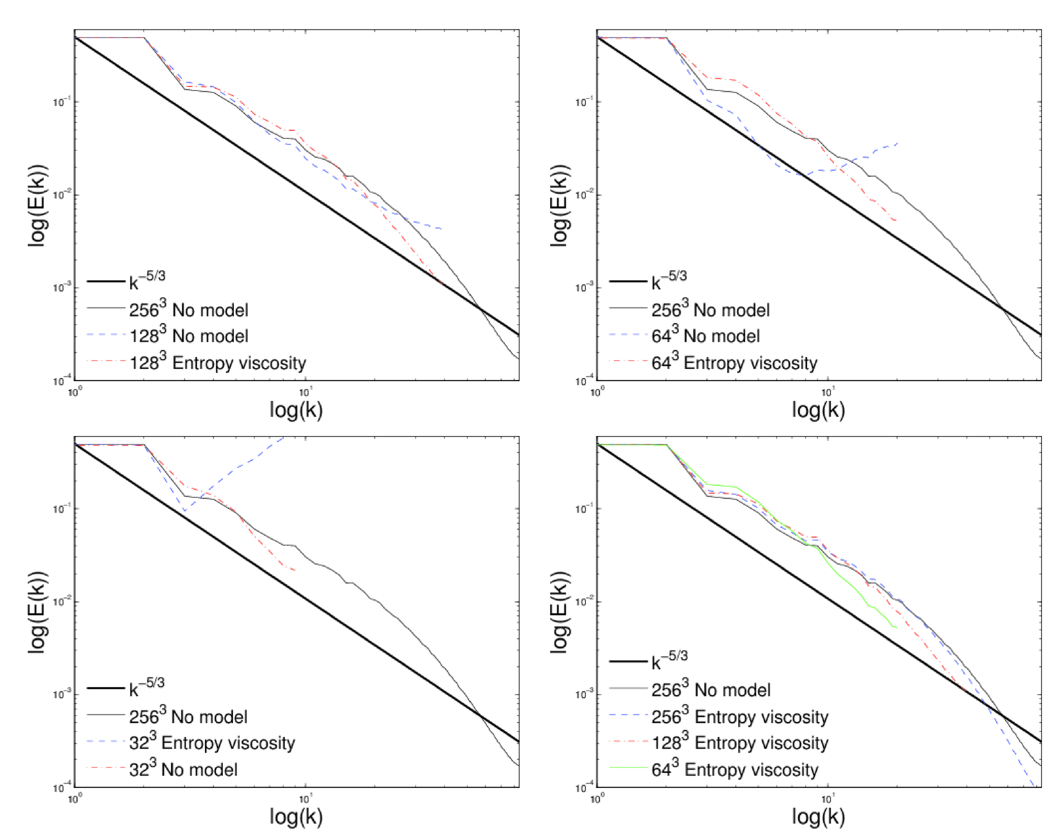
Entropy Residual:

$$\partial_t\left(\frac{u^2}{2}\right) + \partial_x\left(\frac{u^3}{3}\right) = \tfrac{1}{4}H' - \tfrac{1}{3}H' = -\tfrac{1}{12}\delta\left(x - \tfrac{1}{2}t\right) < 0.$$

Observations:

- The **entropy residual** is supported *exactly* in the shock.
- The **PDE residual** does not even detect the shock.

Energy Spectra



Numerical Methods

The following approach was used for the simulations:

- Fully-explicit RK4 for time evolution
- BDF2 used to calculate $\partial_t|\boldsymbol{u}|^2$
- 3D periodic box via pseudo-spectral method [5]
- 2D channel flow via finite-difference ADI scheme

Vortex Tubes

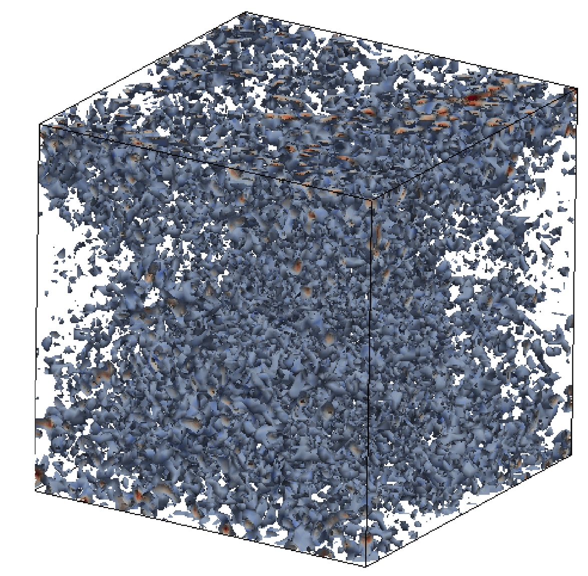


Figure 1: 64^3 No Model

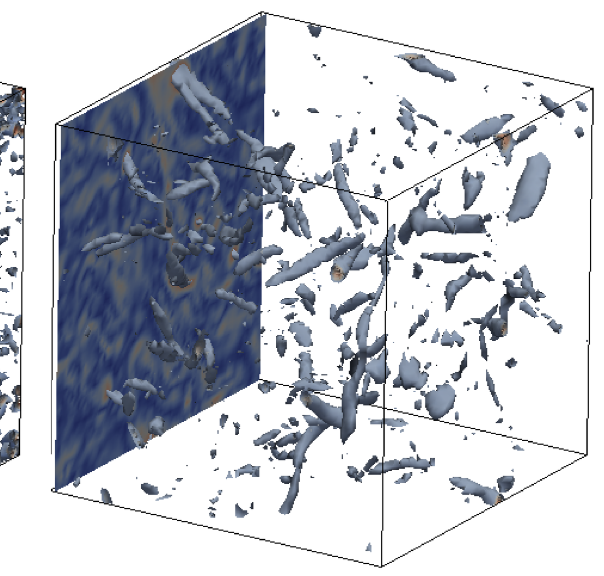


Figure 2: 64^3 entropy-viscosity

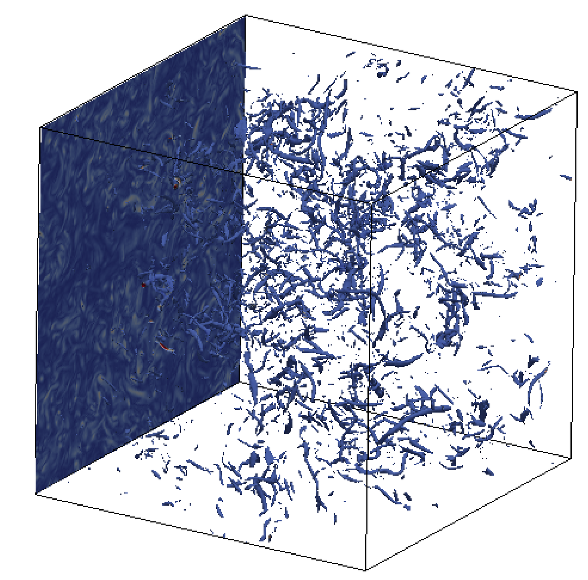


Figure 3:

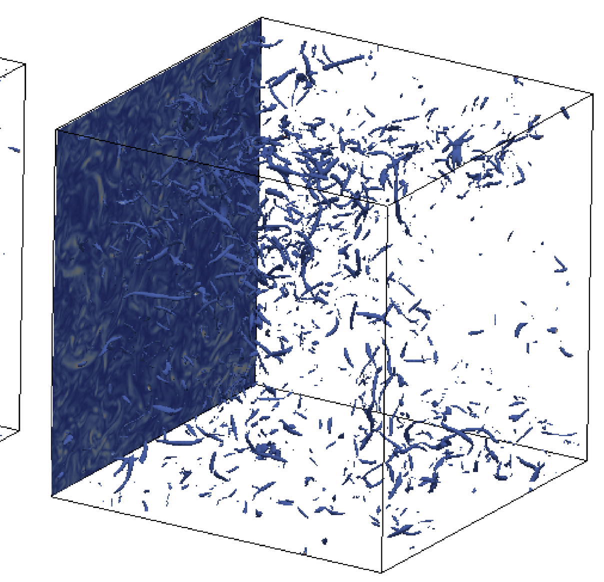


Figure 4: 256^3 entropy-viscosity

References

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[5] M. A. Taylor, S. Kurien, and G. L. Eyink. Recovering isotropic statistics in turbulence simulations: the Kolmogorov 4/5th law. *Phys. Rev. E (3)*, 68(2):026310, 8, 2003.

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