FULL NAME: Key — Exam 1

MATH 447, Differential Equations — Dr. Adam Larios — No calculators or notes

Answers without full, proper justification will not receive full credit.

Possibly useful formulas: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

1. **(4 points)** Let Q be a unitary matrix. Show that for any $\mathbf{x}, \mathbf{y}$, $(Qx, Qy) = (x, y)$, where $(\cdot,\cdot)$ denotes the inner-product.

$$(Qx, Qy) \underset{\text{adjoint property}}{=} (x, Q^* Q y) \underset{\text{since } Q \text{ is unitary}}{=} (x, Iy) = (x, y)$$

2. **(5 points)** Let A be a positive-definite matrix. Show that its eigenvalues are positive.

Let x, λ be an eigenpair for A, i.e. $Ax = \lambda x$, $x \neq \vec{0}$. Then

$$0 \underset{A \text{ is pos. def.}}{<} (Ax, x) = (\lambda x, x) = \lambda (x, x) = \lambda \|x\|^2$$

$\Rightarrow \lambda > 0$

3. **(10 points)** Let A be an $n \times n$ matrix. Show that $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$. (This was a homework problem 3.3(a))

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \max_{x \neq 0} \frac{\sqrt{n}\|Ax\|_\infty}{\|x\|_2}$$

since $\|\vec{z}\|_2 \leq \sqrt{n}\|\vec{z}\|_\infty$ for vector norms

$$\leq \max_{x \neq 0} \frac{\sqrt{n}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n} \max_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{n}\|A\|_\infty$$

since $\|\vec{z}\|_\infty \leq \|\vec{z}\|_2$ for vector norms

4. **(10 points)** Show that $\|A\|_2 = \|A\|_F$ if and only if $\mathrm{rank}(A) = 1$. (Hint: Consider the SVD of A.)

Let $\sigma_1, \ldots, \sigma_r$ be the singular values of A. Recall:

- $r = \mathrm{rank}(A)$
- $\|A\|_2 = \sigma_1$
- $\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$

If $\sigma_1 = \|A\|_2 = \|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$, then we must have $\sigma_2 = \sigma_3 = \cdots = \sigma_r = 0$. Thus $\mathrm{rank}(A) = 1$.

5. **(12 points)** Let A be a non-singular matrix.

(a) Show that A^*A is **self-adjoint** (i.e., Hermitian) and **positive-definite**.

$(A^*A)^* = A^*(A^*)^* = A^*A^{**} = A^*A$, so A^*A is self-adjoint

$(A^*Ax,x) \underset{\uparrow}{=} (Ax,Ax) = \|Ax\|^2 > 0$

adjoint property

since A is non-singular

(b) Find a Cholesky decomposition of A^*A (Hint: Use QR-factorization.)

$A=QR \Rightarrow A^*A = (QR)^*(QR) = R^*Q^*QR = R^*IR = R^*R$.

Let $L=R^*$. R upper-tri $\Rightarrow L$ lower tri. $\Rightarrow A^*A = LL^*$, which is Cholesky.

6. **(10 points)** Let $\mathbf{q}$ be a unit vector (i.e., $\|\mathbf{q}\|_2 = 1$). Define a Householder matrix via $H = I - \alpha\mathbf{q}\mathbf{q}^*$, where I is the identity matrix and $\alpha > 0$. (Note: You don't have to understand the Householder algorithm to do these problems.)

(a) For which values of $\alpha > 0$ is H unitary?

$H^*H = (I-\alpha qq^*)^*(I-\alpha qq^*) = (I-\alpha q^{**}q^*)(I-\alpha qq^*) = (I-\alpha qq^*)(I-\alpha qq^*)$

$= I - 2\alpha qq^* + \alpha^2 q\underbrace{q^*q}_{q^*q=\|q\|^2=1}q^* = I + (-2\alpha+\alpha^2)qq^* \overset{?}{=} I$

want this to be zero. Need $-2\alpha+\alpha^2=0 \Rightarrow \alpha=0$ or 2.
$\alpha>0 \Rightarrow \boxed{\alpha=2}$

(b) Is H a projection? Show why or why not.

Note that $H^*=H$ (see above).

Thus $H^2 = H^*H = I + (-2\alpha+\alpha^2)qq^*$

Thus $H^2=H$ if and only if $-2\alpha+\alpha^2 = -\alpha \Rightarrow \alpha^2=\alpha \Rightarrow \alpha=0$ or 1.

$\Rightarrow H$ is a projection only if $\alpha=0$ or $\alpha=1$.

7. **(15 points)** Recall the ℓ^1-norm of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, given by $\|\mathbf{x}\|_1 = \sum_{i=1}^m |x_i|$. Prove that it is a norm by showing that it satisfies the axioms of being a norm.

- $\|\vec{x}\|_1 = 0 \Rightarrow \sum_{i=1}^m |x_i| = 0 \Rightarrow$ all $x_i=0$ for all $i \Rightarrow \vec{x}=(0,\ldots,0)=\vec{0}$.

  Also $\vec{x}=\vec{0} \Rightarrow \|\vec{x}\|_1 = \sum_{i=1}^m |0| = 0$.

- $\|\alpha\vec{x}\|_1 = \sum_{i=1}^m |\alpha x_i| = \sum_{i=1}^m |\alpha||x_i| = |\alpha|\sum_{i=1}^m |x_i| = |\alpha|\,\|\vec{x}\|_1$

- $\|\vec{x}+\vec{y}\|_1 = \sum_{i=1}^m |x_i+y_i| \underset{\uparrow}{\leq} \sum_{i=1}^m (|x_i|+|y_i|) = \sum_{i=1}^m |x_i| + \sum_{i=1}^m |y_i| = \|\vec{x}\|_1 + \|\vec{y}\|_1$

  triangle inequality for numbers

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8. **(12 points)** Let A and B be $m \times m$ matrices. Let $\|\cdot\|$ be a (vector) norm on $\mathbb{C}^m$, and let $\|\cdot\|_*$ be the induced (matrix) norm on $m \times m$ matrices. Show that

$$\|AB\|_* \leq \|A\|_* \|B\|_*$$

$\|AB\vec{x}\| \leq \|A\|_* \|B\vec{x}\| \leq \|A\|_* \|B\|_* \|\vec{x}\|$. Thus $\|A\|_* \|B\|_*$ is an upper bound for $\frac{\|AB\vec{x}\|}{\|\vec{x}\|}$ for all $\vec{x} \neq \vec{0}$.

By definition of induced norm

Thus,

$\|AB\|_* \leq \|A\|_* \|B\|_*$.

9. **(10 points)** Let A an upper-triangular matrix with entries a_{ij}. Consider solving the problem $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ by following back-substitution algorithm:

1 division (1 op) $x_m = b_m / a_{m,m}$

1 mult, 1 div (2 ops) $x_{m-1} = (b_{m-1} - a_{m-1,m} x_m) / a_{m-1,m-1}$

2 mults, 1 div (3 ops) $x_{m-2} = (b_{m-2} - a_{m-2,m-1} x_{m-1} - a_{m-2,m} x_m) / a_{m-2,m-2}$

$\vdots \quad \vdots$

$(m-1)$ mults, 1 div (m ops) $x_1 = (b_1 - a_{12} x_2 - \cdots - a_{1,m-1} x_{m-1} - a_{1,m} x_m) / a_{11}$

operations

Count **exactly** the number of expensive operations (multiplications and divisions) that are involved in this computation. Your final answer should depend only on m.

Total operations, $1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{2}$

10. **(12 points)** Let P be an orthogonal projection, and let $\mathbf{y} = P\mathbf{x}$ and $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Show that $\mathbf{z}$ and $\mathbf{y}$ are orthogonal.

$\boxed{P^* = P, \quad P^2 = P}$

$(z, y) = z^* y = (x - y)^* y = x^* y - y^* y = x^*(Px) - (Px)^* Px$

$= x^* Px - x^* P^* Px$

$P^* = P$: $\; = x^* Px - x^* P Px$

$= x^* Px - x^* P^2 x$

$P^2 = P$: $\; = x^* Px - x^* Px$

$= 0$ So z and y are orthogonal.