Computing Gorenstein Colength

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**Abstract**

Given an Artinian local ring $R$, we define (in [1]) its Gorenstein colength $g(R)$ to measure how closely we can approximate $R$ by a Gorenstein Artin local ring. In this paper, we show that $R = T/\mathfrak{b}$ satisfies the inequality $g(R) \leq \lambda(R/\mathrm{soc}(R))$ in the following two cases: (a) $T$ is a power series ring over a field of characteristic zero and $\mathfrak{b}$ an ideal that is the power of a system of parameters or (b) $T$ is a 2-dimensional regular local ring with infinite residue field and $\mathfrak{b}$ is primary to the maximal ideal of $T$.

In the first case, we compute $g(R)$ by constructing a Gorenstein Artin local ring mapping onto $R$. We further use this construction to show that an ideal that is the $n$th power of a system of parameters is directly linked to the $(n-1)$st power via Gorenstein ideals. A similar method shows that such ideals are also directly linked to themselves via Gorenstein ideals.
Keywords: Gorenstein colength; Gorenstein linkage.

# 1 Introduction

Let us first recall the definition of Gorenstein colength and review some of its basic properties from [1] in this section.

**Definition 1.1.** Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring. Define the Gorenstein colength of $R$, denoted $g(R)$ as:
$g(R) = \min\{\lambda(S) - \lambda(R) : S$ is a Gorenstein Artin local ring mapping onto $R\}$,
where $\lambda(\_)$ denotes length.

The main questions one would like to answer are the following:

**Question 1.2.**
a) How does one intrinsically compute $g(R)$?
b) How does one construct a Gorenstein Artin local ring $S$ mapping onto $R$ such that $\lambda(S) - \lambda(R) = g(R)$?

In order to answer Question 1.2(a), we prove the following inequalities in [1], which give bounds on $g(R)$.

$$\lambda(R/(\omega^*(\omega))) \leq \min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \text{ is an ideal in } R, \mathfrak{a} \simeq \mathfrak{a}^\vee\} \leq g(R) \leq \lambda(R),$$

**Fundamental Inequalities**

where $\omega$ is the canonical module of $R$, $\omega^*(\omega) = \langle f(\omega) : f \in \mathrm{Hom}_R(\omega, R)\rangle$ is the trace ideal of $\omega$ in $R$ and $\mathfrak{a}^\vee = \mathrm{Hom}_R(\mathfrak{a}, \omega)$.

A natural question one can ask in this context is Question 3.10 in [1], which is the following:

**Question 1.3.** Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring and $\mathfrak{a}$ an ideal in $R$ such that $\mathfrak{a} \simeq \mathfrak{a}^\vee$. Does there exist a Gorenstein Artinian local ring $S$ mapping onto $R$, such that $\lambda(S) - \lambda(R) = \lambda(R/\mathfrak{a})$?

The socle of $R$, soc(R), is a direct sum of finitely many copies of $\mathsf{k}$, hence it is isomorphic to $\mathrm{soc}(R)^\vee$. Hence a particular case of the above question is the following:

**Question 1.4.** Is there a Gorenstein Artin local ring $S$ mapping onto $R$ such that $\lambda(S) - \lambda(R) = \lambda(R/\mathrm{soc}(R))$?

A weaker question one can ask is the following:

**Question 1.5.** Is $g(R) \leq \lambda(R/\mathrm{soc}(R))$?

We answer Question 1.5 in two cases in this paper. In section 3, we show that if $T$ is a power series ring over a field and $\mathfrak{d} = (f_1, \ldots, f_d)$ is an ideal generated by a system of parameters, then $g(T/\mathfrak{d}^n) \geq \lambda(T/\mathfrak{d}^{n-1})$. Further, if the residue field of $T$ has characteristic zero, we construct a Gorenstein Artin local ring $S$ mapping onto $T/\mathfrak{d}^n$ such that $\lambda(S) - \lambda(T/\mathfrak{d}^n) = \lambda(T/\mathfrak{d}^{n-1})$ using a theorem of L. Ried, L. Roberts and M. Roitman proved in [7]. This shows that $g(T/\mathfrak{d}^n) = \lambda(T/\mathfrak{d}^{n-1})$. In particular, this proves that $R = T/\mathfrak{d}^n$ satisfies the inequality in Question 1.5.

In [5], Kleppe, Migliore, Miro-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. In section 4, we use the ideal corresponding to the Gorenstein ring constructed in section 3, to show that $\mathfrak{d}^n$ can be directly liked to $\mathfrak{d}^{n-1}$ and hence to $\mathfrak{d}$ in $(n-1)$ steps.

When $R$ is an Artinian quotient of a two-dimensional regular local ring with an infinite residue field, we use a formula due to Hoskin and Deligne (Theorem 5.6) in order to answer Question 1.5 in section 5.

# 2 Computing $\omega^*(\omega)$

Let $(R, \mathfrak{m}, \mathsf{k})$ be an Artinian local ring with canonical module $\omega$. As noted in [1], maps from $\omega$ to $R$ play an important role in the study of Gorenstein colength. In this section, we prove a lemma which helps us compute the trace ideal $\omega^*(\omega)$ of $\omega$ in $R$. We use the following notation in this section.

**Notation:** Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring mapping onto $R$. Let

$$0 \to T^{b_d} \xrightarrow{\phi} T^{b_{d-1}} \to \ldots \to T \to R \to 0 \qquad (\sharp)$$

be a minimal resolution of $R$ over $T$. Then a resolution of the canonical module $\omega$ of $R$ over $T$ is given by taking the dual of the above resolution, i.e., by applying $\mathrm{Hom}_T(\_, T)$ to the above resolution. Hence a presentation of $\omega$ is $T^{b_{d-1}} \xrightarrow{\phi^*} T^{b_d} \to \omega \to 0$. Tensor with $R$ and apply $\mathrm{Hom}_R(-, R)$ to get an exact sequence $0 \longrightarrow \omega^* \longrightarrow R^{b_d} \xrightarrow{\phi \otimes R} R^{b_{d-1}}$. Let $\omega^*$ be generated minimally by $b_{d+1}$ elements. Thus we have an exact sequence $R^{b_{d+1}} \xrightarrow{\psi} R^{b_d} \xrightarrow{\phi \otimes R} R^{b_{d-1}}$, where $\omega^* = \mathrm{im}(\psi)$.

**Lemma 2.1.** *With notation as above, let $\psi$ be given by the matrix $(a_{ij})$. Then the trace ideal of $\omega$, $\omega^*(\omega)$, is the ideal generated by the $a_{ij}$'s.*

The above lemma is a particular case of the following lemma.

**Lemma 2.2.** *Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and $M$ a finitely generated $R$-module. Let $R^n \xrightarrow{B} R^m \longrightarrow M \longrightarrow 0$ be a minimal presentation of $M$. Apply $\operatorname{Hom}_R(\_, R)$ to get an exact sequence $0 \longrightarrow M^* \longrightarrow (R^*)^m \xrightarrow{B^*} (R^*)^n$. Map a free $R$-module, say $R^k$, minimally onto $M^*$ to get an exact sequence $R^k \xrightarrow{A} (R^*)^m \xrightarrow{B^*} (R^*)^n$, where $M^* = \ker(B^*) = \operatorname{im}(A)$. Then the trace ideal of $M$, $M^*(M) = (a_{ij}$ : $a_{ij}$ are the entries of the matrix $A)$.*

*Proof.* Let $m_1, \ldots, m_n$ be a minimal generating set of $M$, $e_1, \ldots, e_m$ be a basis of $R^m$ such that $e_i \mapsto m_i$, and $e_1^*, \ldots, e_m^*$ be the corresponding dual basis of $(R^*)^m$.

Let $f \in M^*$. Write $f = \Sigma_{i=1}^m r_i e_i^* \in (R^*)^m$. Then $f$ acts on $M$ by sending $m_j$ to $r_j$. Hence if $A = (a_{ij})$, then the generators of $M^*$ are $f_j = \Sigma_{i=1}^m a_{ij} e_i^*$, $1 \leq j \leq k$. Thus $f_j(m_i) = a_{ij}$. Thus $M^*(M) = (a_{ij})$. □

**Corollary 2.3.** *With notation as above, let $(T', \mathfrak{m}_{T'}, \mathsf{k})$ be a regular local ring which is a flat extension of $T$ such that $\mathfrak{m}_T T' \subseteq \mathfrak{m}_{T'}$ and let $R' = T' \otimes_T R$. Then $\omega_{R'}^*(\omega_{R'}) = \omega^*(\omega) T'$.*

*Proof.* Since $T'$ is flat over $T$, $R' = T' \otimes_T R$ and $\mathfrak{m}_T T' \subseteq \mathfrak{m}_{T'}$, a minimal resolution of $R'$ over $T'$ is obtained by tensoring ($\sharp$) by $T'$ over $T$. Therefore $\omega_{R'}^*(\omega_{R'})$ is the ideal generated by the entries of the matrix $\psi \otimes_T T'$. Now, by Lemma 2.1, the ideal in $R$ generated by the entries of $\psi$ is $\omega^*(\omega)$. Therefore, $\omega_{R'}^*(\omega_{R'}) = \omega^*(\omega) T'$. □

# 3 Powers of Ideals Generated by a System of Parameters

In this section, the main theorem we prove is the following:

**Theorem 3.1.** *Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field $\mathsf{k}$ of characteristic zero. Let $f_1, \ldots, f_d$ be a system of parameters in $T$ and $R = T/(f_1, \ldots, f_d)^n$. Then $g(R) = \lambda(T/(f_1, \ldots, f_d)^{n-1})$.*

In order to prove this, we first prove the theorem when $f_i = X_i, i = 1, \ldots, d$, and then use the fact that $T$ is flat over $T' = \mathsf{k}[|f_1, \ldots, f_d|]$.

**Theorem 3.2.** *Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field $\mathsf{k}$ with maximal ideal $\mathfrak{m}_T = (X_1, \ldots, X_d)$. Let $R = T/\mathfrak{m}_T^n$ and $\omega$ be the canonical module of $R$. Then $\omega^*(\omega) = \operatorname{soc}(R) = \mathfrak{m}_T^{n-1}/\mathfrak{m}_T^n$.*

*Proof.* In order to prove this, we show that if $\phi \in \operatorname{Hom}(\omega, R)$, then $\phi(\omega) \subseteq \operatorname{soc}(R)$. Since $\operatorname{soc}(R) \subseteq \omega^*(\omega)$, this will prove the theorem.

Note that we can consider $R$ to be the quotient of the polynomial ring $\mathsf{k}[X_1, \ldots, X_d]$ by $(X_1, \ldots, X_d)^n$. Thus change notation so that $T = \mathsf{k}[X_1, \ldots, X_d]$ and $\mathfrak{m}_T = (X_1, \ldots, X_d)$ is its unique homogenous maximal ideal.

The injective hull of $\mathsf{k}$ over $T$, $E_T(\mathsf{k})$, is $\mathsf{k}[X_1^{-1}, \ldots, X_d^{-1}]$, where the multiplication is defined by

$$(X_1^{a_1} \cdots X_d^{a_d}) \cdot (X_1^{-b_1} \cdots X_d^{-b_d}) = \begin{cases} X_1^{a_1 - b_1} \cdots X_d^{a_d - b_d} & \text{if } a_i \leq b_i \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

and extended linearly (e.g., see [6]).

Let $\mathfrak{b} = \mathfrak{m}_T^n$. The canonical module $\omega$ of $R$ is isomorphic to the injective hull of the residue field of $R$. Hence $\omega \simeq \operatorname{Hom}_R(R, E_T(\mathsf{k})) \simeq (0 :_{\mathsf{k}[X_1^{-1},\ldots,X_d^{-1}]} \mathfrak{b})$. Note that $\mathfrak{b} \cdot (X_1^{-a_1} \cdots X_d^{-a_d}) = 0$ whenever $a_i \geq 0$ and $n > \sum a_i$. Since $\lambda(\omega) = \lambda(R)$, we conclude that

$$\omega \simeq \mathsf{k}\text{-span of } \left\{ X_1^{-a_1} \cdots X_d^{-a_d} : a_i \geq 0; n > \sum_{i=1}^{d} a_i \right\}.$$

Observe that $\omega$ is generated by $\{X_1^{-a_1} \cdots X_d^{-a_d} \colon \sum_{i=1}^d a_i = n-1\}$ as an $R$-module. Let $\phi \in \omega^*$. We will now show that $\phi(X_1^{-a_1} \cdots X_d^{-a_d}) \in \operatorname{soc}(R)$ by induction on $a_1$. Let $w = X_1^{-a_1} \cdots X_d^{-a_d}$, $\sum_{i=1}^d a_i = n-1$.

If $a_1 = 0$, then $X_1 \cdot w = 0$. Hence $\phi(w) \in (0 :_R X_1) = \operatorname{soc}(R)$. If not, then $X_1 w = X_2(X_1^{-(a_1-1)} X_2^{-(a_2+1)} \cdots X_d^{-a_d})$. We have $\phi(X_1^{-(a_1-1)} X_2^{-(a_2+1)} \cdots X_d^{-a_d}) \in \operatorname{soc}(R)$ by induction. Thus $X_2 \phi(X_1^{-(a_1-1)} X_2^{-(a_2+1)} \cdots X_d^{-a_d}) = 0$ which yields $X_1 \phi(w) = 0$. But $(0 :_R X_1) = \operatorname{soc}(R)$, which proves that $\phi(\omega) \subseteq \operatorname{soc}(R)$. □

Since we know that $\lambda(R/(\omega^*(\omega)) \leq g(R)$ by the fundamental inequalities, we immediately get the following:

**Corollary 3.3.** *With notation as in Thoerem 3.2, $g(R) \geq \lambda(R/\operatorname{soc}(R))$.*

We prove the reverse inequality in Theorem 3.8 by constructing a Gorenstein Artin ring $S$ mapping onto $R$ such that $\lambda(S) - \lambda(R) = \lambda(R/\operatorname{soc}(R))$. The following theorem of Ried, Roberts and Roitman is used in the construction.

**Theorem 3.4** (Reid,Roberts,Roitman)**.** *Let $\mathsf{k}$ be a field of characteristic zero, $S = \mathsf{k}[X_1, \ldots, X_d]/(X_1^{n_1}, \ldots, X_d^{n_d}) = \mathsf{k}[x_1, \ldots, x_d]$. Let $m \geq 1$ and $f$ be a nonzero homogeneous element in $S$ such that $(x_1 + \cdots + x_d)^m f = 0$. Then $\deg(f) \geq (t - m + 1)/2$, where $t = \sum_{i=1}^d (n_i - 1)$.*

We use the following notation in this section.

**Notation:** Let $\mathsf{k}$ be a field. For any graded ring $S$ (with $S_0 = \mathsf{k}$), by $h_S(i)$ we mean the $\mathsf{k}$-dimension of the $i$th graded piece of the ring $S$ and if $S$ is Artinian, $\operatorname{Max}(S) := \max\{i : h_S(i) \neq 0\}$. All $\mathsf{k}$-algebras in this section are standard graded, i.e., they are generated as a $\mathsf{k}$-algebra by elements of degree 1.

We also need the following basic fact in order to prove Theorem 3.8.

**Remark 3.5.** Let $S = \mathsf{k}[X_1, \ldots, X_d]/(X_1^{n_1}, \ldots, X_d^{n_d})$ be a quotient of the polynomial ring over a field $\mathsf{k}$ and $f$ be a non-zero homogeneous element in $S$ of degree $s$. Then $S/(0 :_S f)$ is Gorenstein and $\operatorname{Max}(S/(0 :_S f)) = \operatorname{Max}(S) - s$.

**Proposition 3.6.** *Let $T = \mathsf{k}[X_1, \ldots, X_d]$ be a polynomial ring over $\mathsf{k}$ and $\mathfrak{m}_T = (X_1, \ldots, X_d)$ be its unique homogeneous maximal ideal. Let $f$ be a homogeneous element and $\mathfrak{c} = (X_1^n, \ldots, X_d^n) :_T f$ be such that $\mathfrak{c} \subseteq \mathfrak{m}^n$. Then the following are equivalent:*
i) $\lambda(\mathfrak{m}_T^n/\mathfrak{c}) = \lambda(T/\mathfrak{m}_T^{n-1})$.
ii) $\operatorname{Max}(T/\mathfrak{c}) = 2(n-1)$.
iii) $\deg(f) = (d-2)(n-1)$.

*Proof.* Since $\operatorname{Max}(T/(X_1^n, \ldots, X_d^n)) = d(n-1)$, (ii) $\Leftrightarrow$ (iii) follows from Remark 3.5.

Let $R = T/\mathfrak{m}_T^n$ and $S = T/\mathfrak{c}$. Since $T/(X_1^n, \ldots, X_d^n)$ is a Gorenstein Artin local ring, so is $S$. Note that $\operatorname{soc}(R) = \mathfrak{m}_T^{n-1}/\mathfrak{m}_T^n$ and $\lambda(S) - \lambda(R) = \lambda(\mathfrak{m}_T^n/\mathfrak{c})$.

The rings $R$ and $S$ are quotients of the polynomial ring $\mathsf{k}[X_1, \dots, X_d]$ by homogeneous ideals. Thus, both $R$ and $S$ are graded under the standard grading. Since $\mathfrak{c} \subseteq \mathfrak{m}_T^n$,

$$h_S(i) = h_R(i) \text{ for } i < n. \qquad (*)$$

Since $S$ is Gorenstein,

$$h_S(i) = h_S(\mathrm{Max}(S) - i). \qquad (**)$$

Using $(*)$ and $(**)$, we see that the Hilbert function of $S$ is:

| degree i | 0 | 1 | 2 | 3 | ... | n-1 |
|---|---|---|---|---|---|---|
| $h_R(i)$ | 1 | d | $\binom{d+1}{2}$ | $\binom{d+2}{3}$ | ... | $\binom{d+n-2}{n-1}$ |
| $h_S(i)$ | 1 | d | $\binom{d+1}{2}$ | $\binom{d+2}{3}$ | ... | $\binom{d+n-2}{n-1}$ |

| degree i | ... | Max(S) - (n-1) | Max(S) - (n-2) | ... | Max(S) - 1 | Max(S) |
|---|---|---|---|---|---|---|
| $h_R(i)$ | ... | 0 | 0 | ... | 0 | 0 |
| $h_S(i)$ | ... | $\binom{d+n-2}{n-1}$ | $\binom{d+n-3}{n-2}$ | ... | d | 1 |

Thus we have

$$\begin{aligned}\lambda(T/\mathfrak{m}_T^{n-1}) &= h_R(n-2) + h_R(n-3) + \ldots + h_R(0)\\ &= h_S(n-2) + h_S(n-3) + \ldots + h_S(0)\\ &= h_S(\mathrm{Max}(S) - (n-2)) + h_S(\mathrm{Max}(S) - (n-3)) + \ldots + h_S(\mathrm{Max}(S))\\ &= \textstyle\sum_{i \geq \mathrm{Max}(S)-(n-2)} h_S(i)\\ &\leq \lambda(S) - \lambda(R) = \lambda(\mathfrak{m}_T^n/\mathfrak{c}).\end{aligned}$$

Moreover, from the above table, equality holds if and only if $\mathrm{Max}(S) - (n-1) = n-1$, proving (i) $\Leftrightarrow$ (ii). ☐

In the following corollary, we show that $f = (X_1 + \cdots + X_d)^{(d-2)(n-1)}$ satisifies the hypothesis of Proposition 3.6.

**Corollary 3.7.** *Let $T = \mathsf{k}[X_1, \dots, X_d]$ be a polynomial ring over $\mathsf{k}$, a field of characteristic zero, and $\mathfrak{m}_T = (X_1, \dots, X_d)$ be its unique homogeneous maximal ideal. Let $\mathfrak{c}_n = (X_1^n, \dots, X_d^n) : l^{(d-2)(n-1)}$, where $l = X_1 + \cdots + X_d$. Then $\mathfrak{c}_n \subseteq \mathfrak{m}_T^n$.*

*Moreover, $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n) = \lambda(T/\mathfrak{m}_T^{n-1})$.*

*Proof.* By Theorem 3.4, if $F$ is a homogeneous element in $T$ such that $l^m F \in (X_1^n, \dots, X_d^n)$, then $\deg(F) \geq (d(n-1) - m + 1)/2$. Therefore, for $m = (d-2)(n-1)$, we see that $\deg(F) \geq n - 1/2$, i.e., $F \in \mathfrak{m}_T^n$. Thus $(X_1^n, \dots, X_d^n) : (X_1 + \cdots X_d)^{(d-2)(n-1)} \subseteq \mathfrak{m}_T^n$.

Moreover, by Proposition 3.6, since $\deg(l^{(d-2)(n-1)}) = (d-2)(n-1)$, $\lambda(\mathfrak{m}_T^n/\mathfrak{c}_n) = \lambda(T/\mathfrak{m}_T^{n-1})$. ☐

**Theorem 3.8.** *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field $\mathsf{k}$ of characteristic zero, with unique maximal ideal $\mathfrak{m}_T = (X_1, \dots, X_d)$. Let $R := T/\mathfrak{m}_T^n$. Then $g(R) \leq \lambda(R/\mathrm{soc}(R)) = \lambda(T/\mathfrak{m}^{n-1})$.*

*Proof.* Let $\mathfrak{c}_n = (X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)}$, where $l = X_1 + \cdots + X_d$. Let $S = T/\mathfrak{c}_n$. Then $S$ is a Gorenstein Artin local ring mapping onto $R$. Note that $R \simeq \mathsf{k}[X_1, \dots, X_d]/(X_1, \dots, X_d)^n$ and $S \simeq \mathsf{k}[X_1, \dots, X_d]/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)})$.

Hence, by Corollary 3.7, $\lambda(S) - \lambda(R) = \lambda(R/\mathrm{soc}(R)) = \lambda(T/\mathfrak{m}_T^{n-1})$. This shows that $g(R) \leq \lambda(R/\mathrm{soc}(R))$. □

**Remark 3.9.** The ring $S$ constructed in the proof of the theorem does not work when $\mathrm{char}(\mathsf{k}) = 2$. For example, when $d = 3$ and $n = 3$, we have $h_R(i) = 1, 3, 6$ and $h_S(i) = 1, 2, 5, 2, 1$.

**Remark 3.10.** Let $S$ be a graded Gorenstein Artin quotient of $T = \mathsf{k}[X_1, \dots, X_d]$, where $\mathsf{k}$ is a field of characteristic zero. We say that $S$ is a compressed Gorenstein algebra of socle degree $t = \mathrm{Max}(S)$, if for each $i$, $h_S(i)$ is the maximum possible given $d$ and $t$, i.e., $h_S(i) = \min\{h_T(i), h_T(t-i)\}$ (e.g., see [3]). Note that the proofs of Proposition 3.6 and Corollary 3.7 show that $S = T/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)})$ is a compressed Gorenstein Artin algebra of socle degree $2n-2$. A similar technique also shows that $S = T/((X_1^n, \dots, X_d^n) :_T l^{(d-2)(n-1)-1})$ is a compressed Gorenstein Artin algebra of socle degree $2n-1$.

In the following remark, we record some key observations which we will use to prove Theorem 3.1.

**Remark 3.11.** Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field $\mathsf{k}$. Let $f_1, \dots, f_d$ be a system of parameters. Then $T' = \mathsf{k}[|f_1, \dots, f_d|]$ is a power series ring and $T$ is free over $T'$ of rank $e = \lambda(T/(f_1, \dots, f_d))$. Thus, if $\mathfrak{b}$ and $\mathfrak{c}$ are ideals in $T'$, then $(\mathfrak{c} :_{T'} \mathfrak{b})T = (\mathfrak{c}T :_T \mathfrak{b}T)$ and $\lambda(T/\mathfrak{b}T) = e \cdot \lambda(T'/\mathfrak{b})$.

Firstly, we construct a Gorenstein Artin ring $S$ mapping onto $R$ such that $\lambda(S) - \lambda(R) = \lambda(T/(f_1, \dots, f_d)^{n-1})$ which proves $g(R) \leq \lambda(T/(f_1, \dots, f_d)^{n-1})$. We do this as follows:

Suppose that $\mathrm{char}(\mathsf{k}) = 0$. Let $\mathfrak{d} = (f_1, \dots, f_d)$, $\mathfrak{c} = (f_1^n, \dots, f_d^n) :_{T'} l^{(d-2)(n-1)}$, where $l = (f_1 + \cdots + f_d)$. We see that since $(f_1^n, \dots, f_d^n) :_{T'} l^{(d-2)(n-1)} \subseteq \mathfrak{d}^n$ in $T'$ by Corollary 3.7, the same holds in $T$ by using Remark 3.11. Moreover, since $\lambda(\mathfrak{d}^n T/\mathfrak{c}T) = e\lambda(\mathfrak{d}^n/\mathfrak{c})$ and $\lambda(T/\mathfrak{d}^{n-1}T) = e\lambda(T'/\mathfrak{d}^{n-1})$, the length condition in Corollary 3.7 gives $\lambda(\mathfrak{d}^n T/\mathfrak{c}T) = \lambda(T/\mathfrak{d}^{n-1}T)$.

This implies that if $R = T/\mathfrak{d}^n T$, then $S = T/\mathfrak{c}T$ is a Gorenstein Artin ring mapping onto $R$ and $\lambda(S) - \lambda(R) = \lambda(\mathfrak{d}^n T/\mathfrak{c}T) = \lambda(T/\mathfrak{d}^{n-1}T)$. Therefore $g(R) \leq \lambda(T/\mathfrak{d}^{n-1}T)$. Thus as a consequence of Theorem 3.8, we have proved

**Theorem 3.12.** *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field $\mathsf{k}$ of characteristic zero, $f_1, \dots, f_d$ be a system of parameters and $\mathfrak{d} = (f_1, \dots, f_d)$. Let $R = T/\mathfrak{d}^n$. Then $g(R) \leq \lambda(T/\mathfrak{d}^{n-1})$.*

In order to prove Theorem 3.1, we know need to show that $g(R) \geq \lambda(T/\mathfrak{d}^{n-1})$. We prove this by first computing the trace ideal $\omega^*(\omega)$ of the canonical module and use the fundamental inequalities. We use the lemmas concerning the computation of $\omega^*(\omega)$ proved in section 2.

**Theorem 3.13.** *Let $T = \mathsf{k}[|X_1, \dots, X_d|]$ be a power series ring over a field $\mathsf{k}$. Let $f_1, \dots, f_d$ be a system of parameters and $R = T/(f_1, \dots, f_d)^n$. Then $\omega^*(\omega) = (f_1, \dots, f_d)^{n-1}/(f_1, \dots, f_d)^n$, where $\omega$ is the canonical module of $R$.*

*Proof.* Let $T' = \mathsf{k}[[f_1, \dots, f_d]]$, $\mathfrak{d} = (f_1, \dots, f_d)^n T'$ and $R' \simeq T'/\mathfrak{d}^n$. By Theorem 3.2, $\omega_{R'}^*(\omega_{R'}) = \mathfrak{d}^{n-1}/\mathfrak{d}^n$. Therefore, since $T$ is free over $T'$, by Corollary 2.3, $\omega^*(\omega) = \mathfrak{d}^{n-1}T/\mathfrak{d}^n T = (f_1, \dots, f_d)^{n-1}/(f_1, \dots, f_d)^n$. □

*Proof of Theorem 3.1.* By Theorem 3.12, $g(R) \leq \lambda(T/(f_1, \ldots, f_d^{n-1}))$. The other inequality follows from Theorem 3.13 which can be seen as follows:

Let $\omega$ be the canonical module of $R$. We know that $g(R) \geq \lambda(R/\omega^*(\omega))$ by the fundamental inequalities. This yields $g(R) \geq \lambda(T/(f_1, \ldots, f_d)^{n-1})$ since $R = T/(f_1, \ldots, f_d)^n$ and $\omega^*(\omega) = (f_1, \ldots, f_d)^{n-1}/(f_1, \ldots, f_d)^n$. This gives us the equality $g(R) = \lambda(T/(f_1, \ldots, f_d)^{n-1})$ proving the theorem. □

**Corollary 3.14.** *Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field $\mathsf{k}$ of characteristic zero. Let $f_1, \ldots, f_d$ be a system of parameters and $R = T/(f_1, \ldots, f_d)^n$. Then $g(R) \leq \lambda(R/\mathrm{soc}(R))$.*

*Proof.* We have $\lambda(R/\mathrm{soc}(R)) \geq \lambda(T/(f_1, \ldots, f_d)^{n-1}) = g(R)$, since $(f_1, \ldots, f_d)^n :_T (X_1, \ldots, X_d) \subseteq (f_1, \ldots, f_d)^n :_T (f_1, \ldots, f_d) = (f_1, \ldots, f_d)^{n-1}$. □

**Remark 3.15.** Let $T = \mathsf{k}[[X, Y]]$, $R = T/(X, Y)^n$ and $S = T/(X^n, Y^n)$. Then $S$ is a Gorenstein Artin local ring mapping onto $R$ such that $\lambda(S) - \lambda(R) = \lambda(T/\mathfrak{m}_T^{n-1}) = \lambda(R/\mathrm{soc}(R))$. This, together with Corollary 3.3, shows that $g(R) = \lambda(R/\mathfrak{m}^{n-1})$ without any assumptions on the characteristic of $\mathsf{k}$. Thus, when $d = 2$, using the technique described in Remark 3.11, we see that the conclusion of Theorem 3.1 is independent of the characteristic of $\mathsf{k}$.

**Remark 3.16.** By taking $\mathfrak{d}$ to be the maximal ideal in Theorem 3.1, we get the following: Let $\mathsf{k}$ be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over $\mathsf{k}$. Let $\mathfrak{m}_T = (X_1, \ldots, X_d)$ be the maximal ideal of $T$ and $R := T/\mathfrak{m}_T^n$. Then

$$g(R) = \lambda(T/\mathfrak{m}_T^{n-1}) = \lambda(R/\mathrm{soc}(R)).$$

This also follows immediately from Theorems 3.2 and 3.8.

**Remark 3.17.** If $R = \mathsf{k}[X_1, \ldots, X_d]/(X_1, \ldots, X_d)^n$, where $\mathsf{k}$ is a field of characteristic zero, it follows from Theorem 3.2 and Theorem 3.8 that $g(R) = \lambda(R/\omega^*(\omega))$. Thus Question 3.9 in [1] has a positive answer, i.e., in this case,

$$\min\{\lambda(R/\mathfrak{a}) : \mathfrak{a} \simeq \mathfrak{a}^\vee\} = g(R).$$

# 4 Applications to Gorenstein Linkage

**Proposition 4.1.** *Let $(S, \mathfrak{m}, \mathsf{k})$ be a graded Gorenstein Artin local ring such that $\deg(\mathrm{soc}(S)) = t$. Let $f \in \mathfrak{m}$ be a homogeneous element in $S$ of degree $s$ and $\mathfrak{c} = (0 :_S f)$. Then $(\mathfrak{c} :_S \mathfrak{m}^n) = \mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c}$.*

*Proof.* Note that $\mathfrak{m}^{(t+1)-(s+n)} \cdot \mathfrak{m}^n \cdot f \subseteq \mathfrak{m}^{t+1} = 0$. Hence $\mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c} \subseteq \mathfrak{c}_n :_S \mathfrak{m}^n$. To prove the other inclusion, let $g$ be a homogeneous form of degree less than $(t+1) - (s+n)$. Then $g \cdot f$ is a homogeneous form of degree $t - n$ or less. If $g \cdot f = 0$, then $g \in \mathfrak{c}$. If $g \cdot f \neq 0$, since $S$ is Gorenstein, there is some element $h \in \mathfrak{m}^n$ such that $(gf) \cdot h$ generates soc(S) and hence is not zero. Thus $gf\mathfrak{m}^n \neq 0$ for $g \notin \mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c}$. Therefore $(\mathfrak{c} :_S \mathfrak{m}^n) \subseteq \mathfrak{m}^{(t+1)-(s+n)} + \mathfrak{c}$, proving the proposition. □

**Corollary 4.2.** *Let $\mathsf{k}$ be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring. Let $\mathfrak{m} = (X_1, \ldots, X_d)$ and $\mathfrak{c}_n = ((X_1^n, \ldots, X_d^n) :_T l^s)$, where $l = (X_1 + \cdots + X_d)$ and $s \geq (d-2)(n-1) - 1$. Then $(\mathfrak{c}_n :_T \mathfrak{m}^n) = \mathfrak{m}^{(d-1)(n-1)-s}$.*

*Proof.* By taking $S = T/(X_1^n, \ldots, X_d^n)$, it follows from Proposition 4.1 that $(\mathfrak{c}_n :_T \mathfrak{m}^n) = \mathfrak{m}^{(d-1)(n-1)-s} + \mathfrak{c}_n$. It remains to prove that $\mathfrak{c}_n \subseteq \mathfrak{m}^{(d-1)(n-1)-s}$.

Let $f$ be a homogeneous element of $T$ such that $f \in \mathfrak{c}$, i.e., $f \cdot l^s \subseteq (X_1^n, \ldots, X_d^n)$. Hence by Theorem 3.4, $deg(f) \geq \frac{(d(n-1)-s+1)}{2} \geq (d-1)(n-1) - s$ by the hypothesis on $s$. This shows that $\mathfrak{c}_n \subseteq m^{(d-1)(n-1)-s}$. □

Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring over a field $\mathsf{k}$. Let $f_1, \ldots, f_d$ be a system of parameters. Let $T' = \mathsf{k}[|f_1, \ldots, f_d|]$, $\mathfrak{d} = (f_1, \ldots, f_d)^n T'$ and $\mathfrak{c}_n = (f_1^n, \ldots, f_d^n) :_{T'} l^s$, where $l = f_1 + \cdots + f_d$ and $s \geq (d-2)(n-1) - 1$. Since, by Corollary 4.2, $(\mathfrak{c}_n :_{T'} \mathfrak{d}^n) = \mathfrak{d}^{(d-1)(n-1)-s}$ in $T'$, the same holds in $T$ by Remark 3.11. Therefore $(\mathfrak{c}_n T :_T \mathfrak{d}^n T) = \mathfrak{d}^{(d-1)(n-1)-s} T$. Thus we see that

**Proposition 4.3.** *Let* $\mathsf{k}$ *be a field of characteristic zero and* $T = \mathsf{k}[|X_1, \ldots, X_d|]$ *be a power series ring. Let* $\mathfrak{d} = (f_1, \ldots, f_d)$*, where* $f_1, \ldots, f_d$ *form a system of parameters. Let* $l = f_1 + \cdots + f_d$ *and* $s \geq (d-2)(n-1) - 1$*. Then* $\mathfrak{c}_n = ((f_1^n, \ldots, f_d^n) :_T l^s)$ *is a Gorenstein ideal such that* $(\mathfrak{c}_n :_T \mathfrak{d}^n) = \mathfrak{d}^{(d-1)(n-1)-s}$.

**Definition 4.4.** *Let* $(T, \mathfrak{m}_T, \mathsf{k})$ *be a regular local ring. An unmixed ideal* $\mathfrak{b} \subseteq T$ *is said to be in the Gorenstein linkage class of a complete intersection (glicci) if there is a sequence of ideals* $\mathfrak{c}_n \subseteq \mathfrak{b}_n$, $\mathfrak{b}_0 = \mathfrak{b}$*, satisfying*
*1)* $T/\mathfrak{c}_n$ *is Gorenstein for every* $n$
*2)* $\mathfrak{b}_{n+1} = (\mathfrak{c}_n :_T \mathfrak{b}_n)$ *and*
*3)* $\mathfrak{b}_n$ *is a complete intersection for some* $n$.

We say that $\mathfrak{b}$ is *linked* to $\mathfrak{b}_n$ via Gorenstein ideals in $n$ steps.

**Remark 4.5.**
1. Let $\mathsf{k}$ be a field of characteristic zero and $T = \mathsf{k}[|X_1, \ldots, X_d|]$ be a power series ring. Let $\mathfrak{d} = (f_1, \ldots, f_d)$, where $f_1, \ldots, f_d$ form a system of parameters. In [5], Kleppe, Migliore, Miro-Roig, Nagel and Peterson show that $\mathfrak{d}^n$ can be linked to $\mathfrak{d}^{n-1}$ via Gorenstein ideals in 2 steps and hence to $\mathfrak{d}$ in $2(n-1)$ steps. But in Proposition 4.3, by taking $s = (d-2)(n-1)$, we see that $\mathfrak{d}^n$ can be linked directly via the Gorenstein ideal $(f_1^n, \ldots, f_d^n) :_{T'} l^{(d-2)(n-1)}$ to $\mathfrak{d}^{n-1}$, and hence to $\mathfrak{d}$, a complete intersection, in $n-1$ steps.
2. In a private conversation, Migliore asked if this technique will show that $\mathfrak{d}^n$ is self-linked. We see that this can be done by taking $s = (d-2)(n-1) - 1$ in Proposition 4.3. Thus $\mathfrak{d}^n$ is linked to itself via the Gorenstein ideal $(f_1^n, \ldots, f_d^n) :_{T'} l^{(d-2)(n-1)-1}$.

## A Possible Approach to the Glicci Problem

**The Glicci problem:** Given any homogeneous ideal $\mathfrak{b} \subseteq T := \mathsf{k}[X_1, \ldots, X_d]$, such that $R := T/\mathfrak{b}$ is Cohen-Macaulay, is it true that $\mathfrak{b}$ is glicci?

A possible approach to the glicci problem is the following: Choose $\mathfrak{c}_n \subseteq \mathfrak{b}_n$ to be the closest Gorenstein. The question is: Does this ensure that $\mathfrak{b}_n$ is a complete intersection for some $n$?

**Example 4.6.** Let $T = \mathsf{k}[|X_1, \ldots, X_d|]$, where $\mathrm{char}(\mathsf{k}) = 0$. Let $\mathfrak{d} = (f_1, \ldots, f_d)$ be an ideal generated minimally by a system of parameters. We know by Theorems 3.1 and 3.12 that the ideal $\mathfrak{c}_n = (f_1^n, \ldots, f_d^n) :_T (f_1 + \cdots + f_d)^{(d-2)(n-1)}$ is a Gorenstein ideal closest to $\mathfrak{d}^n$. Now by taking $s = (d-2)(i-1)$ in Proposition 4.3, we see that $\mathfrak{c}_i :_T \mathfrak{d}^i = \mathfrak{d}^{i-1}$, $2 \leq i \leq n$. Thus $\mathfrak{d}^n$ can linked to $\mathfrak{d}$ by choosing a closest Gorenstein ideal at each step.

## 5 The Codimension Two Case

We begin this section by recalling the following result of Serre characterizing Gorenstein ideals of codimension two.

**Remark 5.1.** Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension two. Let $\mathfrak{c}$ be an $\mathfrak{m}_T$ primary ideal such that $S = T/\mathfrak{c}$ is a Gorenstein Artin local ring. Then $S$ is a complete intersection ring, i.e., $\mathfrak{c}$ is generated by 2 elements.

**Notation:** For the rest of this section, we will use the following notation: Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension 2, where that $\mathsf{k}$ is infinite. By $\mu(\_)$, we denote the minimal number of generators of a module and by $e_0(\_)$, we denote the multiplicity of an $\mathfrak{m}_T$-primary ideal in $T$. For an ideal $\mathfrak{b}$ in $T$, by $\bar{\mathfrak{b}}$, we denote the integral closure of $\mathfrak{b}$ in $T$.

**Remark 5.2.** We state the basic facts needed in this section in this remark. Their proofs can be found in [2] (Chapter 14).
1. Let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. We define the order of $\mathfrak{b}$ as $\operatorname{ord}(\mathfrak{b}) = \max\{i : \mathfrak{b} \subseteq \mathfrak{m}_T^i\}$.
Since $\mathfrak{m}_T$ is integrally closed, $\operatorname{ord}(\mathfrak{b}) = \operatorname{ord}(\bar{\mathfrak{b}})$.
2. Let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. Since $\mathsf{k}$ is infinite, a minimal reduction of $\mathfrak{b}$ is generated by 2 elements.
Further, if $\mathfrak{c}$ is a minimal reduction of $\mathfrak{b}$, the multiplicity of $\mathfrak{b}$, $e_0(\mathfrak{b}) = \lambda(T/\mathfrak{c})$.
3. The product of integrally closed $\mathfrak{m}_T$-primary ideals is integrally closed. In particular, if $\mathfrak{b}$ is an integrally closed $\mathfrak{m}_T$-primary ideal, then so is $\mathfrak{b}^n$ for each $n \geq 2$.
4. For an $\mathfrak{m}_T$-primary ideal $\mathfrak{b}$, $\lambda((\mathfrak{b} : \mathfrak{m}_T)/\mathfrak{b}) = \mu(\mathfrak{b}) - 1 \leq \operatorname{ord}(\mathfrak{b})$. Further, if $\mathfrak{b}$ is integrally closed, $\mu(\mathfrak{b}) - 1 = \operatorname{ord}(\mathfrak{b})$.
In particular, this yields $\mu(\mathfrak{b}) \leq \mu(\bar{\mathfrak{b}})$.

**Proposition 5.3.** *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension two and let $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. The closest (in terms of length) Gorenstein ideals contained in $\mathfrak{b}$ are its minimal reductions.*

*Proof.* Let $\mathfrak{c} \subseteq \mathfrak{b}$ be any Gorenstein ideal (and hence a complete intersection by the above remark). It is easy to see that $\lambda(T/\mathfrak{c}) \geq \lambda(T/(f,g))$, where $(f,g) \subseteq \mathfrak{b}$ is a minimal reduction of $\mathfrak{b}$. The reason is that

$$\begin{aligned} \lambda(T/\mathfrak{c}) &= e_0(\mathfrak{c}) \\ &\geq e_0(\mathfrak{b}) \qquad \text{since } \mathfrak{c} \subseteq \mathfrak{b} \\ &= \lambda(T/(f,g)). \end{aligned}$$

As a consequence,

$$\lambda(T/\mathfrak{c}) - \lambda(T/\mathfrak{b}) \geq \lambda(T/(f,g)) - \lambda(T/\mathfrak{b}),$$

$$\text{i.e., } \lambda(\mathfrak{b}/\mathfrak{c}) \geq \lambda(\mathfrak{b}/(f,g)).$$

Thus the closest Gorenstein ideal contained in $\mathfrak{b}$ is a minimal reduction $(f,g)$. ∎

We now prove the following theorem which shows that $g(R) \leq \lambda(R/\operatorname{soc}(R))$ where $R$ is the Artinian quotient of a 2-dimensional regular local ring.

**Theorem 5.4.** *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a regular local ring of dimension 2, with infinite residue field $\mathsf{k}$. Set $R = T/\mathfrak{b}$ where $\mathfrak{b}$ is an $\mathfrak{m}_T$-primary ideal. Then $g(R) \leq \lambda(R/\operatorname{soc}(R))$, i.e., there is a Gorenstein ring $S$ mapping onto $R$ such that $\lambda(S) - \lambda(R) \leq \lambda(R/\operatorname{soc}(R))$.*

In order to prove Theorem 5.4, we use a couple of formulae for $e_0(\mathfrak{b})$ and $\lambda(R)$ (which can be found, for example, in [4]). We need the following notation.

Let $(T, \mathfrak{m})$ and $(T', \mathfrak{n})$ be two-dimensional regular local rings. We say that $T'$ birationally dominates $T$ if $T \subseteq T'$, $\mathfrak{n} \cap T = \mathfrak{m}$ and $T$ and $T'$ have the same quotient field. We denote this by $T \leq T'$. Let $[T' : T]$ denote the degree of the field extension $T/\mathfrak{m} \subseteq T'/\mathfrak{n}$.

Further if $\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal in $T$, let $\mathfrak{b}^{T'}$ be the ideal in $T'$ obtained from $\mathfrak{b}$ by factoring $\mathfrak{b}T' = x\mathfrak{b}^{T'}$, where $x$ is the greatest common divisor of the generators of $\mathfrak{b}T'$. The following theorem ([4], Theorem 3.7) gives a formula for $e_0(\mathfrak{b})$.

**Theorem 5.5.** *Let $(T, \mathfrak{m}_T, \mathsf{k})$ be a two-dimensional regular local ring and $\mathfrak{b}$ be an $\mathfrak{m}_T$-primary ideal. Then*

$$e_0(\mathfrak{b}) = \sum_{T \leq T'} [T' : T] \mathrm{ord}(\mathfrak{b}^{T'})^2.$$

The following formula ([4], Theorem 3.10) is attributed to Hoskin and Deligne.

**Theorem 5.6** (Hoskin-Deligne Formula)**.** *Let $T$, $\mathfrak{b}$ and $R$ be as in Theorem 5.4. Further assume that $\mathfrak{b}$ is an integrally closed ideal. Then,*

$$\lambda(R) = \sum_{T \leq T'} \binom{\mathrm{ord}(\mathfrak{b}^{T'}) + 1}{2} [T' : T].$$

**Corollary 5.7.** *Let $T$, $\mathfrak{b}$ and $R$ be as in the Hoskin-Deligne formula. Then we have the inequality*

$$e_0(\mathfrak{b}) + \mathrm{ord}(\mathfrak{b}) \leq 2\lambda(R).$$

*Proof.* By Theorem 5.5, we have $e_0(\mathfrak{b}) = \sum_{T \leq T'} \mathrm{ord}(\mathfrak{b}^{T'})^2 [T' : T]$.

Using the Hoskin-Deligne formula, we see that

$$\lambda(R) = \sum_{T \leq T'} \frac{\mathrm{ord}(\mathfrak{b}^{T'})^2 + \mathrm{ord}(\mathfrak{b}^{T'})}{2} [T' : T]$$

giving us

$$2\lambda(R) = e_0(\mathfrak{b}) + \sum_{T \leq T'} \mathrm{ord}(\mathfrak{b}^{T'}) [T' : T].$$

Since $T \leq T$ and $\mathfrak{b}^T = \mathfrak{b}$, we get the required inequality. □

**Corollary 5.8.** *Let $T$, $R$ and $\mathfrak{b}$ be as in Theorem 5.4. Then*

$$e_0(\mathfrak{b}) + \mu(\mathfrak{b}) - 1 \leq 2\lambda(T/\mathfrak{b}).$$

**Proof:** Let $\overline{\mathfrak{b}}$ be the integral closure of $\mathfrak{b}$. By the previous corollary, we have $e_0(\overline{\mathfrak{b}}) + \mathrm{ord}(\overline{\mathfrak{b}}) \leq 2\lambda(T/\overline{\mathfrak{b}})$. Since $\overline{\mathfrak{b}}$ is integrally closed, $\mathrm{ord}(\overline{\mathfrak{b}}) = \mu(\overline{\mathfrak{b}}) - 1$. Thus we get $e_0(\overline{\mathfrak{b}}) + \mu(\overline{\mathfrak{b}}) - 1 \leq 2\lambda(T/\overline{\mathfrak{b}})$. Now $e_0(\mathfrak{b}) = e_0(\overline{\mathfrak{b}})$, $\mu(\mathfrak{b}) \leq \mu(\overline{\mathfrak{b}})$ and $\lambda(T/\overline{\mathfrak{b}}) \leq \lambda(T/\mathfrak{b})$, giving the required inequailty. □

**Proof of Theorem 5.4:** For any ideal $\mathfrak{b}$ in $T$, we have $\mu(\mathfrak{b}) - 1 = \lambda((\mathfrak{b} : \mathfrak{m})/\mathfrak{m})$. But $(\mathfrak{b} : \mathfrak{m})/\mathfrak{b} \simeq \mathrm{soc}(R)$. Thus by the previous corollary, we have

$$e_0(\mathfrak{b}) + \lambda(\mathrm{soc}(R)) \leq 2\lambda(R). \qquad (\sharp\sharp)$$

Let $(f, g)$ be a minimal reduction of $\mathfrak{b}$. Then $S := T/(f, g)$ is a complete intersection ring (and hence Gorenstein) mapping onto $R$. Moreover $\lambda(S) = e_0(\mathfrak{b})$. Thus $(\sharp\sharp)$ can be read as $\lambda(S) + \lambda(\text{soc}(R)) \leq 2\lambda(R)$. Rearranging, we get $\lambda(S) - \lambda(R) \leq \lambda(R) - \lambda(\text{soc}(R))$. This proves that $g(R) \leq \lambda(R/\text{soc}(R))$. $\square$

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