Contractive spectral triples for crossed products

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$X = (A, \mathcal{H}, D)$ a spectral triple - this means:
A a (unital) C^*-algebra
$\mathcal{H}$ a Hilbert space, $A \subset B(\mathcal{H})$
D a s. a. operator, compact resolvent (regular): spectrum isolated eigenvalues and $(D - \imath I)^{-1}$ is compact
the set $\mathcal{C}^1(X)$ of a's in A for which $a\, Dom(D) \subset Dom(D)$ and $\|[D, a]\| < \infty$ is dense in A.
In noncommutative geometry, Connes showed that spectral triples not only give a context for K-homology and cyclic cohomology but also encode (noncommutative) metric space information. This has been developed in recent work for the case of group actions by Bellissard, Marcolli and Reihani (BMR) who studied in depth the case where the group $G = \mathbf{Z}$. They show that there are (noncommutative) metric versions of metric space notions, such as (later) *equicontinuous* and *isometric*, using commutators with D (Lipschitz norm).

In BMR, it is shown that there is a natural "dual" spectral triple Y

for the reduced crossed product $A \rtimes_{\alpha,r} \mathbb{Z}$: here

$$Y = (A \rtimes_{\alpha,r} \mathbb{Z}, \mathcal{H} \otimes \ell^2(\mathbb{Z}, \mathbb{C}^2), \widehat{D}).$$

Also, and reminiscent of the formula for the Kasparov product of two unbounded Fredholm modules, $\widehat{D}$ is the 2×2 off-diagonal matrix with entries

$$D \otimes 1 \mp \imath 1 \otimes c$$

where $c(f)(n) = nf(n)$ for $f \in \ell^2(\mathbb{Z})$. Further, there is the canonical dual action of $\widehat{\mathbb{Z}} = \mathbb{T}$ on the crossed product. Then (BMR):

(1) If the action on X is equicontinuous, then Y is isometric;

(2) if the action on X is equicontinuous and X satisfies certain "metric commutant" and compactness conditions, then the Connes metrics induced on the state space of A by both X, Y are equivalent, and give the weak * topology (results of Rieffel, Pavlović and Latrémolière (for the non-unital case) are required for this);

(3) if X is not equicontinuous, it can effectively be replaced by a spectral triple that *is* equicontinuous (using a "metric bundle" construction inspired by Connes-Moscovici).

This gives detailed information about the noncommutative geometry contained in spectral triple crossed products involving the discrete group $\mathbb{Z}$. But in noncommutative geometry, in the work of Connes-Moscovici, one needs to consider spectral triple crossed products involving general locally compact groups, in particular, a general discrete group G or even an étale groupoid. This is what I want to talk about for the rest of the talk. The objective is to obtain an analogue of (1) for this case. So let G be a discrete group acting on A with action α, $X = (A, \mathcal{H}, D)$ a spectral triple. We say that X is *pointwise bounded* if

$$\mathcal{C}_b^1(G,X) = \{a \in \mathcal{C}^1(X) : \alpha_g(a) \in \mathcal{C}^1(X) \text{ for all } g \in G \text{ and } \sup_{g\in G} \|[D,\alpha_g(a)]\| < \infty\}$$

is dense in A. (This is close to "equicontinuous" in the sense of BMR.) X is *isometric* if $\mathcal{C}^1(X) = \mathcal{C}_b^1(G,X)$ and

$$\|[D,\alpha_g(a)]\| = \|[D,a]\|$$

for all $a \in \mathcal{C}^1(X), g \in G$.

There are two immediate difficulties in defining the dual spectral triple Y for general discrete G. (Recall that the BMR theory deals with the case $G = \mathbf{Z}$.) The first is: what should c be in this case in the definition of $\widehat{D}$? There is a natural way to construct c. Suppose that G is finitely generated. Any symmetric generating set S for G determines a word metric on G and this gives a suitable version of c: $c(g)$ is the smallest integer n such that g can be written as a product of n elements of S. The second difficulty is that in the $\mathbb{Z}$ case, we had the dual group $\mathbb{T}$ available to act on the crossed product. This is no longer the case for general G. Instead, as in the Imai-Takai duality theorem, we have to consider the dual coaction

$$\delta : A \rtimes_{\alpha,r} G \to (A \rtimes_{\alpha,r} G) \otimes C_r^*(G)$$

where for $F \in C_c(G,A)$,

$$\delta(F) = \int \widetilde{\pi}(F(s))\widetilde{\lambda}_s \otimes \lambda_s$$

where $(\widetilde{\pi},\widetilde{\lambda})$ is the covariant representation giving the regular representation of $A \rtimes_{\alpha,r} G$ on $\mathcal{H} \otimes \ell^2(G)$ and λ is the left regular representation of G on $\ell^2(G)$.

We need geometric definitions (e.g. of "isometry") for coactions just as we have for actions. It is not immediately clear, I think, what they should be. However, roughly, it is reasonable to think that if we dualize the coaction in some sense, then we should have something like an action (though not of a group). More precisely, let $P_r(G)$ be the state space of $C_r^*(G)$. This is a subsemigroup of $(C_r^*(G))^* \subset B(G)$. If B is a C^*-algebra, $Y = (B, \mathcal{H}', D')$ a spectral triple, and $\delta : B \to B \otimes C_r^*(G)$ a (non-degenerate) coaction, then we can define an *action* of $P_r(G)$ on B by setting, for $\phi \in P_r(G)$,

$$\beta_\phi(b) = S_\phi(\delta(b))$$

where S_ϕ denotes the slice map. The isometry condition in the action case is replaced by the *contractive* condition: the coaction is *contractive* if $\mathcal{C}^1(Y)$ is $P_r(G)$-invariant, and

$$\|[D', \beta_\phi(b)]\| \leq \|[D', b]\|$$

for all $\phi \in P_r(B), b \in \mathcal{C}^1(Y)$.

The theorem is then as follows (with $B = A \rtimes_{\alpha,r} G$ and δ the dual coaction):

Theorem. Let $X = (A, \mathcal{H}, D)$ be a spectral triple, α an action of a discrete group G on A for which X is equicontinuous. Then

$$Y = (A \rtimes_{\alpha,r} G, \mathcal{H} \otimes \ell^2(G, \mathbf{C}^2), \widehat{D})$$

is a spectral triple that is contracting for the dual coaction $\delta : A \rtimes_{\alpha,r} G \to (A \rtimes_{\alpha,r} G) \otimes C_r^*(G)$.

When G is abelian, the dual coaction can be identified with the dual action of $\widehat{G}$ on $A \rtimes_{\alpha,r} G$, and the characters - the extreme points of $P_r(G)$ - then give an *isometric* action (as in the case $G = \mathbf{Z}$ in BMR).