Invariants for operator algebras of topological dynamics

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We begin by letting A be an operator algebra with commutative diagonal (i.e. $\Delta(A) := A \cap A^*$ is commutative), together with a completely contractive homomorphism $\varphi : A \to \Delta(A)$ such that $\varphi^2 = \varphi$ and $\varphi|_{\Delta(A)}$ is the identity. We will let A_0 denote the kernel of φ.

(Our motivating examples here are: directed graph algebras, tensor algebras for multivariate dynamics, and semicrossed products for multivariate dynamics)

Associated to such an algebra we create a directed graph $G(A)$ as follows:

For the vertices we consider the maximal ideal space of $\Delta(A)$. Recall that this is in one-to-one correspondence with nontrivial homomorphisms from $\Delta(A)$ to $\mathbb{C}$ (notationally if $\pi : \Delta(A) \to \mathbb{C}$ we will call the vertex π).

For pairs of vertices π_1, π_2 we consider the collection of completely contractive representations $t : A \to T_2$ where T_2 is the upper triangular 2×2 matrices of the form

$$t(a) = \begin{bmatrix} \pi_2(\varphi(a)) & t_{2,2}(a) \\ 0 & \pi_1(\varphi(a)) \end{bmatrix}$$

where $t_{2,2}$ is a nonzero map. Call this collection $T(\pi_1, \pi_2)$.

We let $K(\pi_1, \pi_2) = \bigcap_{t \in T(\pi_1, \pi_2)} \ker t$.

Now for each possible pair $\pi_1, \pi_2 \in V \times V$ we draw n edges from π_2 to π_1 where $n = \dim(A_0/(A_0 \cap K(\pi_1, \pi_2)))$.

Not enough information (Part 1):

Consider $A := \begin{bmatrix} A(\mathbb{D}) & C(\mathbb{T}) \\ 0 & 0 \end{bmatrix}$ then A is of this form with $\Delta(A) = \mathbb{C}$. Notice that the graph of this algebra is a single vertex and a single edge.

Fix: We have to assume that $\bigcap_{n\geq 1} A_0^n = \{0\}$.

Not enough information (Part 2):

Consider the ideal $I_{1,3}$ in T_3, then T_3 and $T_3/I_{1,3}$ have the same graph.

Fix: Consider "admissible paths" which correspond to representations of A into T_n which along the diagonal correspond to vertices and whose range contains the ideal $I_{1,n}$.

Not enough information (Part 3):

Consider $X = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\}$ with the usual topology, and $f : X \to X$ is given by $f(\frac{1}{n}) = \frac{1}{n+1}$ and $f(0) = 0$. We let $A = C(X) \rtimes_f \mathbb{Z}_+$.

Similarly consider the directed graph G with vertex set X and edges $\{(\frac{1}{n}, \frac{1}{n+1})\} \cup \{(0,0)\}$ and let $B = A(G)$.

Notice that A and B give rise to the same graph but they are very different algebras.

Fix: Topologize the graph.

Part 1: Put the weak-$*$ topology on V.

Part 2: Topologize E (complications, see above).

Partition E via an equivalence relation $\sim$ such that no two edges in an equivalence class share a source. We consider the sets $[e] \times \{s(f) : f \sim e\}$ and we topologize each of these sets via the topology on $\{s(f) : f \sim e\}$, then these sets form a new edge set F, and we consider (V, F, r, s) where $s(([e], x)) = x$ and $r(([e], x)) = r(f)$ where $f \sim e$ and $s(f) = x$. Of course this is rarely going to give rise to a topological graph.

We say a partition is topologically realizable if the range and source maps are continuous with respect to the partition.

Question: If A is an algebra and there are two different partitions of the edge set of A that are both topologically realizable is that "okay"? (i.e. can we have two essentially different topologies on the graph for A)

If yes, is there a canonical choice (Davidson-Roydor).

Not enough information (Part 4):

If $\tau = \{\tau_1, \tau_2, \cdots, \tau_n\}$ are continuous proper self maps of X then the algebras $C(X) \rtimes_\tau \mathbb{F}_n^+$ need not equal $A(X, \tau)$, although they have identical directed graphs (assuming an implicit choice of partitions of the edges).

Fix: Add a labelling to the edges of the graph.