Generalized gauge actions, KMS states, and Hausdorff dimension for higher-rank graphs

Elizabeth Gillaspy
University of Montana

joint with C. Farsi, (S. Kang,) N. Larsen, J. Packer [FGLP18]

NIFAS, 3 November 2018

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry;

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \qquad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \qquad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e:r(e)=v} t_e t_e^*$.

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e : r(e) = v} t_e t_e^*$.

It then follows that $t_e t_f = 0$ unless $s(e) = r(f)$.

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \qquad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e : r(e) = v} t_e t_e^*$.

It then follows that $t_e t_f = 0$ unless $s(e) = r(f)$.

$C^*(E)$ is <u>universal</u> for representations of $\{t_e, t_v\}_{v \in E^0, e \in E^1}$

What are graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e:r(e)=v} t_e t_e^*$.

It then follows that $t_e t_f = 0$ unless $s(e) = r(f)$.

$C^*(E)$ is universal for representations of $\{t_e, t_v\}_{v \in E^0, e \in E^1}$; any collection of partial isometries and projections $\{s_e, s_v\}_{v,e} \subseteq B(\mathcal{H})$ satisfying the above conditions generates a quotient of $C^*(E)$.

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$.

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra.

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra.
- $E^\infty = \{(e_n)_{n\in\mathbb{N}} : s(e_i) = r(e_{i+1}) \ \forall\ i\}$ is a Cantor set;

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra.
- $E^\infty = \{(e_n)_{n\in\mathbb{N}} : s(e_i) = r(e_{i+1}) \ \forall \ i\}$ is a Cantor set; [IK13] connection between KMS states on $C^*(E)$ and Hausdorff structures on E^∞.
- [Mat10, MM14, BCW17] $(E^\infty, \sigma) \sim (F^\infty, \sigma)$ (continuous orbit equivalence), plus a little more, gives $C^*(E) \cong C^*(F)$.

Why are graph C^*-algebras awesome?

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra.
- $E^\infty = \{(e_n)_{n\in\mathbb{N}} : s(e_i) = r(e_{i+1}) \ \forall \ i\}$ is a Cantor set; [IK13] connection between KMS states on $C^*(E)$ and Hausdorff structures on E^∞.
- [Mat10, MM14, BCW17] $(E^\infty, \sigma) \sim (F^\infty, \sigma)$ (continuous orbit equivalence), plus a little more, gives $C^*(E) \cong C^*(F)$.
- [ERRS16] $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ iff a finite number of moves will convert E into F.

Higher-rank graphs (k-graphs)

A k-graph consists of:

- An edge-colored directed graph, with k colors of edges; together with

Higher-rank graphs (k-graphs)

A k-graph consists of:

- An edge-colored directed graph, with k colors of edges; together with
- A factorization rule: for any two colors (red, blue), any red-blue path from v to w is equivalent to a unique blue-red path.

$$\begin{tikzcd}
\bullet \arrow[d, dashed, blue] & v \arrow[l, red] \arrow[d, dashed, blue] \arrow[dl, phantom, "\sim"] \\
w & \bullet \arrow[l, red]
\end{tikzcd}$$

Higher-rank graphs (k-graphs)

A k-graph consists of:

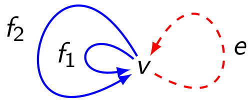
- An edge-colored directed graph, with k colors of edges; together with
- A factorization rule: for any two colors (red, blue), any red-blue path from v to w is equivalent to a unique blue-red path.

$$\begin{tikzcd} \bullet \arrow[d, dashed, blue] & v \arrow[l, red] \arrow[d, dashed, blue] \arrow[dl, phantom, "\sim"] \\ w & \bullet \arrow[l, red] \end{tikzcd}$$

Introduced by Kumjian & Pask in 2000 to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

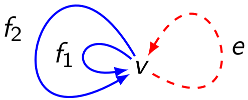
Paths in $E \rightsquigarrow$ k-dimensional rectangles in Λ.

Example and notation



Factorization rule: $f_1 e = e f_2, \quad f_2 e = e f_1.$

Example and notation

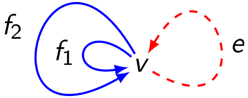


Factorization rule: $f_1 e = e f_2, \quad f_2 e = e f_1.$

For $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ we write

$$\Lambda^n = \{\lambda \in \Lambda : \lambda \text{ has } n_i \text{ edges of color } i\}.$$

Example and notation



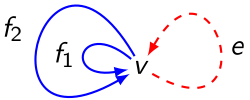
Factorization rule: $f_1 e = ef_2, \quad f_2 e = ef_1.$

For $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$ we write

$$\Lambda^n = \{\lambda \in \Lambda : \lambda \text{ has } n_i \text{ edges of color } i\}.$$

The factorization rule means this is well defined.

Example and notation



Factorization rule: $f_1 e = ef_2, \quad f_2 e = ef_1$.

For $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ we write

$$\Lambda^n = \{\lambda \in \Lambda : \lambda \text{ has } n_i \text{ edges of color } i\}.$$

The factorization rule means this is well defined.

Note that Λ^0 is the vertices of Λ.

Strongly connected k-graphs

A (higher-rank) graph is <u>strongly connected</u> if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\textit{edges of color } i \textit{ from } w \textit{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v\in\Lambda^0}$ of ℓ^1-norm 1.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\textit{edges of color } i \textit{ from } w \textit{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v\in\Lambda^0}$ of ℓ^1-norm 1.

In fact, [HLRS15] a finite Λ is strongly connected iff there is a finite sum of finite products of $\{A_i\}_{1\leq i\leq k}$ which is irreducible.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\textit{edges of color } i \textit{ from } w \textit{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v\in\Lambda^0}$ of ℓ^1-norm 1.

In fact, [HLRS15] a finite Λ is strongly connected iff there is a finite sum of finite products of $\{A_i\}_{1\leq i\leq k}$ which is irreducible.

Note that $A_iA_j = A_jA_i \ \forall \ i, j$ by the factorization rule.

Infinite paths and Cantor sets

An infinite path in a k-graph Λ is (the equivalence class of) an infinite string of composable edges (range but no source) which contains infinitely many edges of any color.

Infinite paths and Cantor sets

An infinite path in a k-graph Λ is (the equivalence class of) an infinite string of composable edges (range but no source) which contains infinitely many edges of any color.

If $|v\Lambda^{e_i}| \geq 2$ for at least one i, the space Λ^∞ of infinite paths in Λ is naturally a Cantor set – compact, perfect, totally disconnected.

Infinite paths and Cantor sets

An infinite path in a k-graph Λ is (the equivalence class of) an infinite string of composable edges (range but no source) which contains infinitely many edges of any color.

If $|v\Lambda^{e_i}| \geq 2$ for at least one i, the space Λ^∞ of infinite paths in Λ is naturally a Cantor set – compact, perfect, totally disconnected.

The collection of sets

$$Z(\lambda) = \{x \in \Lambda^\infty : x = \lambda y\},$$

where $\lambda \in \Lambda$ is a finite path (morphism) in Λ, is a compact open basis for the topology on Λ^∞ making Λ^∞ into a Cantor set.

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞.

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞. For many choices of metric, we get the same (nontrivial) Hausdorff measure on Λ^∞.

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞. For many choices of metric, we get the same (nontrivial) Hausdorff measure on Λ^∞.

Definition

For (X, d) a metric space and $s \in \mathbb{R}_{\geq 0}$, the Hausdorff measure of dimension s of a compact subset Z of X is

$$H^s(Z) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{U_i \in F} (\text{diam } U_i)^s : |F| < \infty, \right.$$
$$\left. \bigcup_i U_i = Z, \ \text{diam } U_i < \epsilon \ \forall i \right\}.$$

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞. For many choices of metric, we get the same (nontrivial) Hausdorff measure on Λ^∞.

Definition

For (X, d) a metric space and $s \in \mathbb{R}_{\geq 0}$, the Hausdorff measure of dimension s of a compact subset Z of X is

$$H^s(Z) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{U_i \in F} (\text{diam } U_i)^s : |F| < \infty, \right.$$
$$\left. \bigcup_i U_i = Z, \text{ diam } U_i < \epsilon \; \forall i \right\}.$$

Moreover, $\exists!\, s \in \mathbb{R} : t < s \Rightarrow H^t(X) = \infty$ and $t > s \Rightarrow H^t(X) = 0$.

We call s the Hausdorff dimension of X.

We study metrics on Λ^∞ which arise from putting "weights" on Λ.

$\mathbb{R}_+$-functors

We study metrics on Λ^∞ which arise from putting "weights" on Λ.

Definition ([McN15])

Let Λ be a higher-rank graph. An $\underline{\mathbb{R}_+\text{-functor}}$ on Λ is a function $y : \Lambda \to \mathbb{R}_{\geq 0}$ such that

$$y(\lambda\nu) = y(\lambda) + y(\nu).$$

Note that this forces $y(v) = 0$ for all vertices v.

$\mathbb{R}_+$-functors

We study metrics on Λ^∞ which arise from putting "weights" on Λ.

Definition ([McN15])

Let Λ be a higher-rank graph. An $\underline{\mathbb{R}_+\text{-functor}}$ on Λ is a function $y : \Lambda \to \mathbb{R}_{\geq 0}$ such that

$$y(\lambda\nu) = y(\lambda) + y(\nu).$$

Note that this forces $y(v) = 0$ for all vertices v.

Proposition (Farsi-G-Larsen-Packer)

For any $\mathbb{R}_+$-functor y on a strongly connected finite k-graph Λ, and any $\beta \geq 0$, the matrices $\{B_i(y,\beta)\}_{1\leq i\leq k} \in M_{\Lambda^0}$ given by

$$B_i(y,\beta)_{v,w} = \sum_{\lambda \in v\Lambda^{e_i}w} e^{-\beta y(\lambda)}$$

have a unique positive common eigenvector $\xi^{y,\beta}$ of ℓ^1-norm 1.

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor.

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor. In fact, in this case, $B_i(y, \beta) = A_i$ for all β.

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor. In fact, in this case, $B_i(y, \beta) = A_i$ for all β.

Define

$$\rho(B(y, \beta)) := (\rho(B_1(y, \beta)), \rho(B_2(y, \beta)), \ldots, \rho(B_k(y, \beta))).$$

Examples and Notation for $\mathbb{R}_+$-functors

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor. In fact, in this case, $B_i(y, \beta) = A_i$ for all β.

Define

$$\rho(B(y, \beta)) := (\rho(B_1(y, \beta)), \rho(B_2(y, \beta)), \ldots, \rho(B_k(y, \beta))).$$

Then, for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, define

$$\rho(B(y, \beta))^n := \rho(B_1(y, \beta))^{n_1} \cdot \rho(B_2(y, \beta))^{n_2} \cdot \cdots \cdot \rho(B_k(y, \beta))^{n_k}.$$

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)} \right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i.

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)}\right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i. Then,

$$d_{y,\beta}(x,z) := w_{y,\beta}(x \wedge z), \qquad \textit{where } x \wedge z = \max\{\lambda : x, z \in Z(\lambda)\},$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology.

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)}\right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i. Then,

$$d_{y,\beta}(x,z) := w_{y,\beta}(x \wedge z), \qquad \textit{where } x \wedge z = \max\{\lambda : x, z \in Z(\lambda)\},$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology. Also, $(\Lambda^\infty, d_{y,\beta})$ has Hausdorff dimension β and Hausdorff measure

$$\mu_{y,\beta}(Z(\lambda)) = H^\beta(Z(\lambda)) = w_{y,\beta}(\lambda)^\beta = e^{-\beta y(\lambda)} \rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)}.$$

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.
Taking $y = 0$ in the previous Theorem gives:

Corollary (FGKLP)

For any finite strongly connected k-graph, and any $\beta \in (0, \infty)$, the function

$$d_\beta(x, z) := \left(\rho(\Lambda)^{-d(x \wedge z)} x^\Lambda_{s(\lambda)}\right)^{1/\beta}$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology.

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.
Taking $y = 0$ in the previous Theorem gives:

Corollary (FGKLP)

For any finite strongly connected k-graph, and any $\beta \in (0, \infty)$, the function

$$d_\beta(x, z) := \left(\rho(\Lambda)^{-d(x \wedge z)} x^\Lambda_{s(\lambda)}\right)^{1/\beta}$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology. Moreover, the ultrametric spaces $(\Lambda^\infty, d_\beta)$ all have the same Hausdorff measure, namely M.

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09]

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09]
  In particular, the symmetries of the KMS_β states are given by Galois groups for large β.

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS_β states are given by Galois groups for large β.
- A sudden change in the number of KMS_t states for (A, α) $\leftrightsquigarrow$ "symmetry breaking"

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS$_\beta$ states are given by Galois groups for large β.
- A sudden change in the number of KMS_t states for (A, α) $\leftrightsquigarrow$ "symmetry breaking"
- For graph algebras and related constructions, KMS states for the gauge action can often be computed by integration against the Hausdorff measure of Λ^∞; cf. [Nes13, Tho14, HLRS15]

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS$_\beta$ states are given by Galois groups for large β.
- A sudden change in the number of KMS_t states for (A, α) $\leftrightsquigarrow$ "symmetry breaking"
- For graph algebras and related constructions, KMS states for the gauge action can often be computed by integration against the Hausdorff measure of Λ^∞; cf. [Nes13, Tho14, HLRS15]

Given an $\mathbb{R}_+$-functor on Λ, we obtain an associated action on $C^*(\Lambda)$, and compute the associated KMS states.

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda\in\Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

From an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$, get an action of $\mathbb{R}$ on $C^*(\Lambda)$:

$$\alpha_t^{y,\beta}(s_\lambda) = e^{ity(\lambda)} \left(\rho(B(y,\beta))^{d(\lambda)}\right)^{it/\beta} s_\lambda = (e^{it})^{y(\lambda)+\frac{1}{\beta}\log\left(\rho(B(y,\beta))^{d(\lambda)}\right)} s_\lambda.$$

$\mathbb{R}_+$-functors, generalized gauge actions, KMS states

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

From an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$, get an action of $\mathbb{R}$ on $C^*(\Lambda)$:

$$\alpha_t^{y,\beta}(s_\lambda) = e^{ity(\lambda)} \left(\rho(B(y,\beta))^{d(\lambda)} \right)^{it/\beta} s_\lambda = (e^{it})^{y(\lambda) + \frac{1}{\beta} \log\left(\rho(B(y,\beta))^{d(\lambda)}\right)} s_\lambda.$$

Generalized gauge actions [Nes13, Tho14, McN15]

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda\in\Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

From an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$, get an action of $\mathbb{R}$ on $C^*(\Lambda)$:

$$\alpha_t^{y,\beta}(s_\lambda) = e^{ity(\lambda)} \left(\rho(B(y,\beta))^{d(\lambda)}\right)^{it/\beta} s_\lambda = (e^{it})^{y(\lambda)+\frac{1}{\beta}\log\left(\rho(B(y,\beta))^{d(\lambda)}\right)} s_\lambda.$$

Generalized gauge actions [Nes13, Tho14, McN15]

Definition

A positive linear map $\phi : C^*(\Lambda) \to \mathbb{C}$ is a KMS state at (inverse) temperature t for $\alpha^{y,\beta}$ if, for all $\lambda, \eta, \nu, \rho \in \Lambda$,

$$\phi(s_\lambda s_\eta^* s_\nu s_\rho^*) = \phi(\alpha_{it}^{y,\beta}(s_\nu s_\rho^*) s_\lambda s_\eta^*).$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

Theorem (Farsi-G-Larsen-Packer, [Tho14])

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. Suppose $\rho(B_i(y,\beta)) > 1$ for some $1 \leq i \leq k$. Then

$$\phi(a) = \int_{\Lambda^\infty} \Phi(a)\, d\mu_{y,\beta}$$

is a KMS state at inverse temperature β for $(C^*(\Lambda), \alpha^{y,\beta})$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

Theorem (Farsi-G-Larsen-Packer, [Tho14])

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. Suppose $\rho(B_i(y, \beta)) > 1$ for some $1 \leq i \leq k$. Then

$$\phi(a) = \int_{\Lambda^\infty} \Phi(a)\, d\mu_{y,\beta}$$

is a KMS state at inverse temperature β for $(C^*(\Lambda), \alpha^{y,\beta})$.

Corollary: $\mu_{y,\beta}$ is quasi-invariant.

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

Theorem (Farsi-G-Larsen-Packer, [Tho14])

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. Suppose $\rho(B_i(y, \beta)) > 1$ for some $1 \leq i \leq k$. Then

$$\phi(a) = \int_{\Lambda^\infty} \Phi(a)\, d\mu_{y,\beta}$$

is a KMS state at inverse temperature β for $(C^*(\Lambda), \alpha^{y,\beta})$.

Corollary: $\mu_{y,\beta}$ is quasi-invariant. (In fact, it's the unique quasi-invariant measure for $\alpha^{y,\beta}$.)

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m - n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t. } d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m - n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m - n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m-n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m-n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda) \cong C(\widehat{Per\,\Lambda})$:

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\mathrm{Per}\,\Lambda = \{m - n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(\mathrm{Per}\,\Lambda) \cong C(\widehat{\mathrm{Per}\,\Lambda})$: the KMS state associated to $z \in \widehat{\mathrm{Per}\,\Lambda} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin \mathrm{Per}\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & \text{else.} \end{cases}$$

Connections with the literature

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ *admits KMS states at inverse temperature t iff* $\alpha^{y,\beta} = \alpha^{y,t}$. *Moreover, the simplex of* KMS_β *states for* $\alpha^{y,\beta}$ *is affinely isomorphic to the state space of* $C^*(Per\,\Lambda)$*: the KMS state associated to* $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ *is*

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

Connections with the literature

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

- Proof uses Christensen's characterization [Chr17] of KMS states for groupoid algebras.

Connections with the literature

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

- Proof uses Christensen's characterization [Chr17] of KMS states for groupoid algebras.
- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.

Connections with the literature

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & \textit{else.} \end{cases}$$

- Proof uses Christensen's characterization [Chr17] of KMS states for groupoid algebras.
- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.
- McNamara [McN15] studied coordinate-wise irreducible k-graphs;

Connections with the literature

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)}\mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

- Proof uses Christensen's characterization [Chr17] of KMS states for groupoid algebras.
- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.
- McNamara [McN15] studied coordinate-wise irreducible k-graphs; in this case, ϕ is the unique KMS_β state.

Connections with the literature

Theorem (Farsi-G-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS$_\beta$ states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & \textit{else.} \end{cases}$$

- Proof uses Christensen's characterization [Chr17] of KMS states for groupoid algebras.
- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.
- McNamara [McN15] studied coordinate-wise irreducible k-graphs; in this case, ϕ is the unique KMS$_\beta$ state.

- J.-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, Selecta Math. **1** (1995), no. 3.
- N. Brownlowe, T.M. Carlsen, and M.F. Whittaker, Graph algebras and orbit equivalence, Ergodic Theory and Dynamical Systems **37** (2017), no. 2, 389–417.
- J. Christensen, Symmetries of the KMS simplex, arXiv:1710.04412, 2017.
- A. Connes, M. Marcolli, and N. Ramachandran, KMS states and complex multiplication, Selecta Math. (N.S.) **11** (2005), no. 3-4, 325–347.
- D. Drinen, Viewing AF-algebras as graph algebras, Proc. Amer. Math. Soc. **128** (2000), 1991–2000.

References II

- S. Eilers, T. Katsura, E. Ruiz, and M. Tomforde, Identifying AF-algebras that are graph C^*-algebras, J. Funct. Anal. **266** (2014), no. 6, 3968–3996.
- S. Eilers, G. Restorff, E. Ruiz, and A. P. W. Sørensen, The complete classification of unital graph C^*-algebras: Geometric and strong, arXiv:1611.07120, 2016.
- C. Farsi, E. Gillaspy, N.S. Larsen, and J. Packer, Generalized gauge actions on k-graph C^*-algebras: KMS states and Hausdorff structure, arXiv:1807.08665, 2018.
- A. an Huef, M. Laca, I. Raeburn, and A. Sims, KMS states on the C^*-algebra of a higher-rank graph and periodicity in the path space, J. Funct. Anal. **268** (2015), 1840–1875.
- M. Ionescu and A. Kumjian, Hausdorff measures and KMS states, Indiana Univ. Math. J. **62** (2013), no. 2, 443–463.

References III

- A. Kumjian, D. Pask, and I. Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. **184** (1998), no. 1, 161–174.
- A. Kumjian, D. Pask, I. Raeburn, and J. Renault, Graphs, groupoids and Cuntz-Krieger algebras, J. Funct. Anal. **144** (1997), 505–541.
- M. Laca, N.S. Larsen, and S. Neshveyev, On Bost-Connes type systems for number fields, J. Number Theory **129** (2009), no. 2, 325–338.
- K. Matsumoto, Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras, Pacific J. Math. **246** (2010), no. 1, 199–225.

- R. McNamara, KMS states of graph algebras with a generalised gauge dynamics, Ph.D. thesis, University of Otago, 2015.
- K. Matsumoto and H. Matui, Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras, Kyoto J. Math. **54** (2014), no. 4, 863–877.
- S. Neshveyev, KMS states on the C^*-algebra of non-principal groupoids, J. Operator Theory **70** (2013), no. 2, 513–530.
- K. Thomsen, KMS weights on groupoid and graph C^*-algebras, J. Funct. Anal. **266** (2014), no. 5, 2959–2988.