NEBRASKA IOWA FUNCTIONAL ANALYSIS SEMINAR

Quasi-invariant measures for generalized approximately proper equivalence relations

---

Ruy Exel

UFSC – UNL

This talk is based on ongoing joint work with Rodrigo Bissacot, Rodrigo Frausino and Thiago Raszeja from the University of São Paulo.

The Problem.

Let

$$A = \{A_{i,j}\}_{i,j}$$

be an $n \times n$ matrix with $A_{i,j} \in \{0, 1\}$ and consider Markov's space

$$\Sigma_A = \left\{ x = x_1 x_2 x_3 ... \in \{1, 2, ..., n\}^{\mathbb{N}},\ A_{x_k, x_{k+1}} = 1,\ \text{for all } k \right\}$$

Equipping $\{1, 2, ..., n\}$ with the discrete topology, it becomes a compact space, so $\{1, 2, ..., n\}^{\mathbb{N}}$ is compact with the product topology, by Tychonov.

One may prove that Σ_A is closed, so Σ_A is compact.

Markov's shift is the map $\sigma : \Sigma_A \to \Sigma_A$, given by

$$\sigma(x_1 x_2 x_3 ...) = x_2 x_3 x_4 ...$$

Given a continuous function $h : \Sigma_A \to \mathbb{R}$, called a potential, and given $\beta > 0$, Ruelle's operator is the linear operator $L_\beta : C(\Sigma_A) \to C(\Sigma_A)$ given by

$$L_\beta(f)|_y = \sum_{\sigma(x)=y} \mathrm{e}^{-\beta h(x)} f(x).$$

The dual Markov operator acts on the space of Borel measures on Σ_A, as follows: given a Borel measure μ on Σ_A, we define $L_\beta^*(\mu)$ to be the measure on Σ_A such that

$$\int_{\Sigma_A} f \, dL_\beta^*(\mu) = \int_{\Sigma_A} L_\beta(f) \, d\mu, \quad \forall f \in C(\Sigma_A)$$

A fundamental problem in Statistical Mechanics is to find probability measures μ on Σ_A such that

$$L_\beta^*(\mu) = \lambda\mu.$$

That is, μ should be an eigen-measure of L_β^*. When $\lambda = 1$, these are called conformal measures.

Existence and uniqueness of conformal measures strongly depends on the matrix A and the regularity of the potential h.

Many people dedicate their life to this and similar problems and, as a result, it is very well understood, although there is still a lot more to be done.

The elements of the set $\{1, 2, ..., n\}$, apearing above, are usually interpreted as possible "states" or "spins" of each particle in a thermodynamics system.

The set of spins is usually finite, as above, but it is also important to analyze systems with an infinite set of spins.

How should we do it?

Seems obvious:

Let I be an infinite set (of spins), e.g,

$$I = \{1, 2, 3, 4, ...\} = \mathbb{N}$$

and let $A = \{A_{i,j}\}_{i,j \in I}$ be an $\underline{\infty \times \infty \text{ matrix}}$ with $A_{i,j} \in \{0, 1\}$. As before, put

$$\Sigma_A = \left\{ x = x_1 x_2 x_3 ... \in I^{\mathbb{N}},\ A_{x_k, x_{k+1}} = 1,\ \text{for all } k \right\}$$

Equipping I with the discrete topology, it is <u>no longer compact</u>, and hence $I^{\mathbb{N}}$ with the product topology is <u>not even locally compact</u>, hence Σ_A might not be locally compact either!

You may carry on if you like, but you will loose many topological tools which require local compactness, such as Riesz, Tietze, Urysohn and Gelfand's Theorem. The available results are therefore mostly in the realm of measure theory.

Our goal is to be able to study infinite state Markov spaces without having to abandon the tools of topology.

Cuntz-Krieger algebras.

Given a separable, infinite dimensional Hilbert space $\mathcal{H}$, and given any positive integer n, we may split $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n,$$

where each $\mathcal{H}_i$ is also separable, infinite dimensional.

Given an $n \times n$ matrix $A = \{A_{i,j}\}_{i,j}$ with $A_{i,j} \in \{0, 1\}$, let us consider, for each $i \leq n$, the subspace

$$\bigoplus_{j\,:\,A_{i,j}=1} \mathcal{H}_j$$

If no row of A is identically zero, these are also separable, infinite dimensional spaces. We may then choose, for every i, an isometric isomorphism

$$S_i : \bigoplus_{j\,:\,A_{i,j}=1} \mathcal{H}_j \to \mathcal{H}_i$$

For example:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \begin{array}{ccccccc} 1 & \quad & 2 & \quad & 3 & \quad & 4 \end{array} \\ \begin{pmatrix} 0 & \oplus & H_2 & \oplus & H_3 & \oplus & H_4 \\ H_1 & \oplus & H_2 & \oplus & H_3 & \oplus & 0 \\ 0 & \oplus & 0 & \oplus & H_3 & \oplus & H_4 \\ H_1 & \oplus & H_2 & \oplus & 0 & \oplus & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \xrightarrow{S_1} \quad H_1 \\ \xrightarrow{S_2} \quad H_2 \\ \xrightarrow{S_3} \quad H_3 \\ \xrightarrow{S_4} \quad H_4 \end{array}$$

Extending each S_i to $\mathcal{H}$ by setting it to be zero on the orthogonal complement of its original domain, we get a family $\{S_1, S_2, \ldots, S_n\}$ of bounded operators on $\mathcal{H}$, satisfying

$$S_i S_i^* S_i = S_i, \qquad \sum_{j=1}^{n} S_j S_j^* = I, \qquad S_i^* S_i = \sum_{j=1}^{n} A_{i,j} S_j S_j^*.$$

Definition. The Cuntz-Krieger algebra, denoted $\mathcal{O}_A$, is a C*-algebra generated by a family of operators $\{S_1, S_2, \ldots, S_n\}$ satisfying the above relations in an "universal way".

For each "word" $\alpha = i_1 i_2 \ldots i_k$, with "letters" $i_j \in \{1, 2, \ldots, n\}$, define

$$S_\alpha = S_{i_1} S_{i_2} \ldots S_{i_k} \in \mathcal{O}_A$$

One may prove that the elements of the form $S_\alpha S_\alpha^*$ are <u>pairwise commuting</u> projections and hence they generate a commutative C*-sub-algebra

$$\mathcal{D}_A \subseteq \mathcal{O}_A.$$

By Gelfand's Theorem, $\mathcal{D}_A$ is isomorphic to $C(X)$, for some compact space X.

In fact it turns out that

$$X = \Sigma_A$$

Under the natural isomorphism $\mathcal{D}_A \simeq C(\Sigma_A)$, each $S_\alpha S_\alpha^*$ is identified with the characteristic function of the cylinder

$$\Big\{(x_1, x_2, x_3, \ldots) \in \Sigma_A : x_i = \alpha_i, \text{ for } i = 1, \ldots, |\alpha|\Big\}.$$

Thus we may view $C(\Sigma_A) = \mathcal{D}_A \subseteq \mathcal{O}_A$, and if we let

$$S = n^{-1/2} \sum_{i=1}^{n} S_i,$$

one may prove that

$$Sf = (f \circ \sigma)S, \quad \forall f \in C(\Sigma_A),$$

where $\sigma : \Sigma_A \to \Sigma_A$ is Markov's shift.

In other words, $\mathcal{O}_A$ encodes the Markov shift in its algebraic structure!

Let us now assume that I is a countably <u>infinite</u> set of indices and

$$A = \{A_{i,j}\}_{i,j \in I}$$

is a matrix of zeros and ones. If A is <u>row-finite</u>, that is, if all rows of A have <u>finitely many of nonzero</u> entries, Kumjian, Pask, Raeburn and Renault [JFA 1997] were able study a generalization of $\mathcal{O}_A$, where Markov's space also plays a role. In fact, when A is row-finite, Σ_A is locally compact, even if $I^{\mathbb{N}}$ is not.

In the general "non row-finite" case, there is no reason for Σ_A to be locally compact.

In the paper "Cuntz-Krieger algebras for infinite matrices" [Crelle 1999], joint with Marcelo Laca, we were able to figure out the general "non row-finite" case.

Given any matrix $A = \{A_{i,j}\}_{i,j\in I}$, for each $i \in I$, let S_i be the bounded operator on the Hilbert space $\ell^2(\Sigma_A)$ given on the orthonormal basis $\{\delta_\omega\}_{\omega\in\Sigma_A}$ by

$$S_i(\delta_\omega) = \begin{cases} \delta_{i\omega}, & \text{if } i\omega \in \Sigma_A, \\ 0, & \text{otherwise.} \end{cases}$$

As before, for each word $\alpha = i_1 i_2 \ldots i_k$, with $i_j \in I$, define

$$S_\alpha = S_{i_1} S_{i_2} \ldots S_{i_k}.$$

It is then possible to prove that the elements of the form $S_\alpha S_\alpha^*$ generate a commutative C*-algebra $\mathcal{D}_A$ of operators on $\ell^2(\Sigma_A)$ whose Gelfand spectrum is necessarily compact and hence <u>cannot be Markov's space</u>, because the latter is <u>not even locally compact!</u>

In other words, $\mathcal{D}_A = C(X_A)$, for some compact space X_A, which could be thought of as an <u>alternative for the badly behaved Markov space</u> Σ_A.

In order to describe the space X_A we shall consider the free group

$$\mathbb{F} = \mathbb{F}_I,$$

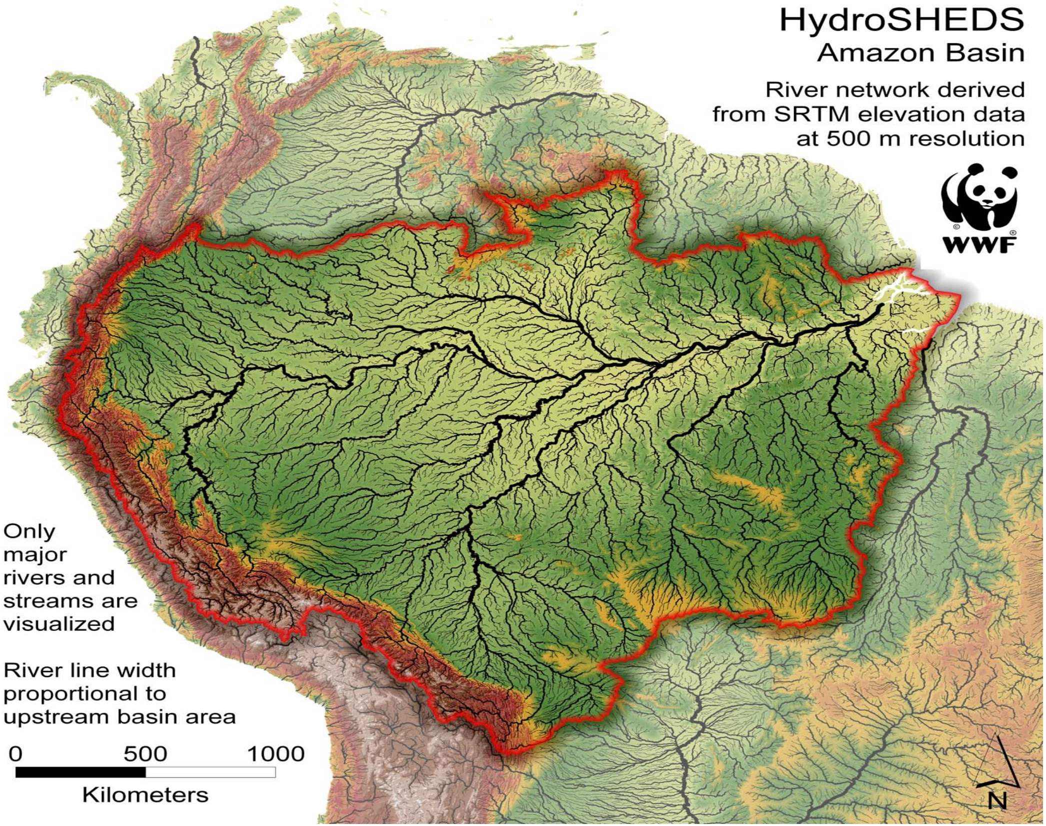
generated by I. For every infinite word ω in Markov's space Σ_A, we will look at the subset

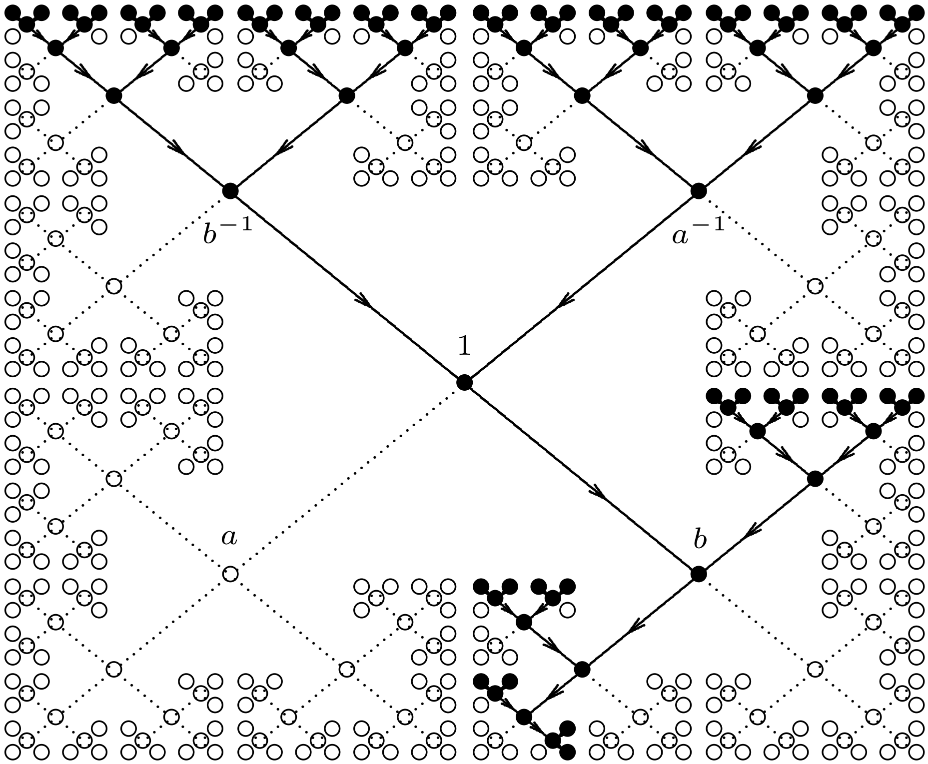
$$\xi_\omega \subseteq \mathbb{F}$$

consisting of

(1) the "river" formed by all prefixes of ω

(2) all elements of the "river basin" of ω


HydroSHEDS
Amazon Basin
River network derived
from SRTM elevation data
at 500 m resolution
WWF
Only
major
rivers and
streams are
visualized
River line width
proportional to
upstream basin area
0
500
1000
Kilometers
N



A picture of $\xi_\omega \subseteq \mathbb{F}$

Here is the Cayley graph of $\mathbb{F}$. The generators point to $\swarrow$ and $\searrow$

The given word ω is the main "river", starting at the group unit. It then recieves a lot of "tributaries", namely all possible rivers which merge into the main river forming an admissble word.

We have therefore defined a map

$$\omega \in \Sigma_A \mapsto \xi_\omega \in \{0,1\}^{\mathbb{F}}.$$

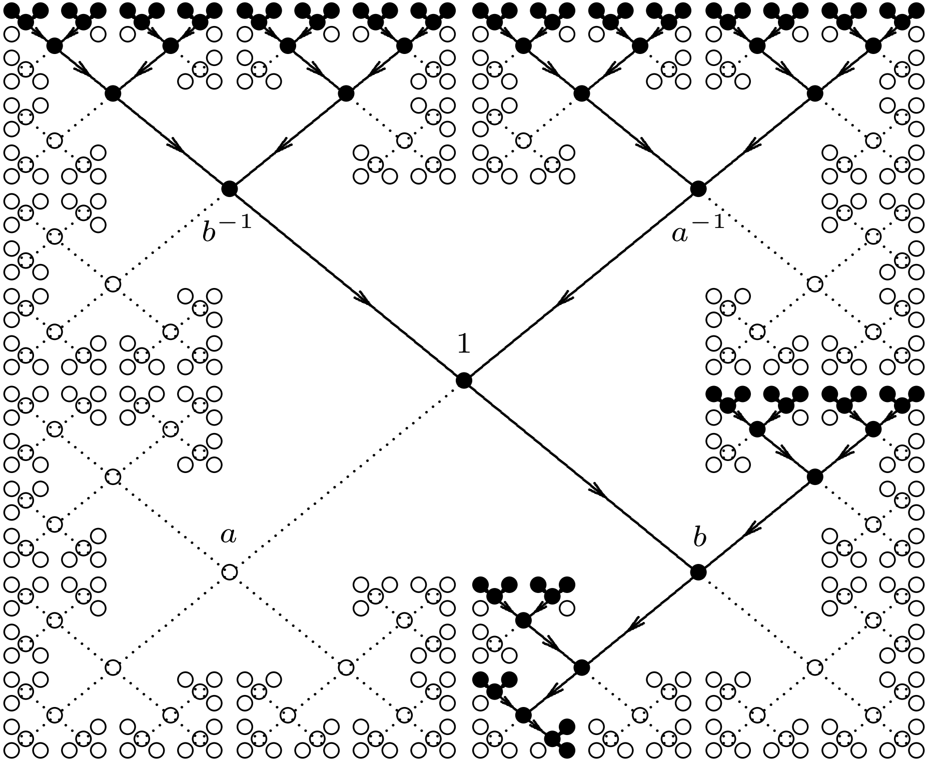
Notice that the range of this map is invariant under left-multiplication by group elements, as long as the translated set includes 1.

Theorem. *The space X_A (the spectrum of $\mathcal{D}_A$) is naturally isomorphic to the closure of $\{\xi_\omega : \omega \in \Sigma_A\}$ within $\{0,1\}^{\mathbb{F}}$.*

X_A is therefore a compactification of Σ_A!

$$\Sigma_A \hookrightarrow X_A \subseteq \{0,1\}^{\mathbb{F}}.$$

The new elements in the closure also look like river basins but the main river may dry up, that is, it may be a finite word! In particular it may dry up at its very source, namely at 1.



b^{-1}
a^{-1}
1
a
b

Let U be the (open) subset of X_A consisting of all river basins ξ which don't dry up at 1. The generalized shift map, which is only defined on U,

$$\sigma : U \subseteq X_A \to X_A$$

is as follows. Given ξ in U, let ω be its main river, which flows at least for a bit, i.e. $|\omega| \geq 1$, since ξ is in U. We may then look at its first edge ω_1, and we put

$$\sigma(\xi) = \omega_1^{-1}\xi \qquad \text{(translation of a subset of } \mathbb{F}\text{)}$$

Proposition. *σ is a local homeomorphism, extending Markov's shift on Σ_A.*

In "Cuntz-like algebras" [Timişoara, 1998] Jean Renault realized that this local homeo encodes all of the relevant information. In particular Renault showed that $\mathcal{O}_A$ is the C*-algebra for the generalized Deaconu-Renault groupoid associated to σ.

Generalized Deaconu-Renault groupoids.

Given a locally compact space X, an open set $U \subseteq X$, and a local homeomorphism

$$\sigma : U \to X,$$

the semi-direct product groupoid is defined as follows:

$$\mathcal{G}_\sigma = \Big\{ (x, n-m, y) : x \in \mathrm{dom}(\sigma^n),\ y \in \mathrm{dom}(\sigma^m),\ \sigma^n(x) = \sigma^m(y) \Big\}$$

Given a continuos potential $h : U \to \mathbb{R}$, we may define a 1-cocycle on $\mathcal{G}_\sigma$ by

$$c(x, n-m, y) = \sum_{i=0}^{n-1} h(\sigma^i(x)) - \sum_{j=0}^{m-1} h(\sigma^j(y)).$$

A 1-cocycle induces a flow, i.e., a strongly continuous one parameter group of automorphisms on the groupoid C*-algebra $C^*(\mathcal{G}_\sigma)$, as follows:

$$\alpha_t(f)|_\gamma = \mathrm{e}^{itc(\gamma)} f(\gamma), \quad \forall\, f \in C_c(\mathcal{G}_\sigma), \quad \forall\, \gamma \in \mathcal{G}_\sigma.$$

It is a problem of fundamental importance to find the probability measures μ on X, such that the associated state

$$\varphi_\mu(f) = \int_X f(x)\, d\mu(x), \quad \forall\, f \in C_c(\mathcal{G}_\sigma),$$

is a β-KMS state on $C^*(\mathcal{G}_\sigma)$.

This was shown by Renault to be equivalent to the fact that μ is quasi-invariant with Radon Nikodym derivative equal to $\mathrm{e}^{-\beta c}$.

By this we mean the following: define measures ν_r and ν_s on $\mathcal{G}_\sigma$ by

$$\int_{\mathcal{G}_\sigma} f\, d\nu_r = \int_X \sum_{r(\gamma)=x} f(\gamma)\, d\mu(x), \quad \text{and} \quad \int_{\mathcal{G}_\sigma} f\, d\nu_s = \int_X \sum_{s(\gamma)=x} f(\gamma)\, d\mu(x).$$

One says that μ is quasi-invariant if $\nu_r \sim \nu_s$. In that case the Radon Nikodym derivative $d\nu_r/d\nu_s$ is a (measurable) 1-cocycle on $\mathcal{G}_\sigma$. Renault says that the above state φ_μ is a β-KMS state on $C^*(\mathcal{G}_\sigma)$ iff $\nu_r \sim \nu_s$ and $d\nu_r/d\nu_s = \mathrm{e}^{-\beta c}$.

Theorem A. *Given $\sigma : U \subseteq X \to X$, and $h : X \to \mathbb{R}$, as above, the following conditions are equivalent for any probability measure μ on X:*

(i) φ_μ is a <u>β-KMS state</u> on $C^*(\mathcal{G}_\sigma)$,

(ii) μ is <u>quasi-invariant</u> with Radon Nikodym derivative $d\nu_r/d\nu_s = \mathrm{e}^{-\beta c}$,

(iii) μ is <u>conformal</u>, i.e, $L_\beta^*(\mu) = \mu|_U$, where $L_\beta : C_c(U) \to C(X)$ is Ruelle's operator

$$L_\beta(f)|_y = \sum_{\sigma(x)=y} \mathrm{e}^{-\beta h(x)} f(x)$$

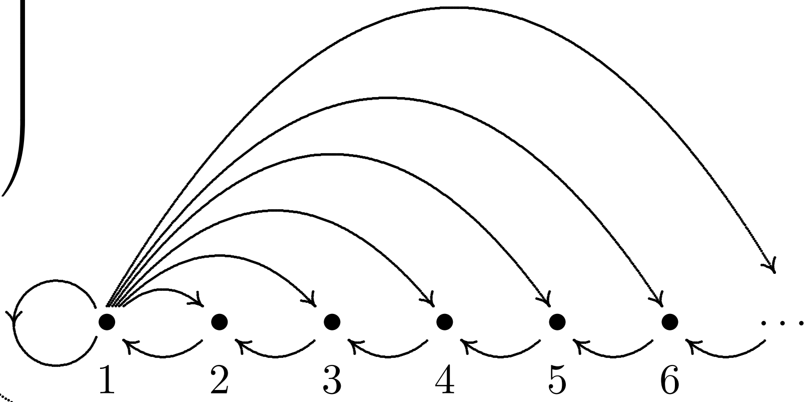
(iv) μ satisfies the <u>Denker-Urbanski</u> condition $\dfrac{d\mu \odot \sigma}{d\mu} = \mathrm{e}^{\beta h}$, where $\mu \odot \sigma$ is the unique measure on U such that

$$(\mu \odot \sigma)(E) = \mu(\sigma(E)),$$

for every Borel set E, such that σ is injective on E.

EXAMPLE
THE RENEWAL SHIFT

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



Notice that this matrix is not row-finite!

The spectrum of the renewal shift is easy to describe: there is only one river basin ξ^0 with a totally dry river and for every finite admissble word ω ending in 1, we get another river basin with a finite river, namely $\xi_\omega = \omega\xi^0$.

Moreover $X_A = \Sigma_A \cup Y_A$, where

$$Y_A = \{\xi^0\} \cup \Big\{\xi_\omega : \omega \text{ is a } \underline{\text{finite}} \text{ admissible word ending in "1"}\Big\}.$$

Take the potential $h \equiv 1$, choose some "inverse temperature" β, and let us look for <u>conformal measures vanishing on</u> Σ_A.

Since Y_A is countable, any such measure μ is determined by the values

$$c^0 := \mu(\{\xi^0\}), \quad \text{and} \quad c_\omega := \mu(\{\xi_\omega\}).$$

The Denker-Urbanski condition becomes

$$c^0 = e^\beta c_1, \quad \text{and} \quad c_{\sigma(\omega)} = e^\beta c_\omega,$$

for every admissible ω ending in 1, where $\sigma(\omega)$ is the shift, deleting the first letter of the finite word ω.

It is then easy to see that a solution must be given by

$$c^0 = \frac{1}{K}, \quad \text{and} \quad c_\omega = \frac{e^{-\beta|\omega|}}{K},$$

where the normalization constant K is given by

$$\begin{aligned} K &= 1 + \sum_{|\omega|>0} e^{-\beta|\omega|} \\ &= 1 + \sum_{n=1}^{\infty} 2^{n-1} e^{-n\beta} \\ &= 1 + \sum_{n=1}^{\infty} 2^{-1} e^{n(\ln(2)-\beta)}, \end{aligned}$$

which converges iff $\beta > \ln(2)$.

MORAL: for ∞-state-Markov shifts, there may be conformal measures which cannot be seen within Σ_A. To see them, one must pass from Σ_A to X_A.

We would now like to look at the sub-groupoid of $\mathcal{G}_\sigma$ formed by the elements of the form $(x, 0, y)$. Notice that such an element lies in $\mathcal{G}_\sigma$, iff there exists some n, such that $x, y \in \mathrm{dom}(\sigma^n)$, and $\sigma^n(x) = \sigma^n(y)$. This highlights an equivalence relation R_n on $\mathrm{dom}(\sigma^n)$ according to which

$$(x, y) \in R_n \iff \sigma^n(x) = \sigma^n(y).$$

An equivalence relation R on a topological space X is said to be proper if the quotient space is Hausdorff and the quotient map is a local homeomorphism.

In that case R is an étale groupoid with the topology inherited from the product topology on $X \times X$.

An equivalence relation is said to be approximately proper if it is the union of an increasing family of proper relations. J. Renault has extensively studied these for compact X [ETDS, 2005], but the situation here is different in two important respects:

(1) we must work with non compact X,

(2) each R_n lives on a different set, namely $\mathrm{dom}(\sigma^n)$.

Definition. Let X be a locally compact space. A generalized approximately proper equivalence relation on X, is a pair

$$\mathcal{R} = (\{U_n\}_{n\in\mathbb{N}},\ \{R_n\}_{n\in\mathbb{N}}),$$

where each U_n is an open subset of X, with

$$X = U_0 \supseteq U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$$

and each R_n is a proper equivalence relation on U_n, such that

(i) R_0 is the diagonal in $X \times X$, and

(ii) $R_n \cap (U_n \times U_m) \subseteq R_m$, for every $n \leq m$.

It follows that the R_n's are increasing in the sense that if $n \leq m$, then the restriction of R_n to U_m is contained in R_m,

Also every U_m is invariant under every R_n in the sense that

$$U_n \ni x \underset{R_n}{\sim} y \in U_m \ \Rightarrow\ x \in U_m.$$

Given U_n and R_n as above, we have that $R := \cup_i R_i$ is an equivalence relation on X which becomes an étale groupoid with the inductive limit topology.

Suppose we have continuous functions $k_n : U_n \to \mathbb{R}$, for $n \geq 1$, such that

$$(x, y) \in (U_n \times U_n) \cap R_{n-1} \;\Rightarrow\; k_n(x) = k_n(y).$$

One may then define a cocycle d_n on each R_n by

$$d_n(x, y) = \sum_{i=1}^{n} k_i(x) - k_i(y), \quad \forall\, (x, y) \in R_n.$$

These admit a common extension d to R, and we once again want to detemine the KMS sates on $C^*(R)$ or the quasi-invariant measures μ on X.

Given that R is the union of the R_n, this is the same as saying that $\mu|_{U_n}$ is quasi-invariant for each R_n.

The crucial point is then to understand quasi-invariant measures for a single proper equivalence relation on a locally compact space.

Let U be a locally compact topological space and let R be a proper equivalence relation on U. Given a continuous function $h : U \to \mathbb{R}$, consider the 1-cocycle

$$c(x, y) = h(x) - h(y), \quad \forall\, (x, y) \in R.$$

For $\beta > 0$, define

$$E_\beta(f)|_y = \sum_{x\,:\,(x,y)\in R} \mathrm{e}^{\beta h(x)} f(x), \quad \forall\, f \in C_c(U).$$

Notice that the above sum is finite because f has compact support and each equivalence class is discrete. However, the partition function

$$\zeta(y) = \sum_{x\,:\,(x,y)\in R} \mathrm{e}^{\beta h(x)}$$

may very well take on the value ∞. This is a crucial difference with the compact case, where equivalence classes are all finite.

Fortunately we are saved by Lebesgue's theory of integration which does not worry too much about functions taking values in $[0, \infty]$.

Theorem B. *Given a locally compact space U, a proper equivalence relation R on U, and a continuous potential $h : U \to \mathbb{R}$, the following conditions are equivalent for any probability measure μ on U:*

(i) $\varphi_\mu(f) = \int_U f(x)\, d\mu(x)$ defines a β-KMS state on $C^*(R)$,

(ii) μ is quasi-invariant with Radon Nikodym derivative $d\nu_r/d\nu_s = \mathrm{e}^{\beta(h(y)-h(x))}$,

(iii) $\int_U f E_\beta(g)\, d\mu = \int_U E_\beta(f) g\, d\mu$, for every $f, g \in C_c(U)$,

(iv) $\int_U f\, d\mu = \int_U E_\beta(f\zeta^{-1})\, d\mu$, for every non-negative f in $C_c(U)$,

(v) there exits a positive measure ν on U, such that

$$\int_U \zeta\, d\nu = 1, \quad \text{and} \qquad \int_U f\, d\mu = \int_U E_\beta(f)\, d\nu, \quad \forall\, f \in C_c(U).$$

Moreover, if the conditions above are satisfied, then

$$\mu\{x \in U : \zeta(x) = \infty\} = 0.$$

Regarding condition (iv) of the above Theorem, namely

$$\int_U f\, d\mu = \int_U E_\beta(f\zeta^{-1})\, d\mu,$$

let

$$F_\beta(f)|_y := E_\beta(f\zeta^{-1})|_y = \sum_{(x,y)\in R} f(x) \underbrace{\mathrm{e}^{\beta h(x)} \Big/ \sum_{(z,x)\in R} \mathrm{e}^{\beta h(z)}}_{\text{Gibbs function}}$$

Then F_β is a conditional expectation and we see that (iv) becomes the usual DLR (Dobrushin–Lanford–Ruelle) condition

$$\mu = F_\beta^*(\mu)$$

Returning to the Deaconu-Renault groupoid $\mathcal{G}_\sigma$ for a given $\sigma : U \subseteq X \to X$, the subgroupoid formed by the triples $(x, 0, y)$ turns out to be the groupoid for the generalized AP equivalence relation where

$$U_n = \mathrm{dom}(\sigma^n), \quad \text{and} \quad R_n = \{(x, y) : \sigma_n(x) = \sigma_n(y)\}$$

If c is the cocycle on $\mathcal{G}_\sigma$ defined by a potential $h : X \to \mathbb{R}$, as above, then the restriction of c to R coincides with the cocycle given by the family of potentials

$$k_n : x \in U_n \mapsto h(\sigma^n(x)) \in \mathbb{R}.$$

It therefore follows that the flow defined by c on $C^*(\mathcal{G}_\sigma)$ leaves $C^*(R)$ invariant and its restriction to $C^*(R)$ coincides with the flow defined by the k_n.

KMS states on $C^*(\mathcal{G}_\sigma)$ therefore restrict to KMS states on $C^*(R)$, and quasi-invariant measures for $\mathcal{G}_\sigma$ are obviously quasi-invariant for R.

Therefore the conditions of Theorem B hold for every measure satisfying the conditions of Theorem A. Highlighting one condition from each we have:

Corolary.

$$\textit{Conformal} \Rightarrow \textit{DLR}$$

In other words, every conformal measure on X satisfies the generalized DLR condition (Theorem B.iv), namely: for every n, and for every f in $C_c(U_n)$,

$$\int_{U_n} f\,d\mu = \int_{U_n} \sum_{(x,y)\in R_n} \frac{\mathrm{e}^{\beta h_n(x)}}{\zeta_n(x)} f(x)\,d\mu(y),$$

where

$$h_n(x) = \sum_{i=0}^{n-1} h(\sigma^i(x)), \quad \text{and} \quad \zeta_n(y) = \sum_{(x,y)\in R_n} \mathrm{e}^{\beta h_n(x)}.$$

REFERENCES

[1] R. Bissacot, R. Exel, R. Frausino and T. Raszeja, “Conformal measures on generalized Renault-Deaconu groupoids”, preprint, 2018, arXiv:1808.00765.

[2] R. Bissacot, R. Exel, R. Frausino and T. Raszeja, “Conformal and DLR measures on Markov subshifts with infinitely many states”, *preprint,* 2018.

[3] R. Exel and M. Laca, “Cuntz-Krieger algebras for infinite matrices”, *J. reine angew. Math.,* **512** (1999), 119-172, arXiv:funct-an/9712008.

[4] R. Exel and M. Laca, “Partial dynamical systems and the KMS condition”, *Commun. Math. Phys.,* **232** (2003), 223-277, arXiv:math.OA/0006169.

[5] A. Kumjian, D. Pask, I. Raeburn and J. Renault, “Graphs, groupoids, and Cuntz-Krieger algebras”, *J. Funct. Anal.,* **144** (1997), 505-541.

[6] J. Renault, “Cuntz-like algebras”, Operator theoretical methods (Timişoara, 1998), 371–386, Theta Found., Bucharest, 2000.

Thank you!