Symmetries of Cuntz-Pimsner algebras

Valentin Deaconu

University of Nevada, Reno

UNL Functional Analysis Seminar Nov 3, 2018
We recall the concept of Cuntz-Pimsner algebra, crossed product $C^*$-correspondences, and the Hao-Ng theorem.

We give applications to group actions on graph $C^*$-algebras and make the connection with the Doplicher-Roberts algebras of group representations.

We illustrate with some examples, including group actions on Hermitian vector bundles.

We discuss similar results for groupoid actions.
A $C^*$-correspondence from $A$ to $B$ is a Hilbert $B$-module $\mathcal{H}$ with a left multiplication given by $\phi: A \to L_B(\mathcal{H})$.

It can be thought as a multivalued, partially defined morphism $A \to B$.

If $A = B$, the Cuntz-Pimsner algebra $O_A(\mathcal{H})$ is universal for covariant representations $\pi: A \to C, \tau: \mathcal{H} \to C$

$$\tau(a\xi) = \pi(a)\tau(\xi), \quad \pi(\langle \xi, \eta \rangle) = \tau(\xi)^*\tau(\eta)$$

$$\pi(a) = \psi(\phi(a)) \text{ for } a \in J_\mathcal{H} = \phi^{-1}(K_A(\mathcal{H})) \cap (\ker \phi)^\perp,$$

where $\psi: K_A(\mathcal{H}) \to C, \psi(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^*$.

Denote by $(\pi_A, \tau_\mathcal{H})$ the universal representation of $(A, \mathcal{H})$. 
Examples

- If $\alpha \in \text{Aut}(A)$, then $\mathcal{O}_A(\mathcal{H}_\alpha) \cong A \rtimes \alpha \mathbb{Z}$.
- If $E = (E^0, E^1, r, s)$ is a (topological) graph and $A = C_0(E^0), \mathcal{H}_E = C_c(E^1)$, then $\mathcal{O}_A(\mathcal{H}_E) = C^*(E)$.
- In particular we obtain Cuntz algebras $\mathcal{O}_n$ and Cuntz-Krieger algebras $\mathcal{O}_\Lambda$ related to subshifts of finite type.
- If $E \to X$ is a complex vector bundle, $A = C(X), \mathcal{H} = \Gamma(E)$, then $\mathcal{O}_A(\mathcal{H}) = \mathcal{O}_E$ is a (locally trivial) continuous field of Cuntz algebras.
- If $E$ is a line bundle, then $\mathcal{O}_E$ is commutative with spectrum homeomorphic to the circle bundle of $E$.
- For $X = S^{2k}$ and $[E] = n + mt \in K^0(S^{2k}) \cong \mathbb{Z}[t]/(t^2)$ with $n \geq 3$ and $\gcd(n - 1, m) = 1$ we have

$$K_0(\mathcal{O}_E) \cong \mathbb{Z}/(n - 1)^2\mathbb{Z} \neq K_0(C(S^{2k}) \otimes \mathcal{O}_n).$$
A locally compact group $G$ acts on $(A, \mathcal{H})$ by $(\alpha, \beta)$ if

$$\langle \beta_g(\xi), \beta_g(\eta) \rangle = \alpha_g(\langle \xi, \eta \rangle), \quad \beta_g(\xi a) = \beta_g(\xi)\alpha_g(a), \quad \beta_g(a\xi) = \alpha_g(a)\beta_g(\xi).$$

By the universal property we get $\gamma : G \to \text{Aut} O_A(\mathcal{H})$.

For $a \in C_c(G, A), \xi \in C_c(G, \mathcal{H})$ define

$$(a\xi)(s) = \int_G a(t)\beta_t(\xi(t^{-1}s))dt, \quad (\xi a)(s) = \int_G \xi(t)\alpha_t(a(t^{-1}s))dt,$$

$$\langle \xi, \eta \rangle(s) = \int_G \alpha_{t^{-1}}(\langle \xi(t), \eta(ts) \rangle)dt.$$ 

The completion gives $(A \rtimes_{\alpha} G, \mathcal{H} \rtimes_{\beta} G)$. 
Main results

- **Theorem** (Hao-Ng). If $G$ is amenable, then
  \[ O_A(\mathcal{H}) \rtimes_\gamma G \cong O_{A \rtimes_\alpha G}(\mathcal{H} \rtimes_\beta G). \]

  If $G$ is abelian, this is $\hat{G}$-equivariant.

- **Lemma 1.** If $G$ acts on $(A, \mathcal{H})$ and $A \rtimes G$ decomposes as $\bigoplus A_i$ with units $q_i$, then $\mathcal{H} \rtimes G$ decomposes into $\bigoplus q_i(\mathcal{H} \rtimes G)q_j$, a sum of $C^*$-correspondences from $A_i$ to $A_j$.

- **Lemma 2.** If $\mathcal{N}$ is an imprimitivity module from $A$ to $B$ and $\mathcal{H}$ is a $C^*$-correspondence over $A$, then $\mathcal{H}' = \mathcal{N}^* \otimes_A \mathcal{H} \otimes_A \mathcal{N}$ is a $C^*$-correspondence over $B$ and $O_A(\mathcal{H})$ is Morita-Rieffel equivalent to $O_B(\mathcal{H}')$.

- **Theorem** (D). If $G$ compact acts on a discrete graph $E = (E^0, E^1, r, s)$, then $C^*(E) \rtimes G$ is Morita-Rieffel equivalent to a graph algebra.
Proof

- We have
  \[ C_0(E^0) \rtimes G \cong \bigoplus C(Gx) \rtimes G \cong \bigoplus C(G/G_x) \rtimes G \cong \bigoplus M_{|Gx|} \otimes C^*(G_x), \]
  which is Morita-Rieffel equivalent to \( C_0(V) \) via \( \mathcal{N} \), with \( V = \{v_i\} \) at most countable.

- Then \( \mathcal{M} = \mathcal{N}^* \otimes (\mathcal{H}_E \rtimes G) \otimes \mathcal{N} \) over \( C_0(V) \) determines a graph \( F \) with \( F^0 = V \) and edges \( F^1 \) determined by the incidence matrix
  \[ a_{ij} = \dim p_i \mathcal{M} p_j \]
  where \( p_i = \chi_{\{v_i\}} \).

- Then
  \[ C^*(E) \rtimes G \cong \mathcal{O}_{C_0(E^0) \rtimes G}(\mathcal{H}_E \rtimes G) \]
  is Morita-Rieffel equivalent to \( C^*(F) \).
Examples

Let $E$ be

Here $A = C(E^0) = \mathbb{C}^3, \mathcal{H} = C(E^1) = \mathbb{C}^4$ and $G = \mathbb{Z}_2$ acts by

$$
\alpha(a_1, a_2, a_3) = (a_1, a_3, a_2), \quad \beta(\xi_1, \xi_2, \xi_3, \xi_4) = (\xi_2, \xi_1, \xi_4, \xi_3).
$$
Examples

- Then \( A \rtimes_\alpha \mathbb{Z}_2 \subset M_2(A) \) as \[
\begin{bmatrix}
    a & b \\
    \alpha(b) & \alpha(a)
\end{bmatrix}
\]
and \( \mathcal{H} \rtimes_\beta \mathbb{Z}_2 \subset M_2(\mathcal{H}) \) as \[
\begin{bmatrix}
    \xi & \eta \\
    \beta(\eta) & \beta(\xi)
\end{bmatrix}
\] with obvious operations.

- Since \( A \rtimes \mathbb{Z}_2 \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2 \), the crossed product \( \mathcal{H} \rtimes \mathbb{Z}_2 \) decomposes as \( \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus M_2 \) and \( C^*(E) \rtimes \mathbb{Z}_2 \) is given by

\[
\begin{array}{cccc}
\mathbb{C} & \mathbb{C}^2 & M_2 & \mathbb{C}^2 & \mathbb{C} \\
\end{array}
\]
Examples

- Let $E$ be the graph with one vertex and three loops which gives $O_3$.
- The symmetric group $S_3$ acts trivially on $A = \mathbb{C}$ and by the permutation representation $\rho$ on $H = \mathbb{C}^3$.
- $H \rtimes S_3$ decomposes over $A \rtimes S_3 \cong C^*(S_3)$ and $O_3 \rtimes S_3$ is given by

\[
\begin{array}{c}
\text{C} \\
\rightarrow \\
\text{C} \\
\rightarrow \\
\text{C} \\
\rightarrow \\
\text{C} \\
\rightarrow \\
\text{C} \\
\end{array}
\]

- Recall that $\hat{S}_3 = \{\iota, \varepsilon, \sigma\}$, $\rho \sim \iota + \sigma$, and the Doplicher-Roberts algebra $O_\rho$ constructed from intertwiners $(\rho^n, \rho^m)$ is Morita-Rieffel equivalent to the same graph algebra, using the character table of $S_3$. 
Examples

- Consider a hermitian $G$-vector bundle $\mathcal{E} \to X$. If $X$ is a point, then $\mathcal{E}$ is a $G$-module. If $X$ has trivial action, then $\mathcal{E}$ is a family of $G$-modules.
- The group $G$ acts on $\Gamma(\mathcal{E})$ by
  \[(g\xi)(x) = g\xi(g^{-1}x)\]
  and we may study $\mathcal{O}_{\mathcal{E}} \rtimes G \cong \mathcal{O}_{C(X) \rtimes G}(\Gamma(\mathcal{E}) \rtimes G)$
- **Fact 1.** If $G$ compact acts freely on $\mathcal{E} \to X$, then $\mathcal{O}_{\mathcal{E}} \rtimes G$ is a continuous field of Cuntz algebras over $X/G$.
- **Fact 2.** If $G$ acts fiberwise on $\mathcal{E} \to X$ of rank $n$, then $\mathcal{O}_{\mathcal{E}} \rtimes G$ is a continuous field with fibers $\mathcal{O}_n \rtimes G$.
- If $X$ is a manifold and $G$ acts by diffeomorphisms, then $\mathcal{E} = TX \otimes \mathbb{C}$ becomes a $G$-bundle. What is $\mathcal{O}_{\mathcal{E}} \rtimes G$?
Recall that a $C_0(X)$-algebra is a $C^*$-algebra $A$ together with a homomorphism $\theta : C_0(X) \to ZM(A)$ such that $\theta(C_0(X))A = A$.

For each $x \in X$ we define the fiber $A_x$ as $A/\theta(I_x)A$ where

$$I_x = \{ f \in C_0(X) : f(x) = 0 \}.$$

A $C_0(X)$-algebra $A$ gives rise to an upper semicontinuous $C^*$-bundle $\mathcal{A}$ such that $A \cong \Gamma_0(X, \mathcal{A})$.

We say that a groupoid $G$ acts on a $C_0(G^0)$-algebra $A$ if for each $g \in G$ there is an isomorphism $\alpha_g : A_{s(g)} \to A_{r(g)}$ such that

$$\alpha_{g_1g_2} = \alpha_{g_1} \circ \alpha_{g_2} \text{ for } (g_1, g_2) \in G^{(2)}.$$ 

If the groupoid $G$ acts on $X$, then $C_0(X)$ becomes in a natural way a $C_0(G^0)$-algebra and $G$ acts on $C_0(X)$ by $\alpha_g(f)(x) = f(g^{-1} \cdot x)$.

Groupoid actions on elementary $C^*$-bundles over $G^0$ satisfying Fell’s condition appear in the context of defining the Brauer group $Br(G)$. 
Let $\mathcal{H}$ be a Hilbert module over a $C_0(X)$-algebra $A$. Define the fibers $\mathcal{H}_x := \mathcal{H} \otimes_A A_x$ which are Hilbert $A_x$-modules.

The set $\text{Iso}(\mathcal{H})$ of $\mathbb{C}$-linear isomorphisms between fibers becomes a groupoid with unit space $X$.

We say that $\mathcal{H}$ is a $C_0(X)$-$C^*$-correspondence over $A$ if there is a $*$-homomorphism $\phi : A \to \mathcal{L}_A(\mathcal{H})$ such that

$$\phi(fb)(\xi a) = \phi(b)(\xi fa) \quad \text{for all } f \in C_0(X), \ a, b \in A, \ \xi \in \mathcal{H}.$$

Each fiber $\mathcal{H}_x$ becomes a $C^*$-correspondence over $A_x$.

An action of $G$ on $\mathcal{H}$ is given by a homomorphism $\rho : G \to \text{Iso}(\mathcal{H})$ where $\rho_g : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ with $\rho_g = I$ if $g \in G^0$, such that

$$\langle \rho_g \xi, \rho_g \eta \rangle_{r(g)} = \alpha_g(\langle \xi, \eta \rangle_{s(g)}),$$

for $\xi, \eta \in \mathcal{H}_{s(g)}$ and

$$\rho_g(\xi a) = \rho_g(\xi) \alpha_g(a), \ \rho_g(a \xi) = \alpha_g(a) \rho_g(\xi)$$

for $\xi \in \mathcal{H}_{s(g)}, \ a \in A_{s(g)}$. 

**Valentin Deaconu**

Symmetries of Cuntz-Pimsner algebras
Proposition. Suppose the groupoid $G$ with Haar system $\{\lambda^x\}$ acts on the $C_0(G^0)$-$C^*$-correspondence $\mathcal{H}$ over $A$.

Then the completion of $\Gamma_c(G, r^*\mathcal{H})$ becomes a $C^*$-correspondence $\mathcal{H} \rtimes G$ over $A \rtimes G$, using the inner product

$$\langle \xi, \eta \rangle(g) = \int_G \alpha_h(\langle \xi(h^{-1}), \eta(h^{-1}g) \rangle_{s(h)}) d\lambda^{r(g)}(h),$$

and the multiplications

$$(\xi f)(g) = \int_G \xi(h)\alpha_h(f(h^{-1}g)) d\lambda^{r(g)}(h),$$

$$(f\xi)(g) = \int_G f(h)\rho_h(\xi(h^{-1}g)) d\lambda^{r(g)}(h),$$

where $\xi, \eta \in \Gamma_c(G, r^*\mathcal{H}), f \in \Gamma_c(G, r^*A)$. 

Valentin Deaconu
Symmetries of Cuntz-Pimsner algebras
**Theorem (D).** Suppose $G$ acts on a $C_0(G^0)$-$C^*$-correspondence $\mathcal{H}$ over $A$. Then the Katsura ideal $J_{\mathcal{H}}$ is $G$-invariant and $G$ acts on $\mathcal{O}_A(\mathcal{H})$, which becomes a $C_0(G^0)$-algebra with fibers $\mathcal{O}_{A_x}(\mathcal{H}_x)$ for $x \in G^0$.

Let $G$ amenable act on a $C_0(G^0)$-$C^*$-correspondence $\mathcal{H}$ over $A$. Assume that $J_{\mathcal{H} \rtimes G} \cong J_{\mathcal{H}} \rtimes G$. Then there are maps

$$\Phi : A \rtimes G \to \mathcal{O}_A(\mathcal{H}) \rtimes G, \quad \Phi(f)(g) = \pi_A(f(g)),$$

$$\Psi : \mathcal{H} \rtimes G \to \mathcal{O}_A(\mathcal{H}) \rtimes G, \quad \Psi(\xi)(g) = \tau_\mathcal{H}(\xi(g))$$

which induce an isomorphism

$$\mathcal{O}_{A \rtimes G}(\mathcal{H} \rtimes G) \cong \mathcal{O}_A(\mathcal{H}) \rtimes G.$$ 

**Corollary.** A vector bundle $p : E \to X$ is called a $G$-bundle if $G^0 = X$ and there is a groupoid morphism $G \to \text{Iso}(E)$.

$\mathcal{O}_E \rtimes G$ is the Cuntz-Pimsner algebra of the $C^*$-correspondence $\Gamma(E) \rtimes G$ over $C^*(G)$. 

Valentin Deaconu
Symmetries of Cuntz-Pimsner algebras


