# Rigidity in group von Neumann algebra

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# Summary

- **Group von Neumann algebras**: definitions; examples
- **Classification of group von Neumann algebras**: description of the problems; revisit some older results; (infinite) direct product rigidity; amalgamated free product rigidity; wreath product rigidity; applications to rigidity in $\mathbb{C}^*$-setting
- **Future directions**: some open problems

# Group von Neumann algebras



- (Murray-von Neumann '36)
- $\Gamma$ - countable discrete group

$\rightsquigarrow$ $u : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ - left regular representation

$$u_\gamma(\xi)(\lambda) = \xi(\gamma^{-1}\lambda), \quad \forall \gamma, \lambda \in \Gamma, \xi \in \ell^2\Gamma$$

$\rightsquigarrow$ the von Neumann algebra associated with $\Gamma$ is

$$\mathcal{L}(\Gamma) := \{u_\gamma \mid \gamma \in \Gamma\}'' = \overline{\mathbb{C}[\Gamma]}^{SOT} \subset \mathfrak{B}(\ell^2\Gamma)$$

$\rightsquigarrow$ $T_i \xrightarrow{SOT} T$ iff $\|T_i\eta - T\eta\| \to 0, \forall \eta \in \ell^2\Gamma$

$\rightsquigarrow$ $\tau(x) = \langle x\delta_e, \delta_e\rangle$ normal, state

- (faithful) $\tau(x^*x) = 0 \Leftrightarrow x = 0$
- (tracial) $\tau(xy) = \tau(yx)$

$\rightsquigarrow$ $\mathcal{L}(\Gamma)$ is a finite von Neumann algebra ($v^*v = 1 \Rightarrow vv^* = 1$)

# Group von Neumann Algebras

⇝ $\boxed{\mathcal{L}(\Gamma) \text{ as algebra of "left convolvers":}}$ $\forall \quad \xi, \eta \in \ell^2\Gamma$ define the convolution $\xi * \eta : \Gamma \to \mathbb{C}$

$$\xi * \eta(\gamma) = \sum_{\lambda \in \Gamma} \xi(\gamma\lambda^{-1})\eta(\lambda)$$

⇝ $\|\xi * \eta\|_\infty \leq \|\xi\|_2 \|\eta\|_2, \quad \xi * \delta_\gamma = v_{\gamma^{-1}}(\xi), \quad \delta_\gamma * \eta = u_\gamma(\eta)$

⇝ $D_\xi = \{\eta \in \ell^2\Gamma \,|\, \xi * \eta \in \ell^2\Gamma\}, \quad D'_\xi = \{\eta \in \ell^2\Gamma \,|\, \eta * \xi \in \ell^2\Gamma\}$, ⇝ densely def. linear operators

$$\eta \to L_\xi(\eta) = \xi * \eta \quad : D_\xi \to \ell^2\Gamma$$
$$\eta \to R_\xi(\eta) = \eta * \xi \quad : D'_\xi \to \ell^2\Gamma$$

⇝ $L_\xi, R_\xi$ have closed graphs and $L_\xi R_\xi = R_\xi L_\xi$

$$lconv(\Gamma) = \{L_\xi \,|\, D_\xi = \ell^2\Gamma\} \subset \mathfrak{B}(\ell^2\Gamma)$$
$$rconv(\Gamma) = \{R_\xi \,|\, D'_\xi = \ell^2\Gamma\} \subset \mathfrak{B}(\ell^2\Gamma)$$

⇝ $L_{\xi*\eta} = L_\xi L_\eta, \quad R_{\xi*\eta} = R_\xi R_\eta$

⇝ $lconv(\Gamma) = rconv(\Gamma)' = v(\Gamma)' = u(\Gamma)'' = \mathcal{L}(\Gamma)$ (note if $T \in \mathcal{L}(\Gamma)$ then $T = L_\xi$ where $\xi = T\delta_e$ )

⇝ Fourier expansion $x = \sum_\gamma x_\gamma u_\gamma, \qquad x_\gamma = \tau(x u_{\gamma^{-1}})$

# Group von Neumann Algebras



## Theorem (Murray-von Neumann '36)

*$\mathcal{L}(\Gamma)$ is a $\mathrm{II}_1$ factor ($\mathcal{Z}(\mathcal{L}(\Gamma)) = \mathbb{C}1$) $\Leftrightarrow \forall\, \gamma \neq e$ we have $|\gamma^\Gamma| = \infty$, i.e. $\Gamma$ is icc.*

**Examples:**

- $\mathbb{F}_n$, $n \geq 2$; $\Gamma_1 * \Gamma_2$, $|\Gamma_1| \geq 2, |\Gamma_2| \geq 3$; $PSL_n(\mathbb{Z}), n \geq 2$;
- $\mathfrak{S}_\infty$; $A \wr \Gamma$, $\Gamma$ *infinite;*

## Major theme of study in von Neumann algebras

- Is it possible to develop a "rigidity theory" in this setting?
- How much information does $\mathcal{L}(\Gamma)$ "remember" of the initial group $\Gamma$? Specifically, is it possible to identify a comprehensive list of canonical algebraic properties of $\Gamma$ that are completely recognized by $\mathcal{L}(\Gamma)$?

# Classification of group von Neumann algebras



**Some non-results:**

- (folk) if $\Gamma, \Lambda$ infinite abelian then $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda) \cong \mathcal{L}^{\infty}([0,1])$
- (Connes '76) if $\Gamma$, $\Lambda$ amenable icc then $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda) \cong \overline{\cup_n \mathcal{M}_{2^n}(\mathbb{C})}^{SOT} = \mathcal{R}$ the hyperfinite factor

  **Concrete examples:** $\mathcal{L}(\mathbb{Z} \wr \mathbb{Z}) \cong \mathcal{L}(\mathbb{Z}_2 \wr \mathbb{Z}) \cong \mathcal{L}(\mathfrak{S}_\infty)$
- (Dykema '93) if $\Gamma_i, \Lambda_i$ are infinite amenable then

$$\mathcal{L}(\Gamma_1 * \Gamma_2 * \cdots * \Gamma_n) \cong \mathcal{L}(\Lambda_1 * \Lambda_2 * \cdots * \Lambda_n)$$

- parallel similar results in orbit equivalence, e.g. (Dye '59, Ornstein-Weiss '81, Gaboriau '05)

**Conclusion:** In general, no memory of the classical group invariants such as torsion, rank, generators and relations, etc

# Classification of group of von Neumann algebras



**Some results:**

- (Murray-von Neumann '43) $\mathcal{L}(\mathbb{F}_2) \ncong \mathcal{L}(\mathfrak{S}_\infty \times \mathbb{F}_2)$
- (McDuff '69) Infinitely many non isomorphic group factors
- (Cowling-Haagerup '89) If $\Gamma < Sp(n,1)$, $\Lambda < Sp(m,1)$ lattices and $n \neq m$ then $\mathcal{L}(\Gamma) \ncong \mathcal{L}(\Lambda)$
- (Ozawa '03) if $\Gamma$ icc hyperbolic then $\mathcal{L}(\Gamma) \ncong \mathcal{L}(\Lambda \times \Theta)$, for every $\Lambda, \Theta$ infinite groups

**Using Popa's deformation/rigidity theory:**

- $\mathcal{L}(\Gamma_1 *_\Sigma \Gamma_2 *_\Sigma \cdots *_\Sigma \Gamma_n) \cong \mathcal{L}(\Lambda_1 *_\Omega \Lambda_2 *_\Omega \cdots *_\Omega \Lambda_m)$ implies $n = m$ and there exists $\sigma \in \mathfrak{S}_n$ such that $\mathcal{L}(\Gamma_i) \cong \mathcal{L}(\Lambda_{\sigma_i})$; known when $\Sigma, \Omega$ amenable and $\Gamma_i, \Lambda_j$ are
  1. infinite property (T) groups, (Ioana-Peterson-Popa '05)
  2. non-amenable direct products of infinite groups (C -Houdayer '10)
  3. non-amenable containing infinite normal amenable subgroups and $\Sigma, \Omega$ finite (Ioana '12)
- (Ozawa-Popa '07) If $L(\mathbb{F}_n) \cong L(\Lambda)$ then $\forall$ infinite amenable subgroup $\Sigma < \Lambda$ the normalizer $N_\Lambda(\Sigma)$ is amenable
- (Ioana-Popa-Vaes '10) if $\Gamma$ is non-amenable, $I = \Gamma \wr \mathbb{Z}/\mathbb{Z}$, and $\mathcal{L}(\mathbb{Z}_3 \wr_I (\Gamma \wr \mathbb{Z})) \cong \mathcal{L}(\Lambda)$ then $\mathbb{Z}_3 \wr_I (\Gamma \wr \mathbb{Z}) \cong \Lambda$; other wreath products by (Berbec-Vaes '13)

# Unique prime factorization for group von Neumann algebras



**Theorem (Ozawa-Popa, '03)**

*Let $\Gamma_i, \Lambda_j$ be icc hyperbolic groups (e.g. $\mathbb{F}_k$, $k \geq 2$). If*

$$\mathcal{L}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n) \cong \mathcal{L}(\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_m)$$

*then $n = m$ and there exist scalars $t_1 t_2 \cdots t_n = 1$ and a permutation $\sigma \in \mathfrak{S}_n$ such that*

$$\mathcal{L}(\Gamma_i)^{t_i} \cong \mathcal{L}(\Lambda_{\sigma_i}), \text{ for all } 1 \leq i \leq n$$

- the result still holds for all icc biexact groups; includes:
  - all groups hyperbolic relative to families of amenable subgroups
  - $\mathbb{Z} \wr \Gamma$, for every $\Gamma$ non-elementary hyperbolic
  - $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$
- we cannot do better at the factors level; by Voiculescu's formula $\mathcal{L}(\mathbb{F}_5 \times \mathbb{F}_3) \cong \mathcal{L}(\mathbb{F}_2 \times \mathbb{F}_9)$
- this parallels results of (Monod-Shalom '06) in orbit equivalence

# Direct product rigidity

**Theorem (C - De Santiago-Sinclair, '15)**

*Let $\Gamma_i$ be icc hyperbolic groups (or more generally nonamenable biexact) and $\Lambda$ be an arbitrary group. If*

$$\mathcal{L}(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n) \cong \mathcal{L}(\Lambda)$$

*then $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$ and there exist scalars $t_1 t_2 \cdots t_n = 1$ such that*

$$\mathcal{L}(\Gamma_i)^{t_i} \cong \mathcal{L}(\Lambda_i), \text{ for all } 1 \leq i \leq n$$

- a contrast point with the OE counterpart (Monod-Shalom '06) is no need to assume strong forms of ergodicity on the "target data"

**Corollary**

*Let $\Gamma_i$ be icc biexact nonamenable groups and $\Lambda$ be an arbitrary group such that*

$$\mathbb{C}^*_r(\Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n) \cong \mathbb{C}^*_r(\Lambda).$$

*Then $\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$ where $\Lambda_i$ are icc, nonamenable groups.*

# Infinite direct product rigidity

- Unique prime factorization results for infinite tensor products of solid factors were obtained by (Isono '16); motivated by these results we can ask the following basic question:
Can we get a similar product rigidity result for infinite direct sum groups?
- **Canonical obstruction**:

$$
\begin{aligned}
\mathcal{L}(\oplus_{i\in\mathbb{N}}\Gamma_i) &\cong \bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i)\\
&\cong \bar{\otimes}_{i\in\mathbb{N}}(\mathcal{L}(\Gamma_i)^{1/2}\otimes\mathcal{M}_2(\mathbb{C}))\\
&\cong (\bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i)^{1/2})\bar{\otimes}\mathcal{R}\\
&\cong (\bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i)^{1/2})\bar{\otimes}\mathcal{R}\bar{\otimes}\mathcal{R}\\
&\cong (\bar{\otimes}_{i\in\mathbb{N}}\mathcal{L}(\Gamma_i))\bar{\otimes}\mathcal{R}\\
&\cong \mathcal{L}(\oplus_{i\in\mathbb{N}}\Gamma_i\oplus A),
\end{aligned}
$$

for any $A$ icc amenable group.

# Infinite direct product rigidity

**Theorem (C - Udrea, '18)**

*Let $(\Gamma_i)_{i\in\mathbb{N}}$ be icc, biexact, property (T) groups and let $\Lambda$ be an arbitrary group such that*

$$\mathcal{L}(\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n \oplus \cdots) \cong \mathcal{L}(\Lambda).$$

*Then $\Lambda = (\Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_n \oplus \cdots) \oplus A$ where $A$ is icc amenable. Also there are positive scalars $t_1, t_2, \dots, t_n, \dots$ so that for each $k \in \mathbb{N}$ we have*

$$\mathcal{L}(\Gamma_1)^{t_1} \cong \mathcal{L}(\Lambda_1), \quad \mathcal{L}(\Gamma_2)^{t_2} \cong \mathcal{L}(\Lambda_2), \quad \dots \quad \mathcal{L}(\Gamma_k)^{t_k} \cong \mathcal{L}(\Lambda_k),$$
$$\mathcal{L}(\Gamma_{k+1} \oplus \Gamma_{k+2} \oplus \cdots) \cong \mathcal{L}((\Lambda_{k+1} \oplus \Lambda_{k+2} \oplus \cdots) \oplus A)$$

- the result applies to all $\Gamma_i$'s
  - uniform lattices in $Sp(n,1)$, $n \geq 2$
  - Gromov random groups with density $3^{-1} < d < 2^{-1}$
  - prop (T), hyperbolic relative to finitely generated amenable groups constructed by (Arzhantseva-Minasyan-Osin '07)

# Infinite direct product rigidity in $\mathbb{C}^*$-setting



## Corollary

*Let $(\Gamma_i)_{i\in\mathbb{N}}$ be icc, biexact, property (T) groups and let $\Lambda$ be an arbitrary group such that*

$$\mathbb{C}^*_r(\Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n \oplus \cdots) \cong \mathbb{C}^*_r(\Lambda).$$

*Then $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_n \oplus \cdots$ where $\Lambda_i$ are icc, biexact, property (T) groups.*

*Proof.* $\oplus_{i\in\mathbb{N}}\Gamma_i$ has trivial amenable radical and by (Breuillard-Kalantar-Kennedy-Ozawa '14) $\mathbb{C}^*_r(\oplus_{i\in\mathbb{N}}\Gamma_i)$ has unique trace; thus the isomorphism $\mathbb{C}^*_r(\oplus_{i\in\mathbb{N}}\Gamma_i) \cong \mathbb{C}^*_r(\Lambda)$ preserves the trace and hence lifts to the von Neumann algebras level and it follows from the previous theorem that $\Lambda = \oplus_{i\in\mathbb{N}}\Lambda_i \oplus A$; however by uniqueness of the trace again (BKKO '14) implies that $A = 1$ $\square$

# Amalgamated free product rigidity

## Theorem (C - Ioana '17)

*Let* $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ *be an amalgam satisfying the following:*

- $\Gamma_1 = \Gamma_1^1 \times \Gamma_1^2$, $\Gamma_2 = \Gamma_2^1 \times \Gamma_2^2$ *where* $\Gamma_i^j$, *are icc, non-amenable, biexact groups;*
- $\Sigma$ *is icc amenable and* $[\Sigma : \gamma\Sigma\gamma^{-1} \cap \Sigma] = \infty$ *for every* $\gamma \in \Gamma_i \setminus \Sigma$.

*Assume* $\Lambda$ *is an arbitrary group such that*

$$\mathcal{L}(\Gamma_1 *_\Sigma \Gamma_2) = \mathcal{L}(\Lambda).$$

*Then* $\Lambda = \Lambda_1 *_\Delta \Lambda_2$ *and there exists* $u \in \mathcal{U}(\mathcal{L}(\Gamma))$ *s.t.*

$$\mathcal{L}(\Lambda_1) = u\mathcal{L}(\Gamma_1)u^*, \quad \mathcal{L}(\Lambda_2) = u\mathcal{L}(\Gamma_2)u^*, \quad \mathcal{L}(\Delta) = u\mathcal{L}(\Sigma)u^*.$$

**Examples:**

1. $[(H_1 * \Theta) \times (K_1 * \Omega)] *_{(\Theta\times\Omega)} [(H_2 * \Theta) \times (K_2 * \Omega)]$ $\forall H_i, K_i-$ *biexact*, $\Theta, \Omega-$ *icc, amenable*
2. $[(A \wr H) \times (A \wr H)] *_{(A\wr C \times A\wr C)} [(A \wr H) \times (A \wr H)]$ $\forall H$ *hyperbolic*, $C < H$ *max. inf. amenable*

# $W^*$-superrigidity



**Theorem (C - Ioana '17)**

*Let $\Sigma < \Gamma_0$ be groups satisfying the following*

- $\Gamma_0$ *be icc, non-amenable, biexact;*
- $\Sigma$ *is icc, amenable, and satisfies* $[\Sigma : \Sigma \cap \gamma\Sigma\gamma^{-1}] = \infty$ *for every* $\gamma \in \Gamma_0 \setminus \Sigma$;
- *the centralizer in* $\Gamma_0$ *of any finite index subgroup of* $\Sigma \cap \gamma\Sigma\gamma^{-1}$ *is trivial, for all* $\gamma \in \Gamma_0$.

*Let* $\Gamma := (\Gamma_0 \times \Gamma_0) *_{diag(\Sigma)} (\Gamma_0 \times \Gamma_0)$.
*If* $\Lambda$ *is an any group and* $\theta : \mathcal{L}(\Gamma) \to \mathcal{L}(\Lambda)$ *is any* $*$*-isomorphism then there exist a group isomorphism* $\delta : \Gamma \to \Lambda$ *a unitary* $a \in \mathcal{L}(\Lambda)$, *and a character* $\eta : \Gamma \to \mathbb{T}$ *such that*

$$\theta(u_\gamma) = \eta(\gamma) a v_{\delta(\gamma)} a^*, \quad \forall \gamma \in \Gamma.$$

**Examples**

- there are uncountably many groups $\Gamma$
- $[(\mathfrak{S}_\infty \wr \mathbb{F}_n) \times (\mathfrak{S}_\infty \wr \mathbb{F}_n)] *_{diag(\mathfrak{S}_\infty \wr \mathbb{Z})} [(\mathfrak{S}_\infty \wr \mathbb{F}_n) \times (\mathfrak{S}_\infty \wr \mathbb{F}_n)] \qquad n \geq 2$

# Applications to $\mathbb{C}^*$-superrigidity



**Corollary (C - Ioana '17)**

*Let* $\Sigma < \Gamma_0$ *be groups satisfying the following*

- $\Gamma_0$ *be icc, non-amenable, biexact;*
- $\Sigma$ *is icc, amenable, and satisfies* $[\Sigma : \Sigma \cap \gamma\Sigma\gamma^{-1}] = \infty$ *for every* $\gamma \in \Gamma_0 \setminus \Sigma$;
- *the centralizer in* $\Gamma_0$ *of any finite index subgroup of* $\Sigma \cap \gamma\Sigma\gamma^{-1}$ *is trivial, for all* $\gamma \in \Gamma_0$.

*Let* $\Gamma := (\Gamma_0 \times \Gamma_0) *_{diag(\Sigma)} (\Gamma_0 \times \Gamma_0)$.
*If* $\Lambda$ *is an any group and* $\theta : \mathbb{C}^*_r(\Gamma) \to \mathbb{C}^*_r(\Lambda)$ *is any* $*$*-isomorphism then there exist a group isomorphism* $\delta : \Gamma \to \Lambda$ *a unitary* $a \in \mathcal{L}(\Lambda)$, *and a character* $\eta : \Gamma \to \mathbb{T}$ *such that*

$$\theta(u_\gamma) = \eta(\gamma) a v_{\delta(\gamma)} a^*, \quad \forall \gamma \in \Gamma.$$

- this provide the first examples of non-amenable $\mathbb{C}^*$-superrigid groups; the only other known examples by (Scheinberg '74, Knuby-Raum-Thiel-White '17, Eckhart-Raum '18, Omland '18)

# Wreath product rigidity



**Theorem (C - Udrea '18)**

- *Let $H, \Gamma_1, \Gamma_2$ icc, biexact, property (T) groups;*
- $\Gamma = \Gamma_1 \times \Gamma_2$.

*Let $\Lambda$ be an arbitrary group and let $\theta : \mathcal{L}(H \wr \Gamma) \to \mathcal{L}(\Lambda)$ be a $*$-isomorphism. Then*

- $\Sigma, \Psi$ *icc, property (T) groups,* $A$ *icc amenable, an action* $\Psi \curvearrowright^{\alpha} A$ *such that* $\Lambda = (\Sigma^{(\Psi)} \oplus A) \rtimes_{\beta \oplus \alpha} \Psi$*, where* $\Psi \curvearrowright^{\beta} \Sigma^{(\Psi)}$ *is the Bernoulli action.*

*Also there is a group isomorphism $\delta : \Gamma \to \Psi$, a character $\eta : \Gamma \to \mathbb{T}$, a $*$-isomorphism $\theta_0 : \mathcal{L}(H^{(\Gamma)}) \to \mathcal{L}(\Sigma^{(\Psi)} \oplus A)$, and a unitary $a \in \mathcal{L}(\Lambda)$ such that for all $x \in \mathcal{L}(H^{(\Gamma)})$, $\gamma \in \Gamma$ we have*

$$\theta(xu_\gamma) = \eta(\gamma) a \theta_0(x) v_{\delta(\gamma)} a^*.$$

**Corollary (C - Udrea '18)**

*Let $H, \Gamma_1, \Gamma_2$ be icc, biexact, property (T) groups and let $\Gamma = \Gamma_1 \times \Gamma_2$. If $\Lambda$ is an arbitrary group so that $\mathbb{C}^*_r(H \wr \Gamma) = \mathbb{C}^*_r(\Lambda)$ then $\Lambda = \Sigma \wr \Gamma$, where $\Sigma$ is an icc, property (T) group.*

Ideas behind the proof of the infinite direct product rigidity here

# Questions

## Open Problem

Does infinite product rigidity still hold if one removes the property (T) assumption (i.e. for infinite direct sum of icc non-amenable biexact groups)?

## Open Problem

Are there any instances when the plain free product rigidity holds?

$L(\Gamma_1 * \Gamma_2) = L(\Lambda) \overset{?}{\Rightarrow} \Lambda = \Lambda_1 * \Lambda_2$ and $L(\Lambda_i) \cong L(\Gamma_i)$ for all $i = 1, 2$.

## Open Problem

Identify other algebraic features that are recognizable from von Neumann algebras. Examples: a) fiber products (C- Das '18), b) HNN-extensions, $HNN(\Gamma, \Sigma, \theta)$, c) graph products $\mathcal{G}(\Gamma_v, v \in \mathcal{V})$?

## Open Problem

Is there a hyperbolic group $\Gamma$ s.t. whenever $\Lambda$ is a group satisfying $\mathcal{L}(\Gamma) \cong \mathcal{L}(\Lambda)$ it follows that $\Lambda$ is hyperbolic as well?

THANK YOU!

supplemental2. **Step 1:** let $\{\gamma : \gamma \in \Gamma\}'' = \mathcal{L}(\oplus_{i\in I}\Gamma_i) = M = \mathcal{L}(\Lambda) = \{\lambda : \lambda \in \Lambda\}''$

$\rightsquigarrow$ (Popa-Vaes) co-multiplication along $\Lambda$ i.e. $\Delta : M \to M\bar{\otimes}M$ given by $\Delta(\lambda) = \lambda \otimes \lambda$

$\rightsquigarrow$ $\forall i, j \in I$ we have $\Delta(\mathcal{L}(\Gamma_{I\setminus\{i\}})), \Delta(\mathcal{L}(\Gamma_i)) \subseteq \Delta(M) \subseteq M\bar{\otimes}M = M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}} \oplus \Gamma_j)$ using (Popa-Vaes '12) control of normalizers $\Rightarrow$

$$\Delta(\mathcal{L}(\Gamma_{I\setminus\{i\}})) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}}) \quad \text{or} \quad \Delta(\mathcal{L}(\Gamma_i)) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})$$

However, if $\Delta(\mathcal{L}(\Gamma_i)) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})$ for all $j$ then $\Delta(\mathcal{L}(\Gamma_i)) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus F})$ for all $F \subset I$ finite; it follows that $\Delta(\mathcal{L}(\Gamma_i))$ is amen. rel. to $M\bar{\otimes}1$ forcing $\Gamma_i$ amenable, contradiction; hence $\forall i, \exists j$ so that

$$\Delta(\mathcal{L}(\Gamma_{I\setminus\{i\}})) \prec M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})$$

# Ideas behind the proof — Infinite direct product rigidity

supplemental3. **Step 2:** $\|E_{M\bar{\otimes}\mathcal{L}(\Gamma_{I\setminus\{j\}})}(\Delta(u))\|_2 \geq C > 0$ for all $u \in \mathcal{U}(L(\Gamma_{I\setminus\{i\}}))$

- (ultrapower tech Ioana '11) let $\mathcal{G} = \{\Sigma : \Sigma \leqslant \Lambda\}$ and $\mathcal{J}$ the directed set of all small sets over $\mathcal{G}$
- if $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \not\prec \mathcal{L}(\Sigma)$ for all $\Sigma \in \mathcal{G}$ then for each $S \in \mathcal{J}$ there exists $\lambda_S \in \Lambda \setminus S$ such that

$$\|E_{\mathcal{L}(\Gamma_{I\setminus\{j\}})}(\lambda_S)\|_2 \geq C/2$$

- if $\omega$ is cofinal ultrafilter on $\mathcal{J}$ then $E_{\mathcal{L}(\Gamma_{I\setminus\{j\}})^\omega}(\lambda^\omega) \neq 0$ where $\lambda^\omega = (\lambda_S)_S$
- Assume: $\lambda^\omega \in \mathcal{L}(\Gamma_{I\setminus\{j\}})^\omega \subseteq \mathcal{L}(\Gamma_j)' \cap M^\omega \Rightarrow \mathcal{L}(\Gamma_j) \subseteq \lambda^\omega M(\lambda^\omega)^{-1} \cap M = \mathcal{L}(\lambda^\omega \Lambda(\lambda^\omega)^{-1} \cap \Lambda)$
- $\lambda^\omega \Lambda(\lambda^\omega)^{-1} \cap \Lambda = \cup_n C(\Sigma_n)$ where $\Sigma_n \notin \mathcal{G}$, descending
- for all $\mathcal{G}$ either $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \prec \mathcal{L}(\Sigma)$ for some $\Sigma \in \mathcal{G}$ or $\mathcal{L}(\Gamma_j) \prec \mathcal{L}(\cup_n C(\Sigma_n))$ with $\Sigma_n \notin \mathcal{G}$

# Ideas behind the proof — Infinite direct product rigidity

supplemental4. **Step 3:** $\exists\ \Sigma \leqslant \Lambda$ s. t. $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \prec \mathcal{L}(\Sigma)$ with $C(\Sigma)$ is non-amenable;

$\leadsto$ Assume: $\mathcal{L}(\Gamma_{I\setminus\{i\}}) \subseteq \mathcal{L}(\Sigma)$; splitting prop of tensors $\Rightarrow \mathcal{L}(\Gamma_{I\setminus\{i\}})\bar{\otimes}B = \mathcal{L}(\Sigma)$ with $B \subseteq \mathcal{L}(\Gamma_i)$

$\leadsto$ solidity of $\mathcal{L}(\Gamma_i)$ imply $B$ is atomic, so up to corners we can assume

$$\mathcal{L}(\Gamma_{I\setminus\{i\}}) = \mathcal{L}(\Sigma)$$

$\leadsto$ passing to relative commutants we get

$$\mathcal{L}(\Gamma_i) = \mathcal{L}(\Sigma)' \cap \mathcal{L}(\Lambda) \subseteq \mathcal{L}(vC(\Sigma))$$

where $vC(\Sigma) = \{\lambda \in \Lambda : |\lambda^{\Sigma}| < \infty\}$—virtual centralizer of $\Sigma$

$\leadsto$ let $\mathcal{O}_1, \mathcal{O}_2, \ldots$ all finite orbits under $\Sigma$-conjug.; let $\Omega_k = \langle \mathcal{O}_1, \ldots, \mathcal{O}_k \rangle$ and note $\cup_k \Omega_k = vC(\Sigma)$

$\leadsto$ using property (T) we get

$$\mathcal{L}(\Gamma_i) = \mathcal{L}(\Sigma)' \cap \mathcal{L}(\Lambda) \subseteq \mathcal{L}(\Omega_\ell)$$

hence $\mathcal{L}(\Gamma_{I\setminus\{i\}})\bar{\otimes}\mathcal{L}(\Gamma_i) = \mathcal{L}(\Sigma \vee \Omega_\ell) = \mathcal{L}(\Lambda)$; $\Rightarrow \Sigma \vee \Omega_\ell = \Lambda$ and $vC(\Sigma) = \Omega_\ell$

$\leadsto$ as $\Sigma$ has finite index subgroup commuting with $vC(\Sigma)$ then $\Lambda$ is commensurable to a product group

# Ideas behind the proof — Infinite direct product rigidity

main. **Step 4:** "perturbing" $\Sigma$ and $vC(\Sigma)$ up to finite index and using (Ozawa-Popa '03) we get that $\Lambda = \Sigma_i \oplus vC(\Sigma_i)$ and $t_i > 0$ such that

$$\mathcal{L}(\Gamma_{I\setminus\{i\}})^{t_i} = \mathcal{L}(\Sigma_i) \text{ and there is } \mathcal{L}(\Gamma_i)^{1/t_i} = \mathcal{L}(vC(\Sigma_i))$$

$\rightsquigarrow$ using these product decompositions for every $i$ one derive the desired conclusion

• since the "equalities" above are up to corners and partial conjugacy all the statements above are virtual (i.e. up to finite index/intersection) and their proofs are pretty involved technically; to get to a honest direct product we often have to use factoriality of the original algebras.