A Paley-Wiener Type Theorem for Singular Measures on $\mathbb{T}$

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The Paley-Wiener Theorem

An entire function $F$ can be written in the form

$$F(z) = \int_{-1/2}^{1/2} f(t)e^{-2\pi itz} \, dt$$

for some $f \in L^2(-1/2, 1/2)$ if and only if $F$ satisfies:

1. $F$ is of exponential type $\pi$;
2. $F(t) \in L^2(\mathbb{R})$.

Alternative characterization: $F$ satisfies

1. $$\sum_{n \in \mathbb{Z}} |F(n)|^2 < \infty;$$
2. $$F(z) = \sum_{n \in \mathbb{Z}} F(n)sinc(z - n).$$
Fix a singular measure $\mu$ on $(-1/2, 1/2)$. Question: when can an entire function $F$ be written as

$$F(z) = \int_{-1/2}^{1/2} f(t) e^{-2\pi i tz} \, d\mu(t)$$

for some $f \in L^2(\mu)$?

We provide a characterization using a sampling theory idea and an interpolation idea.
Motivating Question

Strichartz: given a compact set $K$, determine when an entire function $F$ can be written as

$$F(z) = \int_K e^{-2\pi izt} d\sigma(t)$$

for some complex measure $\sigma$ supported on $K$?

We know when a function $F : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of a measure: Bochner-Schoenberg-Eberlein (BSE) conditions.

These conditions cannot tell the support of the measure.
Fourier Series for Singular Measures
Kaczmarz Algorithm

Given $\{\varphi_n\}_{n=0}^{\infty} \subset H$ and $\langle x, \varphi_n \rangle$, can we recover $x$? Note: yes if ONB/frame.

$x_0 = \langle x, \varphi_0 \rangle \varphi_0$

$x_n = x_{n-1} + \langle x - x_{n-1}, \varphi_n \rangle \varphi_n$.

If $\lim_{n \to \infty} \| x - x_n \| = 0$ for all $x$, then the sequence $\{\varphi_n\}_{n=0}^{\infty}$ is said to be effective. If so, we define

$g_n = e_n - \sum_{j=0}^{n-1} \langle e_n, e_j \rangle g_j$

and obtain

$x = \sum \langle x, g_i \rangle \varphi_i$. 
A sequence $\{\phi_n\}$ is stationary if $\langle \phi_{n+k}, \phi_{m+k} \rangle = \langle \phi_n, \phi_m \rangle$.

Theorem (Kwapien & Mycielski, 2001)

If $\{\phi_n\}_{n=1}^{\infty} \subset H$ is a stationary sequence with dense span, then it is an effective sequence if and only if its spectral measure is either Lebesgue measure or purely singular.
Theorem (Herr & W., 2015)

If $\mu$ is a singular Borel probability measure on $(-1/2, 1/2)$, then the sequence $\{e^{2\pi inx}\}_{n=0}^{\infty}$ is effective in $L^2(\mu)$. As a consequence, any element $f \in L^2(\mu)$ possesses a Fourier series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi inx},$$

where the sum converges in the $L^2(\mu)$ norm. The Fourier coefficients $c_n$ are given by

$$c_n = \int_{-1/2}^{1/2} f(x)\overline{g_n(x)} \, d\mu(x),$$

where $\{g_n\}_{n=0}^{\infty}$ is the auxiliary sequence of $\{e^{2\pi inx}\}_{n=0}^{\infty}$ in $L^2(\mu)$. 
**Inversion Lemma**

**Lemma (Herr & W., 2015)**

There exists a sequence \( \{\alpha_n\}_{n=0}^{\infty} \) such that

\[
g_n(x) = \sum_{j=0}^{n} \alpha_{n-j} e^{2\pi ijx}.
\]

The sequence is given by

\[
\frac{1}{\mu_+(z)} = \sum_{n=0}^{\infty} \alpha_n z^n
\]

where

\[
\mu_+(z) = \int_{-1/2}^{1/2} \frac{1}{1 - e^{-2\pi it} z} d\mu(t).
\]
The Paley-Wiener Theorem via a Sampling Criteria
Theorem (H. & W., 2015)

Let \( \mu \) be a singular Borel probability measure on \((-1/2, 1/2)\). Let \( \{\alpha_n\}_{n=0}^{\infty} \) be the sequence of scalars induced by \( \mu \) by the Inversion Lemma. Suppose \( F : \mathbb{R} \to \mathbb{C} \) is of the form

\[
F(y) = \int_{-1/2}^{1/2} f(x) e^{-2\pi iyx} \, d\mu(x)
\]

for some \( f \in L^2(\mu) \). Then

\[
F(y) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \hat{\mu}(y - n),
\]

where the series converges uniformly in \( y \).
Since the \( \{g_n\} \) form a Parseval frame, we obtain the following variation.

**Theorem (H. & W., 2015)**

Let \( \mu \) be a singular Borel probability measure on \((-1/2, 1/2)\). Let \( \{\alpha_n\}_{n=0}^{\infty} \) be the sequence of scalars induced by \( \mu \) by the Inversion Lemma. Suppose \( F : \mathbb{R} \rightarrow \mathbb{C} \) is of the form

\[
F(y) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i y x} \, d\mu(x)
\]

for some \( f \in L^2(\mu) \). Then

\[
F(y) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \left( \sum_{l=0}^{n} \alpha_{n-l} \hat{\mu}(y - l) \right),
\]

where the series converges uniformly in \( y \).
The Paley-Wiener Theorem for $\mu$

**Theorem (W., 2017)**

Let $\mu$ be a singular Borel probability measure on $(-1/2, 1/2)$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be the sequence of scalars induced by $\mu$ by the Inversion Lemma. The entire function $F$ has the form

$$F(z) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i tz} \, d\mu(t)$$

for some $f \in L^2(\mu)$ if and only if $F$ satisfies

1. $\sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} \alpha_{n-j} F(j) \right|^2 < \infty$;

2. for all $z \in \mathbb{C}$,

$$F(z) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \left( \sum_{l=0}^{n} \alpha_{n-l} \hat{\mu}(z-l) \right).$$
Proof

Necessity: Apply Fourier transform to

\[ f = \sum_{n=0}^{\infty} \langle f, g_n \rangle g_n. \]

Sufficiency: Define \( f \in L^2(\mu) \) by

\[ f = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \alpha_{n-j} F(j) \right) g_n \]

then apply Fourier transform to obtain

\[ \hat{f}(z) = F(z). \]
The Paley-Wiener Theorem via an Interpolation Criteria
There is a 1-to-1 correspondence between the nonconstant inner functions $b$ in $H^2$ and the nonnegative singular Borel measures $\mu$ on $\mathbb{T} \equiv [0, 1)$ given by

$$\text{Re} \left( \frac{1 + b(z)}{1 - b(z)} \right) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi).$$

We will say that $b$ corresponds to $\mu$, and that $\mu$ corresponds to $b$. The space

$$\mathcal{H}(b) = H^2 \ominus bH^2.$$

is backward-shift invariant (Beurling's theorem).
Given a measure $\mu$ on $(-1/2, 1/2)$, the normalized Cauchy transform is the operator $V_\mu$ from $L^2(\mu)$ to the set of functions on $\mathbb{C} \setminus \mathbb{T}$ given by

$$V_\mu f(z) = \frac{\int_{-1/2}^{1/2} f(x) \frac{1}{1 - ze^{-2\pi ix}} d\mu(x)}{\int_{-1/2}^{1/2} \frac{1}{1 - ze^{-2\pi ix}} d\mu(x)}.$$

Clark showed that if $\mu$ is a singular Borel probability measure and $b$ is its corresponding inner function, then $V_\mu$ maps $L^2(\mu)$ unitarily onto $\mathcal{H}(b)$. 
Recall that $f \in H^2(\mathbb{D})$ if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{2\pi ix})|^2 dx < \infty$$

Then there exists a function $f^* \in L^2(\mathbb{T})$ such that $f(re^{2\pi ix}) \rightarrow f^*(e^{2\pi ix})$ in the norm. (Abel summation).

If we replace Lebesgue measure by $\mu$ on $\mathbb{T}$, then for $f \in H^2(\mathbb{D})$, we say $f^*$ is the $L^2(\mu)$-boundary of $f$ if $f(re^{2\pi ix}) \rightarrow f^*(e^{2\pi ix})$ in the $L^2(\mu)$-norm.
Re-Expression of the Normalized Cauchy Transform

Theorem (H.&W., 2015)

Let $\mu$ be a singular Borel probability measure, and $\{g_n\}_{n=0}^{\infty}$ the auxiliary sequence of $\{e_n\}_{n=0}^{\infty}$ in $L^2(\mu)$. Then for $f \in L^2(\mu)$,

$$V_\mu f(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu z^n.$$

Thus, every function $F \in \mathcal{H}(b)$ is of the form $F(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu z^n$. Since $f = \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu e^{2\pi i nx}$ and $F(re^{2\pi i x}) := \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu re^{2\pi i nx}$, Abel summability shows that $\lim_{r \to 1^-} \|F(re^{2\pi i x}) - f(x)\|_\mu = 0$, and so $f$ is an $L^2(\mu)$ boundary function of $F$. 
Lemma

Suppose $\mu$ is a singular Borel probability measure on $\mathbb{T}$, $b$ is the inner function on $\mathbb{D}$ associated to $\mu$ via the Herglotz representation, and suppose $\{a_n\}_{n=0}^\infty \subset \mathbb{C}$. The following conditions are equivalent:

(i) there exists a function $f \in L^2(\mu)$ with the property that

$$a_n = \int_{\mathbb{T}} f(x) e^{-2\pi i nx} \, d\mu(x);$$

(ii) the following inclusion holds:

$$G_a(z) := \frac{\sum_{n=0}^\infty a_n z^n}{\mu_+(z)} \in \mathcal{H}(b).$$
For an entire function $F$ of exponential type, we use $h_F$ to denote the Phragmén-Lindelöf indicator function.

**Theorem (W. 2017)**

Suppose $\mu$ is a singular Borel probability measure with support in $[\alpha, \beta] \subset [-1/2, 1/2]$ where $\beta - \alpha < 1$. Let $b$ be the inner function associated to $\mu$ via the Herglotz Representation. The entire function $F$ is the Fourier transform $\hat{f}$ for some $f \in L^2(\mu)$ if and only if

(i) $F$ is of exponential type;

(ii) the indicator function of $F$ satisfies $h_F(\frac{\pi}{2}) \leq 2\pi \beta$ and $h_F(-\frac{\pi}{2}) \leq -2\pi \alpha$;

(iii) the following inclusion holds:

$$G_F(z) := \sum_{n=0}^{\infty} \frac{F(n)z^n}{\mu_+(z)} \in \mathcal{H}(b)$$

i.e. the function $G_F$ is in the kernel of the Toeplitz operator $T_b$. 
Proof

1. $G_F \in \mathcal{H}(b)$ implies that $F(n)$ can be interpolated by $\hat{f}$; i.e. there exists a $f \in L^2(\mu)$ such that $\hat{f}(n) = F(n)$;

2. the support of $\mu$ implies

$$h_{\hat{f}}\left(\frac{\pi}{2}\right) \leq 2\pi \beta \text{ and } h_{\hat{f}}\left(-\frac{\pi}{2}\right) \leq -2\pi \alpha;$$

3. Carlson’s theorem applies: If $g$ is exponential type on the RHP, $g(n) = 0$ for $n \in \mathbb{N}$, and $h_g\left(\frac{\pi}{2}\right) + h_g\left(-\frac{\pi}{2}\right) < 2\pi$, then $g \equiv 0$.

4. applying Carlson’s theorem to $F - \hat{f}$, we obtain that $F(z) = \hat{f}(z)$ for all $z$. 
The Paley-Wiener Theorem for \( \mu \), yet again

**Theorem (W. 2017)**

Suppose \( \mu \) is a singular Borel probability measure on \((-1/2, 1/2)\), and let \( b \) be the inner function associated to \( \mu \) by the Herglotz Representation. The entire function \( F \) is the Fourier transform \( \hat{f} \) for some \( f \in L^2(\mu) \) if and only if

(i) \(|F(z)| \leq \varepsilon(|z|)e^{\pi|z|} \text{ with } \varepsilon(r) = o(1);\)

(ii) the following inclusions hold:

\[
G_+(z) := \sum_{n=0}^{\infty} \frac{F(n)z^n}{\mu_+(z)} \in \mathcal{H}(b), \quad G_-(z) := \sum_{n=0}^{\infty} \frac{F(-n)z^n}{\mu_+(z)} \in \mathcal{H}(b);
\]

(iii) the \( L^2(\mu) \)-boundaries of \( G_+ \) and \( G_- \) satisfy the relationship

\[
\overline{G_+^*} = G_-^*.
\]
First version:

1. Condition (iii) says that \( \{F(n)\} \) can be interpolated by some \( \hat{f} \), so \( F(n) = \hat{f}(n) \) for \( n \in \mathbb{N}_0 \);

2. Condition (i) and (ii) says that \( F(z) = \hat{f}(z) \) for all \( z \) by Carlson’s theorem.

Second version is similar, but Carlson’s theorem does not apply. We use a generalization of Carlson’s theorem from Boas’ book.
A No-Go Result

Denote: \( PW(\mu) = \{ \hat{f} \mid f \in L^2(\mu) \} \), \( \mathcal{E}_\tau \) the collection of all entire functions of exponential type at most \( \tau \).

**Theorem**

Suppose \( PW(\mu) = \mathcal{E}_\tau \cap L^2(w) \) for some \( \tau \in (0, \pi] \) and some weight or measure \( w \) on \( \mathbb{R} \) with \( \| f \|_\mu \simeq \| \hat{f} \|_w \). Then there exists a Riesz basis of the form

\[
\{ \omega_n e^{2\pi i \lambda_n x} \}_{n \in \mathbb{Z}} \subset L^2(\mu) \tag{2}
\]

for some sequence \( \{ \lambda_n \} \subset \mathbb{R} \) and \( \omega_n > 0 \).
Proof

1. Under the hypotheses, $PW(\mu)$ is a de Branges space;
2. de Branges spaces contain orthogonal bases of kernel functions (on real axis);
3. these kernel functions correspond to weighted exponentials in $L^2(\mu)$;
4. very few measures possess Riesz bases of exponentials.
The End
Thank you!
Selected Works Cited


