A Paley-Wiener Type Theorem for Singular Measures on $\mathbb{T}$

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Iowa-Nebraska Functional Analysis Seminar
April 7, 2018

The Paley-Wiener Theorem

An entire function F can be written in the form

$$F(z) = \int_{-1/2}^{1/2} f(t) e^{-2\pi itz} dt$$

for some $f \in L^2(-1/2, 1/2)$ if and only if F satisfies:

$$\left.\begin{array}{l} \text{1. } F \text{ is of exponential type } \pi; \\ \text{2. } F(t) \in L^2(\mathbb{R}). \end{array}\right\} \mathcal{E}_\pi \cap L^2(\mathbb{R})$$

Alternative characterization: F satisfies

1. $$\sum_{n\in\mathbb{Z}} |F(n)|^2 < \infty;$$
2. $$F(z) = \sum_{n\in\mathbb{Z}} F(n) sinc(z-n).$$

Singular Measures

Fix a singular measure μ on $(-1/2, 1/2)$. Question: when can an entire function F be written as

$$F(z) = \int_{-1/2}^{1/2} f(t) e^{-2\pi i t z} d\mu(t)$$

for some $f \in L^2(\mu)$?

We provide a characterization using a sampling theory idea and an interpolation idea.

Motivating Question

Strichartz: given a compact set K, determine when an entire function F can be written as

$$F(z) = \int_K e^{-2\pi i z t} d\sigma(t)$$

for some complex measure σ supported on K?

We know when a function $F : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of a measure: Bochner-Schoenberg-Eberlein (BSE) conditions.

These conditions cannot tell the support of the measure.

Fourier Series for Singular Measures

Kaczmarz Algorithm

Given $\{\varphi_n\}_{n=0}^{\infty} \subset H$ and $\langle x, \varphi_n \rangle$, can we recover x? Note: yes if ONB/frame.

$$\begin{aligned} x_0 &= \langle x, \varphi_0 \rangle \varphi_0 \\ x_n &= x_{n-1} + \langle x - x_{n-1}, \varphi_n \rangle \varphi_n . \end{aligned}$$

If $\lim_{n\to\infty} \|x - x_n\| = 0$ for all x, then the sequence $\{\varphi_n\}_{n=0}^{\infty}$ is said to be effective. If so, we define

$$g_n = e_n - \sum_{j=0}^{n-1} \langle e_n, e_j \rangle g_j$$

and obtain

$$x = \sum \langle x, g_i \rangle \varphi_i .$$

Stationary Sequences

A sequence $\{\varphi_n\}$ is stationary if $\langle \varphi_{n+k}, \varphi_{m+k} \rangle = \langle \varphi_n, \varphi_m \rangle$.

Theorem (Kwapien & Mycielski, 2001)

If $\{\phi_n\}_{n=1}^{\infty} \subset H$ is a stationary sequence with dense span, then it is an effective sequence if and only if it's spectral measure is either Lebesgue measure or purely singular.

Fourier Series

Theorem (Herr & W., 2015)

If μ is a singular Borel probability measure on $(-1/2, 1/2)$, then the sequence $\left\{e^{2\pi inx}\right\}_{n=0}^{\infty}$ is effective in $L^2(\mu)$. As a consequence, any element $f \in L^2(\mu)$ possesses a Fourier series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{2\pi inx},$$

where the sum converges in the $L^2(\mu)$ norm. The Fourier coefficients c_n are given by

$$c_n = \int_{-1/2}^{1/2} f(x)\overline{g_n(x)}\, d\mu(x),$$

where $\{g_n\}_{n=0}^{\infty}$ is the auxiliary sequence of $\left\{e^{2\pi inx}\right\}_{n=0}^{\infty}$ in $L^2(\mu)$.

Inversion Lemma

Lemma (Herr & W., 2015)

There exists a sequence $\{\alpha_n\}_{n=0}^{\infty}$ *such that*

$$g_n(x) = \sum_{j=0}^{n} \overline{\alpha_{n-j}} e^{2\pi i j x}.$$

The sequence is given by

$$\frac{1}{\mu_+(z)} = \sum_{n=0}^{\infty} \alpha_n z^n$$

where

$$\mu_+(z) = \int_{-1/2}^{1/2} \frac{1}{1 - e^{-2\pi i t} z} d\mu(t).$$

The Paley-Wiener Theorem via a Sampling Criteria

A Shannon Sampling Formula

Theorem (H. & W., 2015)

Let μ be a singular Borel probability measure on $(-1/2, 1/2)$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be the sequence of scalars induced by μ by the Inversion Lemma. Suppose $F : \mathbb{R} \to \mathbb{C}$ is of the form

$$F(y) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i y x} \, d\mu(x)$$

for some $f \in L^2(\mu)$. Then

$$F(y) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \widehat{\mu}(y - n),$$

where the series converges uniformly in y.

An Alternative Sampling Formula

Since the $\{g_n\}$ form a Parseval frame, we obtain the following variation.

Theorem (H. & W., 2015)

Let μ be a singular Borel probability measure on $(-1/2, 1/2)$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be the sequence of scalars induced by μ by the Inversion Lemma. Suppose $F : \mathbb{R} \to \mathbb{C}$ is of the form

$$F(y) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i y x} \, d\mu(x)$$

for some $f \in L^2(\mu)$. Then

$$F(y) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \left(\sum_{l=0}^{n} \alpha_{n-l} \widehat{\mu}(y - l) \right),$$

where the series converges uniformly in y.

The Paley-Wiener Theorem for μ

Theorem (W., 2017)

Let μ be a singular Borel probability measure on $(-1/2, 1/2)$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be the sequence of scalars induced by μ by the Inversion Lemma. The entire function F has the form

$$F(z) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i t z} \, d\mu(t)$$

for some $f \in L^2(\mu)$ if and only if F satisfies

1. $$\sum_{n=0}^{\infty} \left| \sum_{j=0}^{n} \alpha_{n-j} F(j) \right|^2 < \infty;$$

2. *for all $z \in \mathbb{C}$,*

$$F(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} F(j) \right) \left(\sum_{l=0}^{n} \alpha_{n-l} \widehat{\mu}(z-l) \right).$$

Proof

Necessity: Apply Fourier transform to

$$f = \sum_{n=0}^{\infty} \langle f, g_n \rangle g_n.$$

Sufficiency: Define $f \in L^2(\mu)$ by

$$f = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \alpha_{n-j} F(j) \right) g_n$$

then apply Fourier transform to obtain

$$\hat{f}(z) = F(z).$$

The Paley-Wiener Theorem via an Interpolation Criteria

Herglotz Representation Theorem and the space $\mathcal{H}(b)$

Theorem

There is a 1-to-1 correspondence between the nonconstant inner functions b in H^2 and the nonnegative singular Borel measures μ on $\mathbb{T} \equiv [0, 1)$ given by

$$Re\left(\frac{1+b(z)}{1-b(z)}\right) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi).$$

We will say that b corresponds to μ, and that μ corresponds to b. The space

$$\mathcal{H}(b) = H^2 \ominus bH^2.$$

is backward-shift invariant (Beurling's theorem).

Normalized Cauchy Transform

Given a measure μ on $(-1/2, 1/2)$, the normalized Cauchy transform is the operator V_μ from $L^2(\mu)$ to the set of functions on $\mathbb{C} \setminus \mathbb{T}$ given by

$$V_\mu f(z) = \frac{\displaystyle\int_{-1/2}^{1/2} \frac{f(x)}{1 - ze^{-2\pi ix}}\, d\mu(x)}{\displaystyle\int_{-1/2}^{1/2} \frac{1}{1 - ze^{-2\pi ix}}\, d\mu(x)}.$$

Clark showed that if μ is a singular Borel probability measure and b is its corresponding inner function, then V_μ maps $L^2(\mu)$ unitarily onto $\mathcal{H}(b)$.

Boundary Representations

Recall that $f \in H^2(\mathbb{D})$ if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(re^{2\pi i x})|^2 dx < \infty$$

Then there exists a function $f^\star \in L^2(\mathbb{T})$ such that $f(re^{2\pi i x}) \to f^\star(e^{2\pi i x})$ in the norm. (Abel summation).

If we replace Lebesgue measure by μ on $\mathbb{T}$, then for $f \in H^2(\mathbb{D})$, we say $f^\star$ is the $L^2(\mu)$-boundary of f if $f(re^{2\pi i x}) \to f^\star(e^{2\pi i x})$ in the $L^2(\mu)$-norm.

Re-Expression of the Normalized Cauchy Transform

Theorem (H.&W., 2015)

Let μ be a singular Borel probability measure, and $\{g_n\}_{n=0}^{\infty}$ the auxiliary sequence of $\{e_n\}_{n=0}^{\infty}$ in $L^2(\mu)$. Then for $f \in L^2(\mu)$,

$$V_\mu f(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu z^n.$$

Thus, every function $F \in \mathcal{H}(b)$ is of the form $F(z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu z^n$. Since $f = \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu e^{2\pi inx}$ and $F(re^{2\pi ix}) := \sum_{n=0}^{\infty} \langle f, g_n \rangle_\mu re^{2\pi inx}$, Abel summability shows that $\lim_{r\to 1^-} \left\| F(re^{2\pi ix}) - f(x) \right\|_\mu = 0$, and so f is an $L^2(\mu)$ boundary function of F.

An Interpolation Problem

Lemma

Suppose μ is a singular Borel probability measure on $\mathbb{T}$, b is the inner function on $\mathbb{D}$ associated to μ via the Herglotz representation, and suppose $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$. The following conditions are equivalent:

(i) *there exists a function $f \in L^2(\mu)$ with the property that*

$$a_n = \int_{\mathbb{T}} f(x) e^{-2\pi i n x} \, d\mu(x); \tag{1}$$

(ii) *the following inclusion holds:*

$$G_a(z) := \frac{\sum_{n=0}^{\infty} a_n z^n}{\mu_+(z)} \in \mathcal{H}(b).$$

The Paley-Wiener Theorem for μ, redux

For an entire function F of exponential type, we use h_F to denote the Phragmén-Lindelöf indicator function.

Theorem (W. 2017)

Suppose μ is a singular Borel probability measure with support in $[\alpha, \beta] \subset [-1/2, 1/2]$ where $\beta - \alpha < 1$. Let b be the inner function associated to μ via the Herglotz Representation. The entire function F is the Fourier transform $\hat{f}$ for some $f \in L^2(\mu)$ if and only if

(i) *F is of exponential type;*

(ii) *the indicator function of F satisfies $h_F(\frac{\pi}{2}) \leq 2\pi\beta$ and $h_F(-\frac{\pi}{2}) \leq -2\pi\alpha$;*

(iii) *the following inclusion holds:*

$$G_F(z) := \frac{\sum_{n=0}^{\infty} F(n) z^n}{\mu_+(z)} \in \mathcal{H}(b)$$

i.e. the function G_F is in the kernel of the Toeplitz operator $T_{\overline{b}}$.

Proof

1. $G_F \in \mathcal{H}(b)$ implies that $F(n)$ can be interpolated by $\hat{f}$; i.e. there exists a $f \in L^2(\mu)$ such that $\hat{f}(n) = F(n)$;
2. the support of μ implies
$$h_{\hat{f}}(\frac{\pi}{2}) \leq 2\pi\beta \text{ and } h_{\hat{f}}(-\frac{\pi}{2}) \leq -2\pi\alpha;$$
3. Carlson's theorem applies: If g is exponential type on the RHP, $g(n) = 0$ for $n \in \mathbb{N}$, and $h_g(\frac{\pi}{2}) + h_g(-\frac{\pi}{2}) < 2\pi$, then $g \equiv 0$.
4. applying Carlson's theorem to $F - \hat{f}$, we obtain that $F(z) = \hat{f}(z)$ for all z.

The Paley-Wiener Theorem for μ, yet again

Theorem (W. 2017)

Suppose μ is a singular Borel probability measure on $(-1/2, 1/2)$, and let b be the inner function associated to μ by the Herglotz Representation. The entire function F is the Fourier transform $\hat{f}$ for some $f \in L^2(\mu)$ if and only if

(i) $|F(z)| \leq \varepsilon(|z|)e^{\pi|z|}$ *with* $\varepsilon(r) = o(1)$;

(ii) *the following inclusions hold:*

$$G_+(z) := \frac{\sum_{n=0}^{\infty} F(n)z^n}{\mu_+(z)} \in \mathcal{H}(b), \qquad G_-(z) := \frac{\sum_{n=0}^{\infty} \overline{F(-n)}z^n}{\mu_+(z)} \in \mathcal{H}(b);$$

(iii) *the $L^2(\mu)$-boundaries of G_+ and G_- satisfy the relationship*

$$\overline{G_+^\star} = G_-^\star.$$

Outline of Proof

First version:

1. Condition (iii) says that $\{F(n)\}$ can be interpolated by some $\hat{f}$, so $F(n) = \hat{f}(n)$ for $n \in \mathbb{N}_0$;
2. Condition (i) and (ii) says that $F(z) = \hat{f}(z)$ for all z by Carlson's theorem.

Second version is similar, but Carlson's theorem does not apply. We use a generalization of Carlson's theorem from Boas' book.

A No-Go Result

Denote: $PW(\mu) = \{\hat{f} | f \in L^2(\mu)\}$, $\mathcal{E}_\tau$ the collection of all entire functions of exponential type at most τ.

Theorem

Suppose $PW(\mu) = \mathcal{E}_\tau \cap L^2(w)$ for some $\tau \in (0, \pi]$ and some weight or measure w on $\mathbb{R}$ with $\|f\|_\mu \simeq \|\hat{f}\|_w$. Then there exists a Riesz basis of the form

$$\{\omega_n e^{2\pi i \lambda_n x}\}_{n \in \mathbb{Z}} \subset L^2(\mu) \tag{2}$$

for some sequence $\{\lambda_n\} \subset \mathbb{R}$ and $\omega_n > 0$.

Proof

1. Under the hypotheses, $PW(\mu)$ is a de Branges space;
2. de Branges spaces contain orthogonal bases of kernel functions (on real axis);
3. these kernel functions correspond to weighted exponentials in $L^2(\mu)$;
4. very few measures possess Riesz bases of exponentials.

The End

Thank you!

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