Further analysis of the Cartan abelian core

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joint work with Jonathan Brown, Gabriel Nagy, Carla Farsi, Elizabeth Gillaspy, Aidan Sims, and Dana Williams

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Plan

I. Introduction
II. Graph and $k$-graph algebras
III. Uniqueness theorems
IV. Cartan subalgebras
Let $G$ be a graph, $k$-graph, or groupoid and $C^*(G)$ the universal C*-algebra defined from it.
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**Uniqueness Theorems:** Under what conditions is a $*$-homomorphism $\phi : C^*(G) \to B(H)$ injective?

Theorem (Brown-Nagy-R-Sims-Williams) There is a canonical subalgebra $M \subseteq C^*(G)$ from which injectivity lifts.

- $M$ captures the forced periodicity in $G$.
- $M$ is in certain cases a Cartan subalgebra.

**Goal** Analyze the pair $(C^*(G), M)$ in the context of Renault’s theory of Cartan inclusions.

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Graph Algebras

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C∗-algebras defined from directed graphs

Let $E=(E_0,E_1,r,s)$ be a directed graph with vertex set $E_0$, edge set $E_1$, and range and source maps $r,s:E_1 \to E_0$.

A Cuntz-Krieger system associates the pieces of $E$ to operators on a Hilbert space $H$, as follows:

- $E_0 \ni v \mapsto T_v$ mutually orthogonal projections
- $E_1 \ni e \mapsto T_e$ partial isometries $T_e^*T_e:H \to T_eT_e^*$ satisfying the Cuntz-Krieger relations

\begin{align*}
CK_1 & \quad T_{s(e)}T_e^*T_e = T_v, \quad \text{where } v = s(e)
CK_2 & \quad \sum r(e) = w T_eT_e^* = T_w \quad (\text{assuming } 0 < |r(w) - 1| < \infty)
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More generally, for $k \in \mathbb{N}^+$, a $k$-graph is a graded category $\Lambda = (\Lambda^n, n \in \mathbb{N}^k)$ with degree map $d : \Lambda \to \mathbb{N}^k$, $\Lambda^n := d^{-1}(n)$, satisfying the *Unique Factorization Property*:

If $\lambda \in \Lambda^{m+n}$ then there are unique $\mu \in \Lambda^m, \nu \in \Lambda^n$ s.t. $\lambda = \mu\nu$.
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Q: When is the natural map $\pi : C^*(\Lambda) \to C^*(T_\lambda)$ injective?
Examples of Uniqueness Theorems

Coburn's Theorem ('67):

\[ C^*\left(\mathcal{T}_\lambda\right) \sim = \mathcal{C} \]

Cuntz ('77):

\[ \left(\text{n loops}\right) C^*\left(\mathcal{T}_\lambda\right) \sim = \mathcal{O}_n \]

Cuntz-Krieger ('80): Cuntz-Krieger algebras \( \mathcal{O}_{\Lambda} \).

Example where uniqueness fails:

\[ \mathcal{C}^*\left(\mathcal{T}_\lambda\right) = \mathcal{M}_3\left(\mathcal{C}\right) \not\sim \mathcal{C}\left(\mathcal{T}, \mathcal{M}_3\left(\mathcal{C}\right)\right) = \mathcal{C}^*\left(\mathcal{E}\right). \]

The graph has a cycle without entry.

\[ v_1 v_2 E v_3 e_3, 1 e_2, 3 e_1, 2 \]
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\[ e \quad f \]

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\[ \begin{tikzpicture}
  \node (e) at (0,0) {$e$};
  \node (f) at (1,0) {$f$};
  \draw[->] (e) -- (f);
\end{tikzpicture} \]

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Uniqueness theorems for other combinatorial algebras:
Inverse semigroups (LaLonde, Milan), Steinberg algebras (Clark-Exel-Pardo), ultragraph algebras (Gonçalves, Li, Royer).

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To study nonabelian operator algebras, examine nice abelian subalgebras, such as Cartan subalgebras. A brief history:

1971 Vershik: notion of Cartan sub-von Neumann algebra.
1977 Feldman-Moore: Cartan von Neumann pairs arise from measured countable equivalence relations.
1980 Renault's definition of Cartan $C^*$-subalgebra. Corresponds to a nice groupoid with a twist.
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Given a Cartan subalgebra $B \subseteq A$, there exists an étale, 2nd countable, locally compact Hausdorff, topologically principal groupoid $\mathcal{G}$ and a twist $\Sigma$ s.t. $(C^*_r(\mathcal{G}, \Sigma), C_0(\mathcal{G}^{(0)})) \cong (A, B)$. 
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The *path groupoid* of a directed graph $E$ is defined from the infinite path space $E^\infty$ (paths with range, no source).

$$G_E = \{ (\alpha y, m, \beta y) \mid y \in E^\infty, \alpha, \beta \in E^*, m = d(\alpha) - d(\beta) \}$$

$$(x, m, y)(y, m', z) = (x, m + m', z) \quad (x, m, y)^{-1} = (y, -m, x)$$

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The **path groupoid** of a directed graph $E$ is defined from the infinite path space $E^\infty$ (paths with range, no source).

$$G_E = \{(\alpha y, m, \beta y) \mid y \in E^\infty, \alpha, \beta \in E^*, m = d(\alpha) - d(\beta)\}$$

$$(x, m, y)(y, m', z) = (x, m + m', z) \quad (x, m, y)^{-1} = (y, -m, x)$$

Isotropy subgroupoid: $\text{Iso}(G_E) = \{(\alpha y, m, \beta y) \in G_E \mid \alpha y = \beta y\}$
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Unit space:

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G^{(0)}_E = \{ (x, 0, x) \mid x \in E^\infty \}.
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G_E^{(0)} = \{(x, 0, x) \mid x \in E^\infty\}.
$$

**Basis for topology:**

**cylinder sets** $Z(\alpha, \beta) = \{(\alpha y, d(\alpha) - d(\beta), \beta y) \in G_E\}$. 

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$C^*(\mathcal{G})$: For a topological groupoid $\mathcal{G}$, $C^*(\mathcal{G})$ is defined to be a completion of $C_c(\mathcal{G})$, and $C^*_r(\mathcal{G})$ is the image of $C^*(\mathcal{G})$ under the direct sum of the left regular representations.
$C^*(\mathcal{G})$: For a topological groupoid $\mathcal{G}$, $C^*(\mathcal{G})$ is defined to be a completion of $C_c(\mathcal{G})$, and $C^r_*(\mathcal{G})$ is the image of $C^*(\mathcal{G})$ under the direct sum of the left regular representations.

$k$-graph case:

$C^r_*(\mathcal{G}_\Lambda) = C^*(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$ with $C^*(\text{Iso}(\mathcal{G}_\Lambda)^\circ) \cong \mathcal{M}$.
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Q: What is the Weyl groupoid $\mathcal{G}$ of $(C^*(E), \mathcal{M})$?
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- When Condition (L) holds, $\mathcal{G}$ equals the path groupoid $\mathcal{G}_E$. 
\( \mathcal{C}^*(\mathcal{G}) \): For a topological groupoid \( \mathcal{G} \), \( \mathcal{C}^*(\mathcal{G}) \) is defined to be a completion of \( \mathcal{C}_c(\mathcal{G}) \), and \( \mathcal{C}^r(\mathcal{G}) \) is the image of \( \mathcal{C}^*(\mathcal{G}) \) under the direct sum of the left regular representations.

**\( k \)-graph case:**
\[ \mathcal{C}^r(\mathcal{G}_\Lambda) = \mathcal{C}^*(\mathcal{G}_\Lambda) \cong \mathcal{C}^*(\Lambda) \text{ with } \mathcal{C}^*(\text{Iso}(\mathcal{G}_\Lambda) \circ) \cong \mathcal{M}. \]

**Q:** What is the Weyl groupoid \( \mathcal{G} \) of \( (\mathcal{C}^*(E), \mathcal{M}) \)?

- When Condition (L) holds, \( \mathcal{G} \) equals the path groupoid \( \mathcal{G}_E \).
- When (L) fails, the Weyl groupoid *cannot* be the path groupoid \( \mathcal{G}_E \), which fails to be topologically principal:
**$C^\ast(G)$**: For a topological groupoid $G$, $C^\ast(G)$ is defined to be a completion of $C_c(G)$, and $C^r_\ast(G)$ is the image of $C^\ast(G)$ under the direct sum of the left regular representations.

**$k$-graph case:**

$C^r_\ast(G_\Lambda) = C^\ast(G_\Lambda) \cong C^\ast(\Lambda)$ with $C^\ast(\text{Iso}(G_\Lambda)^\circ) \cong \mathcal{M}$.

**Q**: What is the Weyl groupoid $G$ of $(C^\ast(E), \mathcal{M})$?

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  Topological principality requires $(\alpha y, m, \beta y) \in \text{Iso}(G_E) \Rightarrow m = 0$
\[C^*(\mathcal{G})\]: For a topological groupoid \(\mathcal{G}\), \(C^*(\mathcal{G})\) is defined to be a completion of \(C_c(\mathcal{G})\), and \(C^*_r(\mathcal{G})\) is the image of \(C^*(\mathcal{G})\) under the direct sum of the left regular representations.

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- When Condition (L) holds, \(\mathcal{G}\) equals the path groupoid \(\mathcal{G}_E\).
- When (L) fails, the Weyl groupoid cannot be the path groupoid \(\mathcal{G}_E\), which fails to be topologically principal:

Topological principality requires \((\alpha y, m, \beta y) \in \text{Iso}(\mathcal{G}_E) \Rightarrow m = 0\)
which fails here because \((\alpha \lambda \lambda^\infty, 1, \alpha \lambda^\infty) \in \text{Iso}(\mathcal{G}_E)\).
Theorem (Farsi-Gillaspy-R-Sims, 2017) [Description of the Weyl groupoid for the pair \((C^*(E), \mathcal{M})\) for \(E\) a directed graph.]
Theorem (Farsi-Gillaspy-R-Sims, 2017) [Description of the Weyl groupoid for the pair $$(C^*(E), \mathcal{M})$$ for $E$ a directed graph.]

Idea of proof: make all elements in the isotropy subgroupoid into units by removing evidence that they are not and distinguishing them with distinct indices from $\mathbb{T}$. 

Further analysis of the Cartan abelian core
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The general case is in progress. Considerations:
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Theorem (Farsi-Gillaspy-R-Sims, 2017) [Description of the Weyl groupoid for the pair \((C^*(E), \mathcal{M})\) for \(E\) a directed graph.]

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Theorem (Brown-Nagy-R-Sims-Williams, 2016) The cycline subalgebra \(C^*(\text{Iso}(\mathcal{G})^\circ)\) of a groupoid algebra is Cartan iff \(\text{Iso}(\mathcal{G})^\circ\) closed and abelian.

Brown, Li, Yang: Concrete necessary and sufficient conditions on a \(k\)-graph for \(\text{Iso}(\mathcal{G})^\circ\) to be closed. It’s not always!
Think of elements of degree $\varepsilon_i$ as edges of color $i$. A morphism of degree $\varepsilon_i + \varepsilon_j = \varepsilon_j + \varepsilon_i$ has factorizations $\alpha\beta = \beta'\alpha'$, where $d(\alpha') = d(\alpha) = \varepsilon_i$ and $d(\beta) = d(\beta') = \varepsilon_j$. These "commuting squares" determine all factorization rules of the $k$-graph. Example:

\[
\begin{array}{cccc}
\beta & b & g & b \\
\beta & b & h & b \\
\end{array}
\]

Commutation rules

\[
\begin{array}{cccc}
e & b & \beta & r \\
e & r & \beta & b \\
\beta & b & g & r \\
\beta & b & h & r \\
\end{array}
\]

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Further analysis of the Cartan abelian core
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```
Commutation rules
```

- $e_b \beta_r = e_r \beta_b$
- $\beta_b g_r = \beta_r g_b$
- $\beta_b h_r = \beta_r h_b$
- $g_b g_r = g_r g_b$
- $g_b h_r = h_r g_b$
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These “commuting squares” determine all factorization rules of the $k$-graph. Example:

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Example of 2-graph $\Lambda$ with $(\text{Iso}(G_\Lambda))^\circ$ not closed:
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Commutation rules:

\[
\begin{align*}
e_b \alpha_r &= e_r \alpha_b & e_b \beta_r &= e_r \beta_b \\
\beta_b g_r &= \beta_r g_b & \beta_b h_r &= \beta_r h_b, \\
g_b g_r &= g_r g_b & g_b h_r &= h_r g_b, \\
h_b g_r &= g_r h_b & h_b h_r &= h_r h_b, \\
\alpha_b f_r &= \alpha_r f_b & f_b f_r &= f_r f_b
\end{align*}
\]
Example of 2-graph $\Lambda$ with $(\text{Iso}(G_\Lambda))^\circ$ not closed:

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g_b g_r &= g_r g_b & g_b h_r &= h_r g_b, \\
h_b g_r &= g_r h_b & h_b h_r &= h_r h_b, \\
\alpha_b f_r &= \alpha_r f_b & f_b f_r &= f_r f_b
\end{align*}
\]

Here $(e_r(e_b e_r)^\infty, (1, -1), e_b(e_b e_r)^\infty) \in \overline{\text{Iso}(G_\Lambda)^\circ} \setminus \text{Iso}(G)^\circ$.
Thank you!
A groupoid twist is a groupoid extension

$$\mathbb{T} \times \mathcal{G}^{(0)} \leftarrow \Sigma \rightarrow \mathcal{G}$$

defined via a 2-cocycle $\omega : \mathcal{G}^{(2)} \rightarrow \mathbb{T}$ on the set of composable pairs in $\mathcal{G}$, with product topology and $r(z, \gamma) = (1, r(\gamma))$, $s(z, \gamma) = (1, s(\gamma))$, and

$$(s, \eta)(t, \gamma) = (st\omega(\eta, \gamma), \eta\gamma) \quad (z, \eta)^{-1} = (z^{-1}\omega(\eta, \eta^{-1}), \gamma^{-1})$$

For $f, g \in C_c(\mathcal{G}, \Sigma)$, define

$$f \ast g(\gamma) = \int_{\mathcal{G}} f(\eta)g(\eta^{-1}\gamma)\omega(\eta, \eta^{-1}\gamma)d\lambda^{r(\gamma)}(\eta)$$

and

$$f^*(\gamma) = \frac{f(\gamma^{-1})\omega(\gamma, \gamma^{-1})}{\int_{\mathcal{G}} f(\eta^{-1})\omega(\eta, \eta^{-1})d\lambda^{r(\gamma)}(\eta)}$$

Again, $C^*(\mathcal{G}, \Sigma)$ is the completion of $C_c(\mathcal{G}, \Sigma)$ in the usual norm.
To prove $C^*_r(G) \cong C^*(G_E)$:
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1. Identify Cuntz-Krieger $E$-system in $C^*_r(G)$.

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To prove $C_r^*(\mathcal{G}) \cong C^*(\mathcal{G}_E)$:

1. Identify Cuntz-Krieger $E$-system in $C_r^*(\mathcal{G})$.

To each $e \in E^1 \cup E^0$, let

$$t_e = \begin{cases} 
\chi_ZG(e,s(e)) & \text{if } e \notin E^1_0 \\
\sum_{z \in \mathbb{T}} z\chi_ZG_E(e,s(e)) \times \{z\} & \text{if } e \in E^1_0 
\end{cases}$$

where $E^1_0 \subseteq E^1$ consists of exactly one edge from each cycle without entry.
To prove $C^*_r(G) \cong C^*(G_E)$:

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\chi Z_G(e,s(e)) & \text{if } e \notin E^1_o \\
\sum_{z \in \mathbb{T}} z \chi Z_{G_E}(e,s(e)) \times \{z\} & \text{if } e \in E^1_o
\end{cases}$$

where $E^1_o \subseteq E^1$ consists of exactly one edge from each cycle without entry.

2. Prove that $\pi : C^*(E) \to C^*(t_\alpha)$ is injective – for this we use GIUT.

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To prove $C_r^*(G) \cong C^*(G_E)$:

1. **Identify Cuntz-Krieger $E$-system in $C_r^*(G)$**.

To each $e \in E^1 \cup E^0$, let

$$t_e = \begin{cases} 
\chi_{Z_G}(e,s(e)) & \text{if } e \notin E_1^0 \\
\sum_{z \in T} z \chi_{Z_{G_E}}(e,s(e)) \times \{z\} & \text{if } e \in E_1^0
\end{cases}$$

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**Step 1.** $C_0(G^{(0)}) \subseteq C^*(\{t_\alpha\})$ (use Stone-Weierstrass on the $G_U$ part and then a compactness argument)
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Step 1. $C_0(G^{(0)}) \subseteq C^*(\{t_\alpha\})$ (use Stone-Weierstrass on the $G_U$ part and then a compactness argument)

Step 2. $C_c(G) \subseteq C^*(\{t_\alpha\})$ (use the above and the fact that the $C^*$-algebra is closed under convolution).
Goal: \((C^*_r(G), C_0(G^{(0)})) \cong (C^*(G_E), C^*((\text{Iso}(G_E))^\circ))\)
Goal: \( (C^r_r(G), C^*_0(G^{(0)})) \cong (C^*(G_E), C^*((\text{Iso}(G_E))^\circ)) \)

Last page sketched \( C^*_r(G) \cong C^*(G_E) \).
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To prove \(M := C^*((\text{Iso}(G_E))^\circ) = C_0(G^{(0)})\):
Goal: \((C^*_r(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \cong (C^*(\mathcal{G}_E), C^*((\text{Iso}(\mathcal{G}_E))^\circ))\)

Last page sketched \(C^*_r(\mathcal{G}) \cong C^*(\mathcal{G}_E)\).

To prove \(\mathcal{M} := C^*((\text{Iso}(\mathcal{G}_E))^\circ) = C_0(\mathcal{G}^{(0)})\):

Recall: \(\mathcal{G}^{(0)} = (U \times \mathbb{T}) \cup K\), where

\[
U = \{\alpha \lambda^\infty \mid \alpha \in E^*, \lambda \text{ is a cycle without entry in } E\}
\]

\[
K = E^\infty \setminus U
\]
Goal: \((C^*_r(G), C_0(G^{(0)})) \cong (C^*(GE), C^*((\text{Iso}(GE))^°))\)

Last page sketched \(C^*_r(G) \cong C^*(GE)\).

To prove \(\mathcal{M} := C^*((\text{Iso}(GE))^°) = C_0(G^{(0)})\):

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K = E^\infty \setminus U
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Any irreducible representation of \(C^*((\text{Iso}(G))^°)\) factors through an irrep. of exactly one fiber algebra \(C^*((\text{Iso}(GE))^°)_x, \ x \in E^\infty\).
Goal: \((C^r_r(G), C_0^{(0)}(G)) \cong (C^*(G_E), C^*((\text{Iso}(G_E))^\circ))\)

Last page sketched \(C^r_r(G) \cong C^*(G_E)\).

To prove \(\mathcal{M} := \text{C}^*((\text{Iso}(G_E))^\circ) = C_0^{(0)}(G)\):

Recall: \(G^{(0)} = (U \times T) \cup K\), where

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U = \{ \alpha \lambda^\infty \mid \alpha \in E^*, \lambda \text{ is a cycle without entry in } E \} \\
K = E^\infty \setminus U
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Any irreducible representation of \(\text{C}^*((\text{Iso}(G))^\circ)\) factors through an irrep. of exactly one fiber algebra \(\text{C}^*((\text{Iso}(G_E))^\circ)_x, x \in E^\infty\).

Fibers over \(x \in K\) are singletons; each fiber over an \(x \in U\) is isomorphic to \(\text{C}^*(\mathbb{Z})\). Hence
Goal: \((C^*_r(G), C_0(G^{(0)})) \cong (C^*(G_E), C^*((\text{Iso}(G_E))^\circ))\)

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Recall: \(G^{(0)} = (U \times \mathbb{T}) \cup K\), where

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U = \{ \alpha \lambda^\infty \mid \alpha \in E^*, \lambda \text{ is a cycle without entry in } E \}
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K = E^\infty \setminus U
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Any irreducible representation of \(C^*((\text{Iso}(G))^\circ)\) factors through an irrep. of exactly one fiber algebra \(C^*((\text{Iso}(G_E))^\circ)_x\), \(x \in E^\infty\). Fibers over \(x \in K\) are singletons; each fiber over an \(x \in U\) is isomorphic to \(C^*(\mathbb{Z})\). Hence

\[
C^*((\text{Iso}(G_E))^\circ)^\Lambda = (\text{Iso}(G_E))^\circ = (U \times \mathbb{T}) \cup K.
\]
Abstract Uniqueness Theorem (Brown-Nagy-R)
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Let $A$ be a C*-algebra and $M \subset A$ a C*-subalgebra. Suppose there is a set $S$ of pure states on $M$ satisfying
Abstract Uniqueness Theorem (Brown-Nagy-R)
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Abstract Uniqueness Theorem (Brown-Nagy-R)
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Abstract Uniqueness Theorem (Brown-Nagy-R)
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Then a *-homomorphism $\Phi : A \to B$ is injective iff $\Phi|_M$ is injective.

Our proof of the main theorem applies the AUT to the set $S$ of pure states of $C^*_r(\text{Iso}(G)^\circ)$ that factor through some $C^*_r(G_u^u)$ with $G_u^u = \text{Iso}(G)_u^\circ$ (where $G_u^u = \text{Iso}(G) \cap r^{-1}(u)$).
Brown, Li, Yang

*Cartan Subalgebras of Topological Graph algebras and k-graph $C^*$-algebras*

[arxiv](https://arxiv.org/)

J.H. Brown, G. Nagy, and S. Reznikoff

*A generalized Cuntz-Krieger uniqueness theorem for higher-rank graphs.*

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