Compression, Matrix Range and Completely Positive Map

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Iowa-Nebraska Functional Analysis Seminar
November 5, 2016
\( \mathcal{H}, \mathcal{K} : \) Hilbert space. If \( \dim \mathcal{H} = n < \infty, \mathcal{H} \cong \mathbb{C}^n. \)

\( \mathcal{B}(\mathcal{H}, \mathcal{K}) : \) bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K}. \) If \( \dim \mathcal{H} = m \) and \( \dim \mathcal{K} = n, \mathcal{B}(\mathcal{H}, \mathcal{K}) \cong M_{m,n}. \) \( \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H}) \) and \( M_n = M_{n,n}. \)

\( A \in \mathcal{B}(\mathcal{H}) \) is said to be self-adjoint if \( A = A^*. \) \( \mathcal{B}(\mathcal{H})_{\text{sa}} \) will denote the space of self-adjoint operators in \( \mathcal{B}(\mathcal{H}). \)

Every \( A \in \mathcal{B}(\mathcal{H}) \) has a self-adjoint decomposition \( A = A_1 + iA_2, \)
\( A_1, A_2 \in \mathcal{B}(\mathcal{H})_{\text{sa}}. \)

If \( \dim \mathcal{H} = n, \mathcal{B}(\mathcal{H})_{\text{sa}} = H_n, \) the set of \( n \times n \) Hermitian matrices.

\( S \subseteq \mathbb{R}^n, \mathbb{C}^n \) is said to be convex if for all \( x, y \in S, \) the line segment \( \{ tx + (1 - t)y : 0 \leq t \leq 1 \} \subseteq S. \)
Suppose $A \in B(\mathcal{H})$ and $\mathcal{K}$ is a norm closed subspace of $\mathcal{H}$. Let $P_\mathcal{K} \in B(\mathcal{H}, \mathcal{K})$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Then $B = P_\mathcal{K} A|_\mathcal{K} \in B(\mathcal{K})$ is called a compression of $A$ to $\mathcal{K}$ and $A$ is a dilation of $B$ to $\mathcal{H}$.

Let $A \in M_n$ and $1 \leq m \leq n$. Then $B \in M_m$ is a compression of $A$ if and only if there exists $V \in M_{n,m}$ such that $V^* V = I_m$ and $B = V^* A V$.

For $A \in B(\mathcal{H})$ and $m \geq 1$, let

$$W_m(A) = \{ B \in M_m : B \text{ is a compression of } A \}$$

For $m = 1$, $W_1(A) = \{ \langle Ax, x \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1 \}$ is the numerical range of $A$, usually denoted by $W(A)$.

By the Toeplitz- Hausdorff Theorem, $W(A)$ is convex. For $m > 1$, $W_m(A)$ is usually not convex.
Theorem 1 (Sz.-Nagy and Foias)

$A \in B(\mathcal{K})$ is a contraction ($\|A\| \leq 1$) if and only if $A$ has a unitary dilation $U \in B(\mathcal{H})$ such that

$$A^k = P_{\mathcal{K}} U^k|_{\mathcal{K}} \text{ for all } k \geq 1.$$ 

Given $A \in B(\mathcal{H})$, the **numerical radius** of $A$ is given by

$$w(A) = \sup\{ |z| : z \in \mathcal{W}(A) \}.$$ 

$w(A)$ is a norm on $B(\mathcal{H})$ and satisfies $w(A) \leq \|A\| \leq 2w(A)$.

Theorem 2 (Sz.-Nagy and Foias)

$A \in B(\mathcal{K})$ satisfies $w(A) \leq 1$ if and only if there is a unitary $U \in B(\mathcal{H})$ such that

$$A^k = 2 \ P_{\mathcal{K}} U^k|_{\mathcal{K}} \text{ for all } k \geq 1.$$
Compression of linear operators

Theorem 3 (Ando, Arveson)

$A \in B(\mathcal{K})$ satisfies $w(A) \leq 1$ if and only if $A$ is a compression of

$$
\begin{pmatrix}
0 & 2I_{\mathcal{H}} \\
0 & 0
\end{pmatrix}
$$

for some $\mathcal{H}$.

Note: $W\left(\begin{pmatrix}
0 & 2I_{\mathcal{H}} \\
0 & 0
\end{pmatrix}\right) = W\left(\begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}\right) = \{z \in \mathbb{C} : |z| \leq 1\}$

$$
w(A) \leq 1 \iff W(A) \subseteq W\left(\begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}\right)
$$

Let $A = A_1 + iA_2$ be the self-adjoint decomposition of $A \in B(\mathcal{K})$. Then

$$
W(A) \cong W(A_1, A_2) = \{ (\langle A_1 x, x \rangle, \langle A_2 x, x \rangle) : x \in \mathcal{K}, \langle x, x \rangle = 1 \} \subset \mathbb{R}^2
$$

Given $A_1, \cdots, A_p \in B(\mathcal{K})_{sa}$, define the joint numerical range

$$
W(A_1, \cdots, A_p) = \{ (\langle A_1 x, x \rangle, \cdots, \langle A_p x, x \rangle) : x \in \mathcal{K}, \langle x, x \rangle = 1 \} \subset \mathbb{R}^p
$$
A ∈ \mathcal{B}(\mathcal{H}) is said to be positive \((A \geq 0)\) if \(\langle Ax, x \rangle \geq 0\) for all \(x \in \mathcal{H}\).

An operator system \(S\) of a \(C^*\)-algebra \(A\), is a norm-closed self-adjoint \((S = S^*)\) subspace \(S\) of \(A\) containing \(1_A\).

A linear map \(\Phi : S \to B\) is positive on \(S\) if \(A \geq 0 \Rightarrow \Phi(A) \geq 0\)

\(\Phi_k : M_k(S) \to M_k(B), \ \Phi_k((A_{ij})) = (\Phi(A_{ij}))\)

\(\Phi\) is \(k\)-positive if \(\Phi_k\) is positive.

\(\Phi\) is completely positive if \(\Phi\) is \(k\)-positive for all \(k \geq 1\).

Theorem 4 (Arveson’s Extension Theorem)

Let \(A\) be a unital \(C^*\)-algebra and \(S\) be an operator system of \(A\). Then every completely positive map from \(S\) to a \(C^*\)-algebra \(B\) can be extended to a completely positive map from \(A\) to \(B\).
Suppose $W(A_1, A_2) \subseteq W(B_1, B_2)$.

If $c_0 l + c_1 B_1 + c_2 B_2 \geq 0$, then we have

$$\langle (c_0 l + c_1 B_1 + c_2 B_2) x, x \rangle \geq 0 \text{ for all } x \in \mathcal{H} \text{ with } \langle x, x \rangle = 1$$

$$\Rightarrow c_0 + (c_1, c_2) \cdot (b_1, b_2) \geq 0 \text{ for all } (b_1, b_2) \in W(B_1, B_2)$$

$$\Rightarrow c_0 + (c_1, c_2) \cdot (a_1, a_2) \geq 0 \text{ for all } (a_1, a_2) \in W(A_1, A_2)$$

$$\Rightarrow \langle (c_0 l + c_1 A_1 + c_2 A_2) x, x \rangle \geq 0 \text{ for all } x \in \mathcal{K} \text{ with } \langle x, x \rangle = 1$$

Therefore, the map

$$\Phi(c_0 l + c_1 B_1 + c_2 B_2) = (c_0 l + c_1 A_1 + c_2 A_2) \quad (1)$$

is positive.

**Remark:** If $A = A_1 + iA_2$ and $B = B_1 + iB_2$. Then (1) is equivalent to

$$\Phi(c_0 l + c_1 B + c_2 B^*) = (c_0 l + c_1 A + c_2 A^*)$$.
Theorem 5 (Stinespring’s dilation theorem)

Let $\mathcal{A}$ be a unital $C^*$-algebra, and let $\Phi : \mathcal{A} \to B(\mathcal{H})$ be a linear map. Then $\Phi$ is completely positive if and only if there exist a Hilbert space $\mathcal{K}$, a unital $C^*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$, and a bounded operator $V \in B(\mathcal{H}, \mathcal{K})$ such that

$$\Phi(T) = V^* \pi(T) V \quad \text{for all} \ T \in \mathcal{A}. $$

Note: $\Phi$ is unital if and only if $V^* V = I_{\mathcal{H}}$.

Therefore, $A$ is a compression of $B \otimes I_{\mathcal{H}}$ for some $\mathcal{H}$ if and only if $A = \Phi(B)$ for some unital completely positive map $\Phi$. 

Reformulation of Theorem 3

Let $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = B_1 + iB_2$, $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

**Theorem 3a**

Suppose $A_1, A_2 \in \mathcal{B}(\mathcal{H})_{sa}$. Then $W(A_1, A_2) \subseteq W(B_1, B_2)$ if and only if the map

$$\Phi(c_0 I + c_1 B_1 + c_2 B_2) = c_0 I_{\mathcal{H}} + c_1 A_1 + c_2 A_2$$

is completely positive.

**Theorem 3b**

Suppose $A \in \mathcal{B}(\mathcal{H})$. Then the map

$$\Phi(c_0 I + c_1 B + c_2 B^*) = c_0 I_{\mathcal{H}} + c_1 A + c_2 A^*$$

is positive on $\text{span}(I_2, B, B^*)$ if and only if it is completely positive.
Another Proof of Theorem 3b

**Theorem 6 (Choi)**

Let $S_2$ be the space of $2 \times 2$ complex symmetric matrices. Then every positive map $\Phi : S_2 \to B(\mathcal{H})$ is completely positive.

Choi proves the above theorem for finite dimensional $\mathcal{H}$. The infinite dimensional case can be proven from the finite dimensional case.

**Theorem 3c**

Let $B \in M_2$ and $A \in B(\mathcal{H})$. Then the map

$$\Phi(c_0 I_2 + c_1 B + c_2 B^*) = c_0 I_K + c_1 A + c_2 A^*$$

is positive on $\text{span}(I_2, B, B^*)$ if and only if it is completely positive.

**Proof.** Every $B \in M_2$ is unitarily similar to a symmetric matrix $S$. Let $S = \text{span}(I_2, S, S^*)$. If $S = S_2$, the result follows from Theorem 6. If $S \neq S_2$, then $S \cong \mathbb{C}$ or $\mathbb{C}^2$ and the result follows.
Recall \[
\begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix} = B_1 + iB_2 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} + i \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}.
\]

Let \(B_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}\). \(B_1, B_2, B_3\) are known as the \textbf{Pauley matrices}.

\[\text{Conjecture 1 (Extension of Theorem 3a)}\]

Suppose \(A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H})_{sa}\). Then \(W(A_1, A_2, A_3) \subseteq W(B_1, B_2, B_3)\) if and only if the map

\[
\Phi(c_0 I_2 + c_1 B_1 + c_2 B_2 + c_3 B_3) = c_0 I_{\mathcal{H}} + c_1 A_1 + c_2 A_2 + c_3 A_3
\]

is completely positive.

1) The conjecture fails for \(\dim \mathcal{H} = 2\). Just take \(A_i = B_i^t\).

2) If \(\dim \mathcal{H} \neq 2\), \(W(A_1, A_2, A_3)\) is convex but

\[W(B_1, B_2, B_3) = \{ w \in \mathbb{R}^3 : \| w \| = 1 \} \]

If \(W(A_1, A_2, A_3) \subseteq W(B_1, B_2, B_3)\), then \(W(A_1, A_2, A_3)\) is a singleton. Therefore, all \(A_i\) are scalar and the conjecture holds.
Let $B_1, B_2, B_3$ be the Pauley matrices. Set $\hat{B}_i = B_i \oplus B_i^t$ for $i = 1, 2, 3$. Then

$$W(\hat{B}_1, \hat{B}_2, \hat{B}_3) = \{ w \in \mathbb{R}^3 : \|w\| \leq 1 \} \text{ is convex.}$$

Conjecture 2 (Extension of Theorem 3b)

Suppose $A_1, A_2, A_3 \in \mathcal{B}(\mathcal{H})_{sa}$. Let

$$\Phi(c_0 I_2 + c_1 \hat{B}_1 + c_2 \hat{B}_2 + c_3 \hat{B}_3) = c_0 I_K + c_1 A_1 + c_2 A_2 + c_3 A_3$$

Then $\Phi$ is positive on span$(l_2, \hat{B}_1, \hat{B}_2, \hat{B}_3)$ if and only if $\Phi$ is completely positive.

The conjecture holds if dim $\mathcal{H} = n \leq 3$. Let $\Psi : M_2 \to M_n$ be given by $\Psi(X) = \Phi(X \oplus X^t)$. If $\Phi$ is positive and $n \leq 3$, then $\Psi$ is decomposable. There exist completely positive $\psi_1, \psi_2 : M_2 \to M_n$ such that $\psi(X) = \psi_1(X) + \psi_2(X)^t$. Then the result follows.

Question Does the result hold for all $n$?
Theorem 7 (Choi and Li)

Suppose $B \in M_2$ or $B = [b] \oplus B_1 \in M_3$. Then for all $A \in \mathcal{B}(\mathcal{H})$, $W(A) \subseteq W(B)$ if and only if the map

$$\Phi(c_0 I_2 + c_1 B + c_2 B^*) = c_0 I_\mathcal{H} + c_1 A + c_2 A^*$$

is completely positive.

Questions:

1) If $B \in M_3$ satisfies the conclusion in Theorem 7, must $B$ be unitarily similar to $[b] \oplus B_1$?

2) For which subset $S$ of $\mathbb{C}$ can we find $B \in M_n$ such that $W(B) = S$ and satisfies the conclusion in Theorem 7?

3) Does there exist $B \in M_n$ such that $W(B) =$ the square with vertices $\{1, -1, i, -i\}$ and satisfies the conclusion in Theorem 7? $B = \text{diag}(1, -1, i, -i)$ does not work.
Suppose $S$ is an operator system of $M_m$ and $k \geq 1$. For a fixed $\mathcal{H}$, let

$$P_k(S, \mathcal{H}) = \{ \Phi : S \to B(\mathcal{H}) \text{ is } k\text{-positive} \}, \text{ and}$$

$$CP(S, \mathcal{H}) = \{ \Phi : S \to B(\mathcal{H}) \text{ is completely positive} \}.$$ 

Clearly,

$$CP(S, \mathcal{H}) \subset \cdots \subset P_k(S, \mathcal{H}) \subset \cdots \subset P_2(S, \mathcal{H}) \subset P_1(S, \mathcal{H})$$

The previous results shows that we have

$$CP(S, \mathcal{H}) = P_1(S, \mathcal{H})$$

for

1) $S = \text{span}(I, B, B^*)$ with $B \in M_2$ or $B = [b] \oplus B_1 \in M_3$ and any $\mathcal{H}$.

2) $S = \text{span}(I, B_1, B_2, B_3)$ and $\dim \mathcal{H} \leq 3$.

**Question:** When will $CP(S, \mathcal{H}) = P_k(S, \mathcal{H})$?
Let $A \in \mathcal{B}(\mathcal{H})$. For each $m \geq 1$, Arveson defines the matrix range

$$\mathcal{W}_n(A) = \{ \Phi(A) : \Phi \text{ is a unital completely positive map from } \mathcal{B}(\mathcal{H}) \text{ to } M_n \}$$

### Theorem 8 (Arveson)

1) $\mathcal{W}_n(A)$ is C* convex. That is, given $X_1, \ldots, X_k \in \mathcal{W}_n(A)$ and $Z_1, \ldots, Z_k \in M_n$ such that $\sum_{i=1}^{k} Z_i^* Z_i = I_n$, we have

$$\sum_{i=1}^{k} Z_i^* X_i Z_i \in \mathcal{W}_n(A).$$

$\mathcal{W}_n(A)$ is the closure of the smallest C* convex set containing $W_n(A)$.

2) Let $A$ be a normal operator and let $n \geq 1$. Then $\mathcal{W}_n(A)$ is the closure of

$$\left\{ \sum_{i=1}^{r} \lambda_i H_i : r \geq 1, \ H_i \geq 0, \ \lambda_i \in \text{sp}(T) \text{ and } \sum_{i=1}^{r} H_i = I_n \right\}$$

3) For some irreducible operators, the sequence $\{ \mathcal{W}_n(A) \}_{n=1}^{\infty}$ is a complete invariant for unitary similarity.
**Theorem 9** (Choi) Suppose $\Phi : M_n \rightarrow M_m$ is a linear map. Then the following conditions are equivalent:

(a) $\Phi$ is completely positive.

(b) $\Phi$ is $k$-positive for $k = \min(m, n)$.

(c) The Choi matrix $C(\Phi) = (\Phi(E_{ij}))$ is positive semidefinite.

(d) There exist $V_1, \ldots, V_r \in M_{n,m}$ such that

$$\Phi(A) = \sum_{j=1}^{r} V_j^* A V_j. \quad (2)$$

Furthermore, suppose (d) holds. Then we have

(1) The map $\Phi$ is unital ($\Phi(I_n) = I_m$) if and only if $\sum_{j=1}^{r} V_j^* V_j = I_m$.

(2) The map $\Phi$ is trace preserving ($\text{tr} (\Phi(A)) = \text{tr} (A)$) if and only if $\sum_{j=1}^{r} V_j V_j^* = I_n$.

The minimum of $r$ in (2) is called the rank of $\Phi$. 
Joint matrix range

Given \( n, m \geq 1 \), let \( CP(n, m) \) be the set of **unital completely positive maps** from \( M_n \) to \( M_m \). For \( 1 \leq r \leq mn \), let \( CP^r(n, m) \) be the set of \( \Phi \in CP(n, m) \) of rank \( \leq r \). Clearly,

\[
CP^1(n, m) \subset CP^2(n, m) \subset \cdots \subset CP^{mn}(n, m) = CP(n, m)
\]

Let \( A = (A_1, A_2, \ldots, A_p) \in H_n^p \). For each \( m \geq 1 \) and \( 1 \leq r \leq mn \), define

\[
\mathcal{W}^r_m(A) = \{(\Phi(A_1), \ldots, \Phi(A_p)) : \Phi \in CP^r(n, m)\}
\]

We have

\[
\mathcal{W}^1_m(A) \subseteq \mathcal{W}^2_m(A) \subseteq \cdots \subseteq \mathcal{W}^{mn}_m(A) = \mathcal{W}_m(A)
\]

Toeplitz-Haudorff Theorem: \( \mathcal{W}^1_1(A_1, A_2) = \mathcal{W}_1(A_1, A_2) \).

**Question:** When will \( \mathcal{W}^r_m(A) = \mathcal{W}_m(A) \)?

**Note:** \( \mathcal{W}^r_m(A_1, A_2, \ldots, A_p) = \mathcal{W}^1_m(A_1 \otimes I_r, A_2 \otimes I_r, \ldots, A_p \otimes I_r) \).
Joint matrix range

Theorem 10

Suppose $A_1, A_2, \ldots, A_p \in H_n$. Let $1 \leq r \leq mn - 1$. Then

$$\mathcal{W}_m^r(A_1, \ldots, A_p) = \mathcal{W}_m(A_1, \ldots, A_p) \quad (*)$$

if $m^2(p + 1) - 1 < (r + 1)^2 - \delta_{mn,r+1}$.

For example, if $p = k^2 - 1$ and $n > k$, then

$$\mathcal{W}_m^{mk-1}(A_1, \ldots, A_p) = \mathcal{W}_m(A_1, \ldots, A_p)$$

for all $A_1, \ldots, A_p \in H_n$. In this case, one can show that $r = mk - 1$ is the smallest number for $(*)$ to hold. Putting $m = r = 1$, we have

$$\mathcal{W}_1^1(A_1, \ldots, A_p) = \mathcal{W}_1(A_1, \ldots, A_p)$$

if

$$p < 2^2 - \delta_{n,2} = 4 - \delta_{n,2}.$$ 

Therefore, $W(A_1, A_2, A_3)$ is convex if $n \geq 3$. 

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Compression, Matrix Range and Completely Positive Map
Recall that for $\mathbf{A} = (A_1, A_2, \ldots, A_p) \in H_n^p$, 

$$\mathcal{W}_m^1(\mathbf{A}) \subseteq \mathcal{W}_m^2(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{W}_m^{mn}(\mathbf{A}) = \mathcal{W}_m(\mathbf{A})$$

Let $S = \text{span}(I_n, A_1, A_2, \ldots, A_p)$ and $\mathcal{H} = \mathbb{C}^m$. Define 

$$\mathcal{P}_k(\mathbf{A}) = \{(\Phi(A_1), \ldots, \Phi(A_p)) : \Phi \in P_k(S, \mathcal{H})\}$$

we have 

$$\mathcal{W}_m(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{P}_k(\mathbf{A}) \subseteq \cdots \subseteq \mathcal{P}_2(\mathbf{A}) \subseteq \mathcal{P}_1(\mathbf{A})$$

**Note:** $\mathcal{P}_k(\mathbf{A}) = \mathcal{W}_m(\mathbf{A}) \iff P_k(S, M_m) = CP(S, M_m)$

For $n \geq 3$, $p = 3$ we have 

$$\mathcal{W}_1^1(\mathbf{A}) = \mathcal{P}_1(\mathbf{A})$$
Compression of linear operators

Recall that for $A \in B(H)$ and $m \leq \dim H$,

$$W_m(A) = \{ B \in M_m : B \text{ is a compression of } A \}$$

**Theorem 11 (Fan and Pall)**

Suppose $A \in H_n$ has eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$ and $1 \leq m \leq n$. Then $W_m(A)$ consists of all $B \in H_m$ with eigenvalues $b_1 \geq b_2 \geq \cdots \geq b_m$ satisfying the following inequalities:

$$a_i \geq b_i \geq a_{n-m+i} \quad \text{for all } 1 \leq i \leq m$$

In particular, $B \in W_{n-1}(A)$ if and only if

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq b_{n-1} \geq a_n$$

Suppose $A \in M_n$ is normal with eigenvalues $a_1, a_2, \ldots, a_n$. If $a_1, a_2, \ldots, a_n \in \mathbb{C}$ are collinear, then there exist $c \in \mathbb{C}$ and $\theta \in \mathbb{R}$ such that $e^{i\theta}A + cl_n \in H_n$ and we have

$$W_m(e^{i\theta}A + cl_n) = e^{i\theta}W_m(A) + cl_m$$
Theorem 12 (Fan and Pall)

Let $A \in M_n$ and $B \in M_{n-1}$ be normal matrices with eigenvalues $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_{n-1}$, respectively. Suppose $a_1, a_2, \ldots, a_q$ are each distinct from $b_1, b_2, \ldots, b_{q-1}$, while $a_i = b_{i-1}$ for $q + 1 \leq i \leq n$. Then $B$ is a compression of $A$ if and only if $a_1, a_2, \ldots, a_q$ are collinear and every segment on this line limited by two adjacent $a_i$'s contains one $b_j$, $1 \leq j \leq q - 1$.

If no three $a_i$'s are collinear, then up to permutation of indices, we must have $a_i = b_i$ for $i = 1, \ldots, n-2$ and $b_{n-1} \in \overline{a_{n-1} a_n}$.
Suppose $A \in M_n$ is normal with non-collinear eigenvalues $a_1, a_2, \ldots, a_n$. Let $D_m(A) = \{\text{diag}(B) : B \in W_m(A)\} \subset \mathbb{C}^m$.

**Theorem 13**

Suppose $A \in M_n$ is normal with non-collinear eigenvalues $a_1, a_2, \ldots, a_n$. Then the following conditions are equivalent:

1) $D_m(A)$ is convex.

2) $W_m(A)$ is convex.

3) $W_m(A)$ is $C^*$-convex. ($\iff W_{m}^{1}(A) = W_{m}(A) = \mathcal{W}_{m}(A)$)

4) Every vertex of $W(A)$ has multiplicity $\geq m$. 
Given $A \in M_n$, $B \in M_m$ and $1 \leq k \leq n$, $m$. $A$ and $B$ is said to have a **common k-dimensional compression** if there exist $U \in M_{m,k}$ and $V \in M_{m,k}$ such that $U^*U = I_k = V^*V$ and $U^*AU = V^*BV$.

For $m = k \leq n$, this is equivalent to the compression of Hermitian matrices studied by Fan and Pall.

**Extension of the result of Fan and Pall**

**Theorem 14** Suppose and $A \in H_n$ and $B \in H_m$ have eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_m$, respectively, and $1 \leq k \leq n$, $m$. Then $B$ and $C$ have a common $k$-dimensional compression if and only if the following inequalities hold:

$$a_i \geq b_{m-k+i} \quad \text{and} \quad b_i \geq a_{n-k+i} \quad \text{for all} \; 1 \leq i \leq k$$
**Theorem 15** Suppose $A$ and $B$ are $3 \times 3$ normal matrices with non-collinear eigenvalues $a_1, a_2, a_3$ and $b_1, b_2, b_3$ respectively. Then

1) $A$ and $B$ have a common 2-dimensional compression $C$, with degenerate $W(C)$ ($\iff C$ is normal), if and only if either

(a) $W(A)$ and $W(B)$ have a vertex in common and the corresponding opposite sides intersect, or

(b) one vertex $v$ of $W(A)$ lies on an edge $s$ of $W(B)$ and the vertex $v'$ in $W(B)$ opposite to $s$ lies on the edge $s'$ in $W(A)$ opposite to $v$. 
2) $A$ and $B$ have a common 2-dimensional compression $C$, with non-degenerate $W(C)$ ($\iff C$ is not normal), if and only if the following conditions are satisfied:

(a) $W(A) \cap W(B)$ is an $m$-sided polygon $P$ with $m \geq 3$.

(b) Every edge of $W(A)$ and $W(B)$ intersects a side of $P$ at more than one point.

(c) For $m = 6$, the diagonals of $P$ are concurrent.

$m = 3$ \hspace{1cm} $m = 4$ \hspace{1cm} $m = 5$ \hspace{1cm} $m = 6$