Operator Algebras as Algebras of Functions

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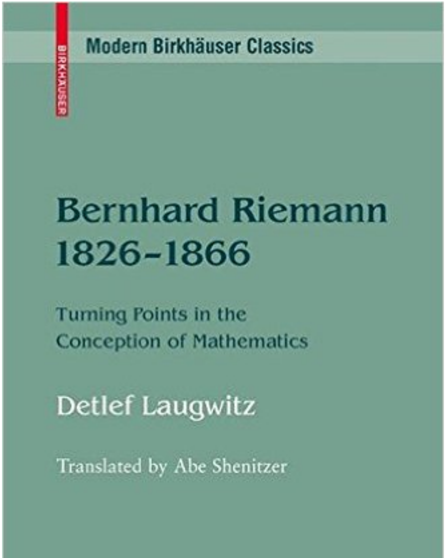
An Old Idea

- R and S two unital rings.
- $X := \text{Hom}(R, S)$
- For $r \in R$, define $\widehat{r} : X \to S$ by

$$\widehat{r}(\varphi) := \varphi(r).$$

- Clearly, $r \to \widehat{r}$ is a homomorphism (i.e., a representation) of R into S-valued functions on X.
- What information do these functions carry?

Detlef Laugwitz [Laugwitz, 2008]



Dedekind and Weber 1882

Theorie der algebraischen Functionen einer Veäderlichen

I. Abtheilung.

§ 1.

Körper algebraischer Functionen.

Eine Variable θ heisst eine *algebraische Function* einer unabhängigen Veränderlichen z, wenn dieselbe einer irreductibeln algebraischen Gleichung

$$(1.)\qquad F(\theta, z) = 0$$

genügt. F bedeutet hierin einen Ausdruck von der Form

$$F(\theta, z) = a_0\theta^n + a_1\theta^{n-1} + \cdots + a_{n-1}\theta + a_n,$$

worin die Coefficienten a_0, a_1, ... a_n ganze rationale Functionen von z ohne gemeinschaftlichen Theiler sind. Die vorausgesetzte Irreductibilität der Glei-

Part I.

§1.
Algebraic function fields.

A variable θ is called an *algebraic function* of an independent variable z if an irreducible algebraic equation

$$(1.) \qquad F(\theta, z) = 0$$

is satisfied. F here means an expression of the form

$$F(\theta, z) = a_0\theta^n + a_1\theta^{n-1} + \ldots + a_{n-1}\theta + a_n,$$

Dedekind and Weber and Algebraic Functions (bis)

- $F(\theta, z) = a_0(z)\theta^n + a_1(z)\theta^{n-1} + \ldots + a_{n-1}(z)\theta + a_n(z)$
- Get n roots $\{\theta_1(z), \theta_2(z), \ldots, \theta_n(z)\}$
- Riemann wanted to picture how these roots behaved as functions of z.
- $\theta_j(z) = \frac{1}{2\pi i}\int_{\gamma_j} \frac{\theta \frac{\partial F}{\partial \theta}(\theta,z)}{F(\theta,z)}\, d\theta$, where γ_j is a curve surrounding a z_j, where $F(\theta, z_j) = 0$ has n distinct roots.

The square root of z

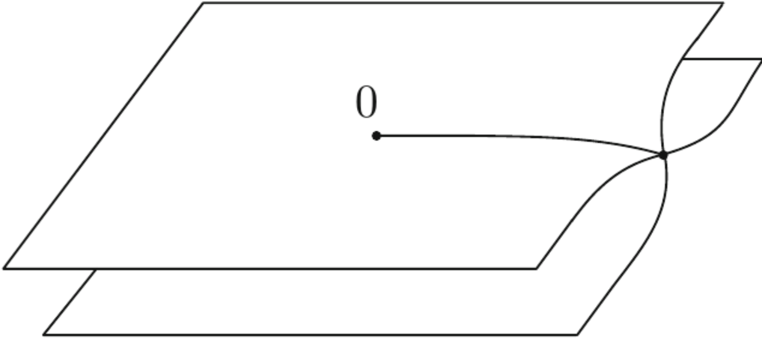


Figure: The Riemann Surface for $\theta^2 - z = 0$

Die Punkte der Riemannschen Fläche

II. Abtheilung.

§ 14.

Die Punkte der *Riemann*schen Fläche.

Die bisherigen Betrachtungen über die Functionen des Körpers Ω waren rein formaler Natur. Alle Resultate waren rationale, d. h. nach den Regeln der Buchstabenrechnung mittelst der vier Species abgeleitete Folgerungen aus der zwischen zwei Functionen in Ω bestehenden irreductibeln Gleichung. Die numerischen Werthe dieser Functionen kamen nirgends in Betracht. Man würde sogar, ohne andere Principien anzuwenden, die formelle Behandlung noch wesentlich weiter treiben können, indem man

The points of a Riemann surface.

Part II.

§14.
The points of the Riemann surface.

The previous observations about the functions in the field Ω were of a purely formal nature. All results were rational, that is, in accordance with the rules of arithmetic with conclusions deduced by means of the four basic operations from the irreducible equation that exists between two functions in Ω. The numerical values of these functions were not considered anywhere. Even without employing other

Definition eines Punktes!

1. ***Definition.*** Wenn ***alle*** Individuen $\alpha, \beta, \gamma, \ldots$ des Körpers Ω durch ***bestimmte*** Zahlwerthe $\alpha_0, \beta_0, \gamma_0, \ldots$ so ersetzt werden, dass

$$\text{(I.)} \quad \alpha_0 = \alpha, \quad \text{falls } \alpha \text{ constant ist, und allgemein}$$

$$\text{(II.)} \quad (\alpha+\beta)_0 = \alpha_0+\beta_0, \qquad \text{(IV.)} \quad (\alpha\beta)_0 = \alpha_0\beta_0,$$

$$\text{(III.)} \quad (\alpha-\beta)_0 = \alpha_0-\beta_0, \qquad \text{(V.)} \quad \left(\frac{\alpha}{\beta}\right)_0 = \frac{\alpha_0}{\beta_0}$$

wird, so soll einem solchen Zusammentreffen bestimmter Werthe ein ***Punkt*** $\mathfrak{P}$ zugeordnet werden (den man sich zur Versinnlichung irgend wie im Raume gelegen vorstellen mag**)), und wir sagen, ***in*** $\mathfrak{P}$ sei $\alpha = \alpha_0$, oder α habe ***in*** $\mathfrak{P}$ den Werth α_0. Zwei Punkte heissen stets und nur dann ver-

Definition of a point!

1. *Definition.* If *all* the individual elements $\alpha, \beta, \gamma, \ldots$ of the field Ω are replaced by *certain* numerical values, $\alpha_0, \beta_0, \gamma_0 \ldots$, such that

(I.) $\alpha_0 = \alpha$, in the case where α is a constant, and in general

$$\text{(II.)}\quad (\alpha+\beta)_0 = \alpha_0 + \beta_0, \qquad \text{(IV.)}\quad (\alpha\beta)_0 = \alpha_0\beta_0,$$

$$\text{(III.)}\quad (\alpha-\beta)_0 = \alpha_0 - \beta_0, \qquad \text{(V.)}\quad \left(\frac{\alpha}{\beta}\right)_0 = \frac{\alpha_0}{\beta_0}$$

then a definite set of values will thus be assigned to a *point* $\mathfrak{P}$ (which one may consider for visualisation as somehow located in space [1]), and we say $\alpha = \alpha_0$ *at* $\mathfrak{P}$ or α has the value α_0 *at* $\mathfrak{P}$. Two points are called distinct if and only if there is a function α in Ω which has different values at the two points.

- Points are homomorphisms!

Dedekind and Weber Recap

- Let $F(\theta, z) = a_0(z)\theta^n + a_1(z)\theta^{n-1} + \ldots + a_{n-1}(z)\theta + a_n(z)$, where $a_k \in \mathbb{C}(S^1)$. Then
- $R = \mathbb{C}(S^1)[F]$
- $S = \mathbb{C} \cup \infty$
- $X := \text{Hom}(R, S)$ is the Riemann surface of F.

The First Hit

INDEPENDENT INTEGRAL BASES AND A CHARACTERIZATION OF REGULAR SURFACES

BY

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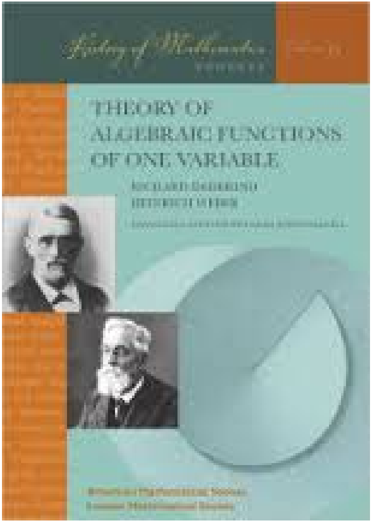
Introduction. The fundamental paper *Theorie der algebraischen Funktionen einer Veränderlichen* of Dedekind and Weber opened a wide field of research. The methods which these authors brought into play in the study of algebraic functions of one variable have lent themselves readily, from the conceptual point of view at least, to generalization to the case of several variables. However, even if one restricts himself to the case of algebraic functions of two variables he finds a sharp line of demarcation in the analogy with the one variable case when he attacks problems of enumeration such as the problem of Riemann-Roch.

John Stillwell [Dedekind & Weber, 2012]



Note

Nowadays, we distinguish between polynomials and polynomial functions. A polynomial of k variables is an element of the free abelian semigroup algebra $\mathbb{C}^{\mathbb{N}^k}$: $F = \sum_{n\in\mathbb{N}^k} a_n X_1^{n_1} \cdots X_k^{n_k}$, where the "indeterminate" X_i is really the characteristic function of $(0, \cdots, 1, \cdots, 0)$. So $R = \mathbb{C}^{\mathbb{N}^k}$, $S = \mathbb{C}$, and $\mathrm{Hom}(R, S) = \mathbb{C}^k$. For $F \in R$,

$$\widehat{F}(z_1, \cdots, z_k) = \sum_{n\in\mathbb{N}^k} a_n z_1^{n_1} \cdots z_k^{n_k}$$

.

Back to Topic and a Halmos Doctrine

- If you want to study a problem about operators on an infinite dimensional Hilbert space, study it first in the finite dimensional setting and answer it there. Then move on to the infinite dimensional setting.

Inspiration From Rings of Operators I

1. The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ((18), chiefly parts I and II). Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalise the theory of unitary group-representations essentially beyond their classical frame have always been blocked by the unsolved questions connected with these problems. Third, various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to a class of abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated.

1997: A Serendipitous Coincidence

I read

- "On the structure of algebras with nonzero radical" by G. Hochschild[Hochschild, 1947]
- "A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$" by M. Pimsner.[Pimsner, 1997]

These led to a program devoted to formulating the theory of finite dimensional algebras in the setting of operator algebras on Hilbert Space.

C^*-correspondences

Definition

Let A be C^*-algebra. A (right) Hilbert C^*-module over A is a module E that is endowed with an A-valued inner product ($\langle \cdot, \cdot \rangle$) such that E is complete in the norm: $\|\xi\| := \|\langle \xi, \xi \rangle\|_A^{\frac{1}{2}}$. If, in addition, E is a left module over A via a $*$-homomorphism $\varphi : A \to \mathcal{L}(E)$, then we say that E is a C^*-correspondence over A.

- $\mathcal{L}(E)$ is the algebra of adjointable continuous, A-linear transformations on E.

Examples of C^*-Correspondences

Example

- $A := \mathbb{C}$ $E := \mathbb{C}^d$
- Directed Graphs (assume finite)
  - $G = \{G^0, G^1, r, s\}$, $r, s : G^1 \to G^0$
  - $A := \mathbb{C}^{G^0}$, $E = E(G) := \mathbb{C}^{G^1}$
  - $(a \cdot \xi \cdot b)(e) := a(r(e))\xi(e)b(s(e))$
  - $\langle \xi, \eta \rangle(v) := \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$
- α an endomorphism of A. $E = A$:
  $a \cdot \xi \cdot b := \alpha(a)\xi b$, $\langle \xi, \eta \rangle := \xi^* \eta$.

The Fock Correspondence

- $E^{\otimes 2} := E \otimes_A E$, $a \cdot \xi \otimes \eta \cdot b := (\varphi(a)\xi) \otimes \eta b$, $\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \langle \xi_2, \varphi(\langle \xi_1, \eta_1 \rangle)\eta_2 \rangle$.
- $\mathcal{F}(E) := \sum_{k=0}^{\infty} E^{\otimes k}$, the left action of A is denoted φ_∞

Example

- $\mathcal{F}(\mathbb{C}) = \ell^2(\mathbb{Z}_+)$
- $\mathcal{F}(\mathbb{C}^d) =$ the full Fock space of $\mathbb{C}^d$.
- $$\mathcal{F}(E(G)) = L^\infty - \sum_{v \in G^0}^{\oplus} \ell^2(vG^*),$$

  $G^* :=$ the path category of G.

The Fock Correspondence (continued)

Example

- $G^* = \{e_{i_1} e_{i_2} \cdots e_{i_n} \mid r(e_{i_{k+1}}) = s(e_{i_k})\}$ and vG^* is all the paths that end at v.

The Tensor Algebra

Definition

The algebraic tensor algebra of E, $\mathcal{T}_{0+}(E)$, is the algebraic subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\{\varphi_\infty(a) \mid a \in A\}$ and $\{T_\xi\}_{\xi \in E}$, where $T_\xi \eta := \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$ is the creation operator determined by ξ. The tensor algebra of E is the closure of $\mathcal{T}_{0+}(E)$ in the norm of $\mathcal{L}(\mathcal{F}(E))$. The Toeplitz algebra of E, $\mathcal{T}(E)$ is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\mathcal{T}_+(E)$.

Examples of Tensor Algebras

Example

- When $A = \mathbb{C} = E$, then $\mathcal{F}(E)$ is naturally isomorphic to $\ell^2(\mathbb{Z}_+)$ under the map

$$\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \to (\xi_1 \xi_2 \cdots \xi_n)\delta_n$$

  and $\mathcal{T}_{0+}(E)$ and $\mathcal{T}_+(E)$ are both generated by T_1. Consequently, $\mathcal{T}_{0+}(E)$ is isomorphic to $\mathbb{C}[X]$ and $\mathcal{T}_+(E)$ is completely isometrically isomorphic to the disc algebra, $A(\mathbb{D})$.
- When $A = \mathbb{C}$ and $E = \mathbb{C}^d$, then $\mathcal{F}(E)$ is isomorphic to $\ell^2(\mathbb{F}_d^+)$ under the map

$$e_{i_1} \otimes e_{i_2} \otimes \cdots e_{i_n} \to \delta_w$$

Examples of Tensor Algebras (continued)

Example

- where $\{e_1, e_2, \cdots, e_d\}$ is a prescribed o.n. basis for E, and $w = i_1 i_2 \cdots i_n$. $\mathcal{T}_{0+}(E)$ is isomorphic to the free algebra on d-generators, and $\mathcal{T}_+(E)$ is completely isometrically isomorphic to Popescu's noncommutative disc algebra $\mathcal{A}_d$.
- When G is a graph, $A = \mathbb{C}^{G^0}$ and $E = \mathbb{C}^{G^1}$, then $\mathcal{T}_{0+}(E)$ is isomorphic to the quiver algebra of G and $\mathcal{T}_+$ is a completion of $\mathcal{T}_{0+}$.

Examples of Tensor Algebras (continued)

Example

- When A is a C^*-algebra and $E = {}_{\alpha}A$ for an endomorphism α, $\mathcal{T}_{0+}(E)$ is the twisted polynomial algebra determined by A and α, $\mathcal{T}_+(E)$ is a semicrossed product studied by Peters in [Peters, 1984].

Noncommutative Affine Space

Rieffel's Induced Representations and Intertwiners

- Ordinary affine n-space should be viewed as $\mathbb{C}^n$ but you must ignore the vector space structure. The polynomial functions are the functions in the algebra generated by $(\mathbb{C}^n)^*$.
- If $\sigma : A \to B(H_\sigma)$ is a C^*-representation, then $E \otimes_\sigma H_\sigma$ is the completed space defined in the (pre)inner product: $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle := \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E) h_2 \rangle_H$.
- The map $X \to X \otimes I_H$, $X \in \mathcal{L}(E)$, is called the representation of $\mathcal{L}(E)$ induced by σ (in the sense of Rieffel) and is denoted σ^E.

Noncommutative Affine Space (Continued)

- $\mathfrak{I}(\sigma^E \circ \varphi, \sigma) := \{X \in B(E \otimes_\sigma H_\sigma, H_\sigma) \mid X\sigma^E \circ \varphi(\cdot) = \sigma(\cdot)X\}$ is the intertwiner space from $\sigma^E \circ \varphi(\cdot)$ to σ, called the noncommutative affine space determined by σ.
- The affine spaces where the elements of $\mathcal{T}_{0+}(E)$ are represented as functions are indexed by the representations of A.
- $\mathfrak{I}((\sigma \oplus \tau)^E \circ \varphi, \sigma \oplus \tau) = \left\{ \begin{bmatrix} \mathfrak{z}_{11} & \mathfrak{z}_{12} \\ \mathfrak{z}_{21} & \mathfrak{z}_{22} \end{bmatrix} \mid \mathfrak{z}_{11} \in \mathfrak{I}(\sigma^E \circ \varphi, \sigma), \mathfrak{z}_{12} \in \mathfrak{I}(\tau^E \circ \varphi, \sigma), \text{etc.}\right.$

Examples

- Let $A = \mathbb{C}$, $E = \mathbb{C}^d$, and let $\sigma : A \to B(\mathbb{C}^n)$ be the representation $\sigma(c) := cI_n$.
- $E \otimes_\sigma \mathbb{C}^n = M_{d,1}(\mathbb{C}^n)$ – Column space over $\mathbb{C}^n$.
- $\mathfrak{I}(\sigma^E \circ \varphi, \sigma) = B(M_{d,1}(\mathbb{C}^n), \mathbb{C}^n) = M_n(\mathbb{C})^d$.
- If A and E come from a graph G, then a representation $\sigma : A \to B(H_\sigma)$ is given by writing $H_\sigma = \sum_{v \in G^0} H_v$, where $H_v = \sigma(\delta_v) H_\sigma$.
- $\mathfrak{I}(\sigma^E \circ \varphi, \sigma) = \bigoplus_{e \in G^1} B(H_{s(e)}, H_{r(e)})$

Noncommutative Polynomial Functions

- For $\mathfrak{z} \in \mathfrak{I}(\sigma^E \circ \varphi, \sigma)$, the *(modified) tensor power* $\mathfrak{z}^{(n)}$ of $\mathfrak{z}$ is the map from $E^{\otimes n} \otimes_\sigma H_\sigma$ to H_σ defined by the formula

$$\mathfrak{z}^{(n)} := \mathfrak{z}(I_E \otimes \mathfrak{z})(I_{E^{\otimes 2}} \otimes \mathfrak{z}) \cdots (I_{E^{\otimes(n-1)}} \otimes \mathfrak{z}),$$

$$\mathfrak{z}^{(0)} = I_{H_\sigma}.$$

- if $\xi \in E^{\otimes n}$, $n \geq 1$, then the *insertion operator* $L_\xi : H_\sigma \to E^{\otimes n} \otimes_\sigma H_\sigma$ is defined by the formula

$$L_\xi h := \xi \otimes h.$$

If $n = 0$, then $E^{\otimes n} = A$ and $L_\xi = \sigma(\xi)$.

Noncommutative Polynomial Functions (bis)

Definition

Let $\sigma : A \to B(H_\sigma)$ and let $F = \sum_{n \geq 0} T_{\xi_n}$, $\xi_n \in E^{\otimes n}$, be an element of $\mathcal{T}_{0+}(E)$. Then the polynomial function $\widehat{F}_\sigma$ on $\mathfrak{I}(\sigma^E \circ \varphi, \sigma)$ with values in $B(H_\sigma)$ determined by F is defined by

$$\widehat{F}_\sigma(\mathfrak{z}) := \sum \mathfrak{z}^{(n)} L_{\xi_n}.$$

Generic Matrices

Example

Let $A = \mathbb{C}$, let $E = \mathbb{C}^d$, and let $\sigma : A \to B(\mathbb{C}^n)$. Then for $F = \sum_{w \in \mathbb{F}_d^+} a_w X^w$,

$$\widehat{F}_\sigma(\mathfrak{z}) = \sum_{w \in \mathbb{F}_d^+} a_w Z^w,$$

where $\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d) \in M_n(\mathbb{C})^d$ and $Z^w = Z_{i_1} Z_{i_2} \cdots Z_{i_{|w|}}$, and $w = i_1 i_2 \cdots i_{|w|}$.

Definition

$\{\sum_{w \in \mathbb{F}_d^+} a_w Z^w\}$ is called the algebra of d generic $n \times n$ matrices.

The Completely Contractive Representations of $\mathcal{T}_+(E)$

Theorem (M & Solel, 1998)

There is a bijection between the completely contractive representations $\rho : \mathcal{T}_+(E) \to B(H_\rho)$ and pairs $(\sigma, \mathfrak{z})$, where $\sigma : A \to B(H_\sigma)$ is a C^*-representation and $\mathfrak{z} \in \mathfrak{I}(\sigma^E \circ \varphi, \sigma)$, with $\|\mathfrak{z}\| \leq 1$. Given ρ, $\sigma := \rho \circ \varphi_\infty$ and $\mathfrak{z}$ is defined by the equation

$$\rho(T_\xi) = \mathfrak{z} L_\xi,$$

where $L_\xi h := \xi \otimes h$, $h \in H$.

- We write $\sigma \times \mathfrak{z}$ for the representation determined by $(\sigma, \mathfrak{z})$.

Representations

- $\mathbb{D}(E,\sigma) := \{\mathfrak{z} \in \mathfrak{I}(\sigma^E \circ \varphi, \sigma) \mid \|\mathfrak{z}\| < 1\}$.

Example

- If $A = \mathbb{C} = E$, and if $\sigma : \mathbb{C} \to \mathbb{C}$, then $\mathbb{D}(E,\sigma) =$ the open unit disc in $\mathbb{C}$, and $\sigma \times \mathfrak{z}(f) = f(\mathfrak{z})$.
- If $A = \mathbb{C}$ and $E = \mathbb{C}^d$, and if $\sigma : \mathbb{C} \to M_n(\mathbb{C})$, then $\mathbb{D}(E,\sigma) = \{\mathfrak{z} \in M_n(\mathbb{C})^d \mid \|\mathfrak{z}\mathfrak{z}^*\| < 1\}$.
- If $A = \mathbb{C}^{G^0}$, $E = \mathbb{C}^{G^1}$, and $\sigma : \mathbb{C}^{G^0} \to B(\mathbb{C}^N)$, then $\mathbb{C}^N = \sum_{v\in G^0}^{\oplus} H_v$, $H_v := \sigma(\delta_v)H$, so σ is completely determined by the numbers $\dim H_v$ (dimension vector) and $\mathbb{D}(E,\sigma)$ is the unit disc in the quiver variety determined by the dimension vector.

The Function Theory

Theorem

For $\sigma : A \to B(H_\sigma)$ *and* $F = \sum_{n\geq 0} T_{\xi_n} \in \mathcal{T}_+(E)$*, the function* $\widehat{F}_\sigma : \overline{\mathbb{D}(E,\sigma)} \to B(H_\sigma)$ *defined by*

$$\widehat{F}_\sigma(\mathfrak{z}) := \sigma \times \mathfrak{z}(F)$$

is continuous, and holomorphic on $\mathbb{D}(E,\sigma)$*. Further,*

$$\widehat{F}_\sigma(\mathfrak{z}) = \sum_{n\geq 0} \mathfrak{z}^{(n)} L_{\xi_n},$$

and the series converges uniformly in norm on concentric balls about $0 \in \mathfrak{I}(\sigma^E \circ \varphi, \sigma)$*.*

An Old Idea Redux

- R and S two unital rings, $X := \mathrm{Hom}(R, S)$.
- Defining $\widehat{r} : X \to S$ by $\widehat{r}(\varphi) := \varphi(r)$ may be "over kill", especially if S is not commutative.
- Let $G \subseteq \mathrm{Aut}(S)$ be a subgroup, e.g. take $G = \mathrm{InnAut}(S)$. Then G acts on X as well as on S:

$$\varphi \cdot g(r) := \varphi(r) \cdot g, \qquad r \in R, \varphi \in X, g \in G.$$

- Further

$$\widehat{r}(\varphi \cdot g) = (\varphi \cdot g)(r) = \varphi(r) \cdot g = \widehat{r}(\varphi) \cdot g.$$

Concomitants

Definition

If a group G acts on two spaces X and Y, say, then a function $f : X \to Y$ is called a G-concomitant in case

$$f(x \cdot g) = f(x) \cdot g, \qquad g \in G, x \in X.$$

- Concomitants are also called covariants, fixed functions, invariant functions, and intertwiners.
- Concomitants really should be thought of as functions from X/G to Y/G. However, this may be a problem if the quotient spaces are "bad".

A general observation

Observation

Suppose G acts on X and Y. Let $\pi_0 : X \to X/G$ be the quotient map. Let $X * Y = (X \times Y)/G$ (product action), and define $\pi : X * Y \to X/G$ by $\pi([x,y]) = \pi_0(x)$. (Then we may think of $X * Y$ as the total space of a fibre bundle over X/G with fibre Y and projection π.) If $f : X \to Y$ is a concomitant then the map $\sigma_f : X/G \to X * Y$ defined by $\sigma_f([x]) = [x, f(x)]$ is a section of this bundle, i.e. $\pi \circ \sigma_f = id_{X/G}$. Conversely, assuming the action of G on X is free (i.e., $x \cdot g = x$ implies $g = e$, for any $x \in X$), every section of this bundle is given by a concomitant.

Application to the Function Theory of Tensor Algebras

- We will restrict the discussion to the setting of d-generic $n \times n$ matrices, where d and n are at least 2.
- In this case, $R = \mathbb{C}\langle X_1, X_2, \cdots, X_d\rangle$, the free algebra on d-generators and $S = M_n(\mathbb{C})$. Every representation of R in S is given by an d-tuple of $n \times n$ matrices. So $X = M_n(\mathbb{C})^d$.
- $\mathrm{Aut} M_n(\mathbb{C}) = PGL(n, \mathbb{C})$. The induced action on X is the diagonal action: $\mathfrak{z} \cdot g = (Z_1 \cdot g, Z_2 \cdot g, \cdot Z_d \cdot g)$. We can think of g as an element in $GL(n, \mathbb{C})$ and $Z \cdot g$ as $g^{-1}Zg$.

Procesi's Theorem [Procesi, 1974]

Theorem

Let $\mathcal{V}(d,n) = \{\mathfrak{z} = (Z_1, Z_2, \cdots, Z_d) \mid Z_1, \cdots, Z_d$ *generate* $M_n(\mathbb{C})\}$. *Then* $\mathcal{V}(d,n)$ *is the bundle space of a holomorphic* $PGL(n,\mathbb{C})$*-principal bundle over an open subset,* $Q(d,n)$*, of the smooth points of the spectrum of the algebra of invariant polynomial functions on* $M_n(\mathbb{C})$*.*

Corollary

Theorem

If $\mathfrak{M}$ *is the associated matrix bundle* $\mathcal{V}(d,n)\times_{PGL(n,\mathbb{C})} M_n(\mathbb{C})$*, then* $\Gamma_h(Q(d,n),\mathfrak{M}) \simeq Hol(\mathcal{V}(d,n), M_n(\mathbb{C}))^{PGL(n,\mathbb{C})}$

- If n or d is greater than 2, then $\Gamma_h(Q(d,n),\mathfrak{M}) \simeq Hol(M_n(\mathbb{C})^d, M_n(\mathbb{C}))^{PGL(n,\mathbb{C})}$ - an integral domain, by a theorem of Amitsur.
- If n or d is greater than 2, then $\mathfrak{M}$ is nontrivial.
- If $n = d = 2$, then $\mathcal{V}(2,2) = \{\mathfrak{z} \mid [Z_1, Z_2]^2 = -\det[Z_1, Z_2] \neq 0\}$. So $\frac{1}{\det[Z_1,Z_2]} \in Hol(\mathcal{V}(2,2), M_2(\mathbb{C}))^{PGL(2,\mathbb{C})}$, but not in $Hol(M_2(\mathbb{C})^2, M_2(\mathbb{C}))^{PGL(2,\mathbb{C})}$. Nevertheless, $\mathfrak{M}$ is nontrivial in this case, and $Hol(\mathcal{V}(2,2), M_2(\mathbb{C}))^{PGL(2,\mathbb{C})}$ is an integral domain.

A Problem

- There is no a priori norm structure on $\Gamma_h(Q(d,n),\mathfrak{M})$.
- Reason: The transition functions describing $\mathfrak{M}$ are $PGL(n,\mathbb{C})$-valued and do not preserve the norms from fibre to fibre.
- Good News: Can always reduce $\mathcal{V}(d,n)$ to a principal $PU(n,\mathbb{C})$.
- Can find a $PU(n,\mathbb{C})$ bundle with bundle space Y over $Q(d,n)$ and a map $f: Y \to \mathcal{V}(d,n)$ that is a homeomorphism onto its range such that $f(y \cdot k) = f(y) \cdot k$ for all $y \in Y$ and $k \in PU(n,\mathbb{C})$.

A Problem (continued)

- Bad news: Let $\mathfrak{M}^* := Y \times_{PU(n,\mathbb{C})} M_n(\mathbb{C})$. Then $\mathfrak{M}^*$ is isomorphic to $\mathfrak{M}$ topological bundles, but the isomorphism does not preserve the holomorphic structure, in general.
- Thus any two reductions give isomorphic C^*-algebra structures to $\Gamma_c(K, \mathfrak{M}^*)$, for any compact subset $K \subseteq Q(d,n)$, but the image of $\Gamma_h(Q(d,n), \mathfrak{M})$ in $\Gamma_c(K, \mathfrak{M}^*)$ depends on the reduction.

Theorem

[Griesenauer et al.,] Let D be a domain in $Q(d,n)$ such that $\overline{D}$ is a Stein compact subset of $Q(d,n)$ and let $\Gamma_h(\overline{D},\mathfrak{M}^*)$ be the closure of the image of $\Gamma_h(Q(d,n),\mathfrak{M})$ in $\Gamma_c(\overline{D},\mathfrak{M}^*)$, where $\mathfrak{M}^*$ is associated to any reduction of $\mathcal{V}(d,n)$. Then the center of $\Gamma_h(\overline{D},\mathfrak{M}^*)$ is $\mathcal{A}(D) := \{f \in C(\overline{D}) \mid f|_D \text{ is holomorphic}\}$ and $\Gamma_h(\overline{D},\mathfrak{M}^*)$ is a rank n^2 Azumaya algebra over $\mathcal{A}(D)$.

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