Rigidity for the von Neuman algebras of products of hyperbolic groups

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von Neuman Algebras

Definition

A *von Neuman algebra* M is a unital, SOT closed, $*$-subalgebra $M \subset B(\mathcal{H})$.

Theorem (MvN36)

$M \subset B(\mathcal{H})$ is a von Neuman algebra iff $M = M''$, where $M' = \{x \in B(\mathcal{H}) : xm = mx \; \forall m \in M\}$.

We focus when M is a *type II_1* factor: M is endowed with unique normal finite faithful tracial state τ, and $M' \cap M \cong \mathbb{C}$.

Group von Neuman Algebras

Let Γ be a (countable) group and $\mathcal{H} = \ell^2\Gamma$ the Hilbert space of all square summable sequences indexed by Γ. The *left regular representation* is defined as $\lambda : \Gamma \to \mathcal{U}(\mathcal{H})$ by

$$u_\gamma(\xi)(x) := \xi(\gamma^{-1}x),\ \forall \gamma, x \in \Gamma, \xi \in \mathcal{H}$$

The *group von Neumann algebra* $L\Gamma$ is defined as

$$L\Gamma := \{u_\gamma\}''_{\gamma\in\Gamma} = \overline{\mathbb{C}[\Gamma]}^{\text{SOT}} \subset B(\mathcal{H})$$

For a discrete countable i.c.c. group $L\Gamma$ is a II_1 factor with $\tau(x) = \langle x\delta_e, \delta_e\rangle$

Hyperbolic i.c.c. groups

Let $\gamma^\Gamma = \{\omega^{-1}\gamma\omega : \omega \in \Gamma\}$.

Definition

Γ (discrete countable) is i.c.c. if $|\gamma^\Gamma| = \infty$ for all $\gamma \in \Gamma \setminus e$.

$L\Gamma$ is a factor iff Γ is i.c.c.
Γ (discrete countable) is hyperbolic if the geodesics on the Cayley graph give rise to a negative curvature.

- $\mathbb{F}_n$ where $n \geq 2$,
- $\pi_1(\Sigma_g)$ with $g \geq 2$.

Pimsner-Popa Index

Definition

Let $P \subset M$ be an inclusion of von Neuman algebras with M a II_1 factor.

$$[M : P]^{-1} = \inf\{\|E_P(x)\|_2^2/\|x\|_2^2 : x \in M_+\}$$

is the Pimsner-Popa index of the inclusion.

Proposition

Let $\Omega \leqslant \Lambda \leqslant \Theta$. If there are projection $p \in L\Omega$, $z \in L\Lambda' \cap L\Theta$ with $pz \neq 0$ so that $[pzL\Lambda p : pzL\Omega p] < \infty$, then $[\Lambda : \Omega] < \infty$.

Amenability and Solidity

Definition

(M, τ) is *amenable* if there exists a state $\phi : B(L^2M) \to \mathbb{C}$ so that $\phi(xm) = \phi(mx)$ and $\phi(m) = \tau(m)$ for all $x \in B(L^2M), m \in M$.

Definition

- M is *solid* if for any diffuse subalgebra $P \subset M$, $P' \cap M$ is amenable.
- M is *strongly solid* if for any diffuse amenable subalgebra $P \subset M$, $N_M(P)'' = \{u \in \mathcal{U}(M) : uPu^* = P\}''$ is amenable.

von Neuman algebras of non-amenable hyperbolic i.c.c groups are solid (N. Ozawa '02)
von Neuman alebras of non-amenable hyperbolic groups are strongly solid (I. Chifan & T. Sinclair '11)

Rigidity

Definition

Two groups Γ, Λ are W^*-equivalent, denoted $\Gamma \cong_{W^*} \Lambda$, if $L\Gamma \cong L\Lambda$

A natural question: Given $\Gamma \cong_{W^*} \Lambda$, what properties of the group does the algebra remember?

Theorem (Co76)

Let Γ be a discrete amenable ICC group. Then $L\Gamma \cong \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite II_1 factor.

Amenable i.c.c. groups

- S_∞ - group of bijections of $\mathbb{N}$ with finite support
- $\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/(2\mathbb{Z})\right) \rtimes \mathbb{Z}$ - the lamplighter group
- $\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \rtimes \mathbb{Z}$

Popa's Deformation/Rigidity

Definition

For $P, Q \subset M$, we say P *intertwines into* Q, denoted $P \preceq_M Q$, if there exists $p \in P$, $q \in Q$, p.i. $v \in M$, an injective $*$-homomorphism $\psi : pPp \to qQq$ so that

$$\psi(x)v = vx \quad \forall x \in pPp$$

Used by A. Ioana, S. Popa, and S. Vaes to provide examples of W^* super-rigid groups, i.e. $\Gamma \cong_{W^*} \Lambda \Rightarrow \Gamma \cong \Lambda$.

Uniqueness of Prime Decomposition

Definition

Given $t > 0$, let $p \in M$ be a projection with $\tau(p) = t$. $M^t := pMp$.

Theorem (OP03)

Let $L(\Gamma_1 \times \cdots \times \Gamma_m) \cong L(\Lambda_1 \times \cdots \times \Lambda_n)$, Γ_i, Λ_j ICC hyperbolic groups, then $n = m$ there exists $t_1, \ldots, t_n$, $\prod t_i = 1$ so that $L(\Lambda_i)^{t_i} \cong L(\Gamma_i)$.

Can we do better? No.

$$M \overline{\otimes} N \cong M^t \overline{\otimes} N^{1/t}$$

$$L(\mathbb{F}_2) \otimes L(\mathbb{F}_9) \cong L(\mathbb{F}_2)^{1/2} \otimes L(\mathbb{F}_9)^2 \cong L(\mathbb{F}_5) \otimes L(\mathbb{F}_3)$$

Removing the symmetric assumption

Theorem

Let $\Gamma = \Gamma_1 \times \cdots \Gamma_n$*,* Γ_i *i.c.c. hyperbolic, and suppose* $L\Gamma \cong L\Lambda$ *for an arbitrary group* Λ*. Then there exists* $\Lambda_1, \ldots, \Lambda_n < \Lambda$ *and* $t_1, \ldots, t_n$ *so that* $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$*,* $\prod t_i = 1$ *and* $L(\Lambda_i) \cong L(\Gamma_i)^{t_i}$*.*

Theorem (Io11)

For $n \geq 2$, let $\Gamma = \prod_{i=1}^{n} \Gamma_i$, Γ_i ICC hyperbolic and $M = L(\Gamma)$. If $L(\Lambda) \cong M^t$ then for every nonempty family of subgroups of Λ, $\mathcal{G}$, there exists $1 \leq \ell \leq n$ so that either

1. $L(\hat{\Gamma}_\ell)^t \preceq_{M^t} L(\Sigma)$ *for some* $\Sigma \in \mathcal{G}$, *or*
2. $L(\Gamma_\ell)^t \preceq_{M^t} L(\cup \Omega_j)$, *where* $\Omega_j = C_\Lambda(\Sigma_j)$, $\Sigma_j < \Lambda$ *a descending sequence of subgroups not in* $\mathcal{G}$.

Let $\mathcal{G}$ be all amenable subgroups Λ' of Λ with $C_\Lambda(\Lambda')$ non amenable.

Corollary

Let $M^t = (L\Gamma)^t \cong L(\Lambda)$ as above. Then there exists a non-amenable subgroup $\Sigma < \Lambda$ with non-amenable centralizer $C_\Lambda(\Sigma)$.

Theorem

Let M, Γ, Λ as above. Then there exists commuting, non-amenable, ICC subgroups $\Sigma_1, \Sigma_2 < \Lambda$ such that $[\Lambda : \Sigma_1\Sigma_2] < \infty$.

Proof-ish

Let $\Gamma = \Gamma_1 \times \Gamma_2$, $M = L\Gamma$.
Take $\Sigma < \Lambda$ s.t. $L\Gamma_1 \preceq L\Sigma$ with $C_\Lambda(\Sigma)$ non-amenable.
Upgrade intertwining to finite index inclusion

$$pL\Gamma_1 p \cong Q \subset qL\Sigma q, \quad [qL\Sigma q : Q] < \infty$$

The following inclusion is also f.i.:

$$rL\Sigma \vee L(C_\Lambda(\Sigma))r \subset rL\Sigma \vee (L\Sigma' \cap M)r \subset rMr$$

Take $\Lambda > \Omega = \{\lambda \in \Lambda : |\lambda|^\Sigma < \infty\}$.

$$rL\Sigma \vee (L\Sigma' \cap M)r \subset rL\Omega\Sigma r \subset rMr \Rightarrow [\Gamma : \Omega\Sigma] < \infty$$

Proof-ish part II_1

Upgrade $\Omega\Sigma$ to honest direct product.

- $|\Omega \cap \Sigma| < \infty$
- $\Omega = \{\lambda \in \Lambda : |\lambda|^{\Sigma} \leq K\}$
- Show each orbit isn't too *large*.
- Pass to finite index subgroup and play with amplification
- Induct?

Theorem

Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$, Γ_i i.c.c. hyperbolic, and suppose $L\Gamma^t \cong L\Lambda$ for an arbitrary group Λ. Then there exists $\Lambda_1, \ldots, \Lambda_n < \Lambda$ and $t_1, \ldots, t_n$ so that $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$, $\prod t_i = t$ and $L(\Lambda_i) \cong L(\Gamma_i)^{t_i}$.

Questions?

Thanks!