Harmonic functions on Bratteli diagrams

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Outline

- Basic settings
- Locally finite connected graphs vs Bratteli diagrams
- Harmonic functions on Bratteli diagrams
- Harmonic functions on trees, Pascal graphs, stationary Bratteli diagrams
- Harmonic functions of finite and infinite energy
- Harmonic functions through Poisson kernel
- Green’s function, dipoles, and monopoles for transient networks

Network basic settings

- *Electrical network* (G, c): $G = (V, E)$ is a locally finite connected graph, $c = c_{xy} = c_{yx} > 0, (xy) \in E$, is a *conductance function*; $c(x) := \sum_{y\sim x} c_{xy}$ is called the *total conductance* at $x \in V$
- *Laplace operator*: $(\Delta u)(x) = \sum_{y\sim x} c_{xy}(u(x) - u(y))$
- *Monopole*: $\Delta w_{x_0}(x) = \delta_{x_0}(x)$; *Dipole:* $\Delta v_{x_1,x_2}(x) = (\delta_{x_1} - \delta_{x_2})(x)$; *Harmonic function*: $\Delta f(x) = 0, \forall x \in V$
- *Hilbert space* $\mathcal{H}_E$ of *finite energy* functions, $(u : V \to \mathbb{R}) \in \mathcal{H}_E$ if

$$||u||^2_{\mathcal{H}_E} = \frac{1}{2} \sum_{(xy)\in E} c_{xy}(u(x) - u(y))^2 < \infty.$$

- *Markov operator*: $P = (p(x, y))_{x,y\in V}$ with *transition probabilities* $p(x, y) := \dfrac{c_{xy}}{c(x)}$. A function f is harmonic iff $Pf = f$
- *Random walk* on $G = (V, E)$ defined by P is *recurrent* if $\forall x \in V$ it returns to x infinitely often with probability 1. Otherwise, it is called *transient*.

Motivational questions

- Are there explicit formulas or algorithms for finding monopoles, dipoles, and harmonic functions for some classes of graphs?
- Under what conditions do these functions have finite (infinite) energy?
- How does the structure of a graph (in particular, a Bratteli diagram) affect the properties of harmonic functions?
- When can a locally finite graph be represented as a Bratteli diagram?
- What are the properties of the random walk defined by the transition matrix P on a Bratteli diagram B?
- Are there interesting examples?

Facts about Laplace operators, harmonic functions, monopoles, dipoles

For (G, c), Δ, and P as above, the following holds:

(i) Δ is an Hermitian, unbounded operator with dense domain in $\mathcal{H}_E$, but it is not self-adjoint, in general;

(ii) $\Delta = c(I - P)$ and $\Delta f = 0 \iff Pf = f$;

(iii) For a harmonic function f,

$$\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{x \in V} c(x)((Pf^2)(x) - f^2(x)),$$

and

$$\|f\|_{\mathcal{H}_E}^2 = -\frac{1}{2} \sum_{x \in V} (\Delta f^2)(x).$$

Facts about Laplace operators, harmonic functions, monopoles, dipoles

(iv) For $x, y \in V$, there exists a vector $v_{xy} \in \mathcal{H}_E$ such that $\langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y)$ ($\forall u \in \mathcal{H}_E$) is called a *dipole.*

(v) A *monopole* at $x \in V$ is an element $w_x \in \mathcal{H}_E$ such that

$$\langle w_x, u \rangle_{\mathcal{H}_E} = u(x), \; u \in \mathcal{H}_E.$$

(vi) Let $x_0 \in V$ be a fixed vertex. Then w_{x_0} is a monopole if and only if it coincides with a finite energy harmonic function h on $V \setminus \{x_0\}$.

(vii) An electrical network is *transient* if and only if there exists a monopole w in $\mathcal{H}_E$.

Bratteli diagrams: definition

Definition

A *Bratteli diagram* is an infinite graph $B = (V, E)$ with the vertex set $V = \bigcup_{i\geq 0} V_i$ and edge set $E = \bigcup_{i\geq 0} E_i$:
1) $V_0 = \{v_0\}$ is a single point;
2) V_i and E_i are finite sets for every i;
3) edges E_i connect V_i to V_{i+1}: there exist a range map r and a source map s from E to V such that $r(E_i) = V_{i+1}, s(E_i) = V_i$, and $s^{-1}(v) \neq \emptyset$; $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

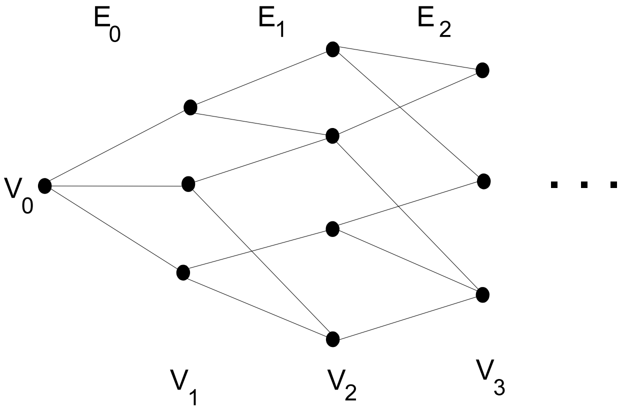
B is *stationary* if it repeats itself below the first level, and B is of *finite rank* if $|V_n| \leq d$ (w.l.o.g. one can assume $|V_n| = d$).

The *incidence matrix* F_n has entries

$$f_{v,w}^{(n)} = |\{e \in E_n : s(e) = w, r(e) = v\}|, \ v \in V_{n+1}, w \in V_n.$$

Every Bratteli diagram is equivalent to a Bratteli diagram with (0,1)-incidence matrices.

Example of a Bratteli diagram



From a graph to a Bratteli diagram

Example (G is not a Bratteli diagram)

Let $G = (V, E)$ be a connected locally finite graph satisfying the property: $\forall x \in V \;\; \exists y_1, y_2$ such that $y_1 \sim x, y_2 \sim x$ and $(y_1 y_2) \in E$. Then G cannot be represented as a Bratteli diagram.

Example ($\mathbb{Z}^d$ is a Bratteli diagram)

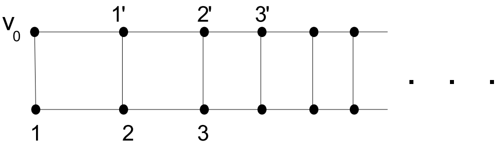
Let $d = 2$ for simplicity. Then we take $(0, 0)$ as $V_0 = \{o\}$, and we set $V_n := \{(x, y) \in \mathbb{Z}^d : |x| + |y| = n\}, n \geq 1$. Then V_n is the n-th level of B. The set of edges E_n between the levels V_n and V_{n+1} is inherited from the lattice. One can take any vertex of $\mathbb{Z}^d$ as the root of the diagram.

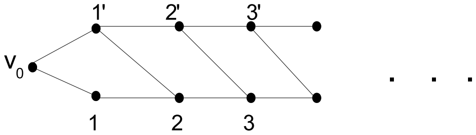
Example (Cayley graph)

Let H be a Cayley graph of a group with finitely generating set $S, S = S^{-1}$. Then H can be represented as a Bratteli diagram if and only if $SS \cap S = \emptyset$.

From a graph to a Bratteli diagram

Example (Infinite “ladder” graph is a Bratteli diagram)





If we add the diagonals in every rectangle, then the “rigid ladder” G is not a Bratteli diagram.

From a graph to a Bratteli diagram

Theorem

A connected locally finite graph $G(V, E)$ has the structure of a Bratteli diagram if and only if:
(i) for every $x \in V, \deg(x) \geq 2$,
(ii) there exists a vertex $x_0 \in V$ such that, for any $n \geq 1$, there are no edges between any vertices from the set
$V_n := \{y \in V : \mathrm{dist}(x_0, y) = n\}$.
(iii) for any vertex $x \in V_n$ there exists an edge $e_{(xy)}$ connecting x with some vertex $y \in V_{n+1}, n \in \mathbb{N}$.

Theorem

Let $G = (V, E)$ be a connected locally finite graph that contains at least one path, ω, without self-intersection. Then G contains a maximal subgraph H that is represented as a Bratteli diagram B such that ω belongs to the path space X_B of B.

Harmonic functions on a Bratteli diagram

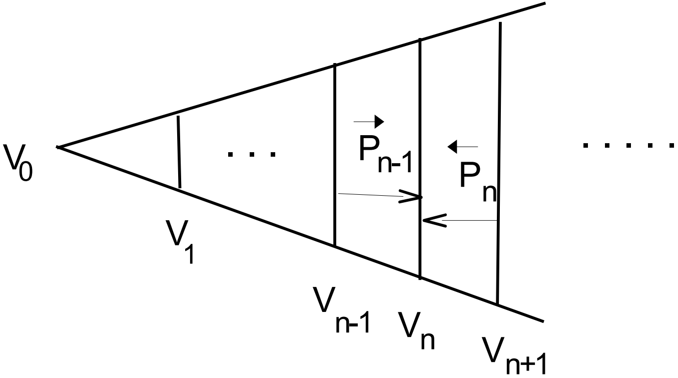
Define the matrices $(\overleftarrow{P}_n)$ and $(\overrightarrow{P}_{n-1})$ for $x \in V_n, z \in V_{n+1}, y \in V_{n-1}$:

$$\overleftarrow{p}^{(n)}_{xz} = \frac{c^{(n)}_{xz}}{c_n(x)}, \quad \overrightarrow{p}^{(n-1)}_{xy} = \frac{c^{(n-1)}_{yx}}{c_n(x)}.$$

The matrix P of transition probabilities has the form

$$P = \begin{pmatrix} 0 & \overleftarrow{P}_0 & 0 & 0 & \cdots & \cdots \\ \overrightarrow{P}_0 & 0 & \overleftarrow{P}_1 & \cdots & \cdots & \\ 0 & \overrightarrow{P}_1 & 0 & \overleftarrow{P}_2 & \cdots & \cdots \\ 0 & 0 & \overrightarrow{P}_2 & 0 & \overleftarrow{P}_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Harmonic functions on a Bratteli diagram



Harmonic functions on a Bratteli diagram

Theorem

(1) Let $(B(V,E),c)$ be a weighted Bratteli diagram with associated sequences of matrices $(\overrightarrow{P}_n)$ and $(\overleftarrow{P}_n)$. Then a sequence of vectors (f_n) ($f_n \in \mathbf{R}^{|V_n|}$) represents a harmonic function $f = (f_n) : V \to \mathbb{R}$ if and only if for any $n \geq 1$

$$f_n - \overrightarrow{P}_{n-1} f_{n-1} = \overleftarrow{P}_n f_{n+1}.$$

(2) The space of harmonic functions, $\mathcal{H}arm$, is nontrivial on a weighted Bratteli diagram (B,c) if and only if there exists a sequence of non-zero vectors $f = (f_n)$, where $f_n \in \mathbb{R}^{|V_n|}$, such that

$$f_n - \overrightarrow{P}_{n-1} f_{n-1} \in \mathrm{Col}(\overleftarrow{P}_n).$$

(3) Suppose that $|V_i| \leq |V_{i+1}|, i = 1, ..., n-1$, and $|V_{n+1}| < |V_n|$ (a "bottleneck" Bratteli diagram). Then $\mathcal{H}arm$ is trivial.

Harmonic functions on a Bratteli diagram

Theorem

(4) If a weighted Bratteli diagram (B, c) is not of "bottleneck" type (that is $|V_n| \leq |V_{n+1}|$ for every n), and, for infinitely many levels n, the strict inequality holds, then the space $\mathcal{H}arm$ is infinite-dimensional.
(5) There are stationary weighted Bratteli diagrams (B, c) such that the space $\mathcal{H}arm$ is finite-dimensional.

Similar approach can be used to prove the existence of monopoles and dipoles on a weighted Bratteli diagram.

Harmonic function on stationary Bratteli diagrams

Theorem

Let (B, c) be a stationary weighted Bratteli diagram with incidence matrix F and $c_{(xy)} = \lambda^n, e = (xy) \in E_n, \lambda > 1$. Suppose that $F = F^T$ and F is invertible. Then any harmonic function $f = (f_n)$ on (B, c) can be found by the formula:

$$f_{n+1}(x) = f_1(x) \sum_{i=0}^{n} \lambda^{-i}$$

where $x \in V$.

Corollary

Let (B, c) be as in the theorem.
(1) The dimension of the space $\mathcal{H}arm$ is $d - 1$ where $d = |V|$.
(2) If $\lambda > 1$, then every harmonic function on (B, c) is bounded.

Harmonic functions on trees

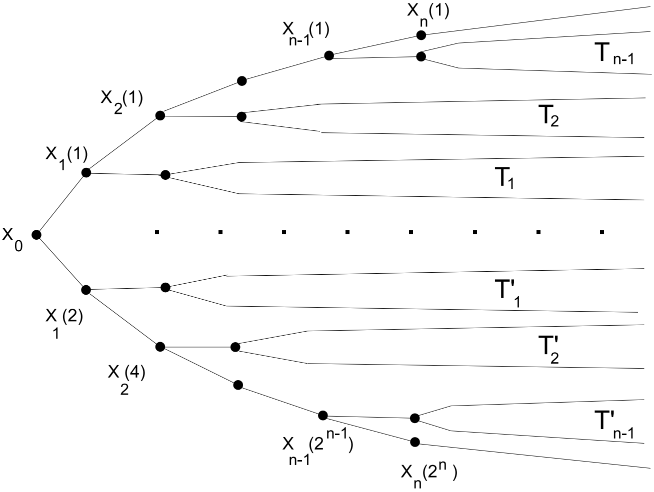
Theorem

Let (T, c) be the weighted binary tree. For each positive $\lambda > 1$ there exists a unique harmonic function $f = f_\lambda$ satisfying the following conditions:
(1) $f(x_0) = 0$;
(2) $f(x_1(1)) = -f(x_1(2)) = \lambda$ and

$$f(x_n(1)) = -f(x_n(2^n)) = \frac{1 + \cdots + \lambda^{n-1}}{\lambda^{n-2}},\ n \geq 2;$$

(3) the function f is constant on each of subtrees T_i and T_i' whose all infinite paths start at the roots $x_i(1)$ and $x_i(2^i)$, respectively, and go through the vertices $x_{i+1}(2)$ and $x_{i+1}(2^{i+1} - 1)$, $i \geq 1$.

Harmonic functions on trees



Harmonic functions on the Pascal graph

The incidence matrix of the Pascal graph is

$$F_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}$$

For $\lambda > 1$, the matrix of transition probabilities is

$$\overleftarrow{P}_n = \begin{pmatrix} \frac{\lambda}{1+\lambda} & \frac{\lambda}{1+\lambda} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{2+\lambda} & \frac{\lambda}{2+\lambda} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\lambda}{2+\lambda} & \frac{\lambda}{2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{\lambda}{1+\lambda} & \frac{\lambda}{1+\lambda} \end{pmatrix}$$

Harmonic function on the Pascal graph

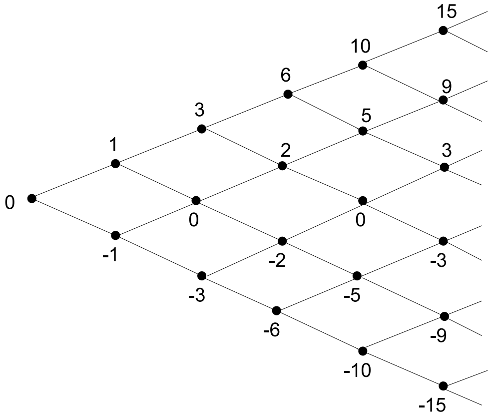
Theorem

Let $c_{(xy)} = 1$. Define $h(0,0) = 0$ and set, for every vertex $v = (n,i)$,

$$h(n,i) := \frac{n(n+1)}{2} - i(n+1),$$

where $0 \leq i \leq n$ and $n \geq 1$. Then $h : V \to \mathbb{R}$ is an integer-valued harmonic function on $(B,1)$ satisfying the symmetry condition $h(n,i) = -h(n,n-i)$.

Harmonic function on the Pascal graph



Harmonic functions of finite and infinite energy

Let (B, c) be a weighted Bratteli diagram. Denote

$$\beta_n = \max\{c(x) : x \in V_n\}, \quad I_1 = \sum_{x \in V_1} c_{ox}(f(x) - f(o)).$$

Theorem

(1) Let f be a harmonic function on a weighted Bratteli diagram (B, c). Then

$$\sum_{n=0}^{\infty} \frac{I_1^2}{\beta_n |V_n|} \leq \|f\|^2_{\mathcal{H}_E}.$$

(2) Suppose that a weighted Bratteli diagram (B, c) satisfies the condition

$$\sum_{n=0}^{\infty} (\beta_n |V_n|)^{-1} = \infty$$

where $V = \bigcup_n V_n$. Then any nontrivial harmonic function has infinite energy, i.e., $\mathcal{H}arm \cap \mathcal{H}_E = \{\mathrm{const}\}$.

Harmonic functions of finite and infinite energy

Example (Binary tree)

Let the conductance function c be defined by $c(e) = \lambda^n$ for all $e \in E_n, n \in \mathbb{N}_0$, and $f_\lambda = (f_n)$ be the symmetric harmonic function. Then

$$\|f_\lambda\|_{\mathcal{H}_E} < \infty \text{ if and only if } \lambda > 1.$$

Example (Pascal graph)

If $c = 1$ (simple random walk), then there is no harmonic function of finite energy on the Pascal graph.

Example (Stationary Bratteli diagram)

For a stationary weighted Bratteli diagram (B, c) with $c_e = \lambda^n, e \in E_n$, $\lambda > 1$, and a harmonic function $f = (f_n)$,

$$\|f\|_{\mathcal{H}_E} < \infty \iff f_1(x) = \text{const}.$$

Integral representation of harmonic functions

Ω_x = the set of paths that starts at x.
$\mathbb{P}_x$ = the Markov measure on Ω_x generated by P
$X_i : \Omega_x \to V$ = the random variable on $(\Omega_x, \mathbb{P}_x)$ such that $X_i(\omega) = x_i$.

$$\tau(V_n)(\omega) = \min\{i \in \mathbb{N} : X_i(\omega) \in V_n\}, \ \ \omega \in \Omega_x.$$

Lemma

Let (B, c) be a transient network, and $W_{n-1} = \bigcup_{i=0}^{n-1} V_i$. Then for every $n \in \mathbb{N}$ and any $x \in W_{n-1}$, there exists $m > n$ such that for $\mathbb{P}_x$-a.e. $\omega \in \Omega_x$

$$\tau(V_{i+1})(\omega) = \tau(V_i)(\omega) + 1, \ \ i \geq m.$$

Integral representation of harmonic functions

For a vector $f_n \in \mathbb{R}^{|V_n|}$, define the function $h_n : X \to \mathbb{R}$:

$$h_n(x) := \int_{\Omega_x} f_n(X_{\tau(V_n)}(\omega))d\mathbb{P}_x(\omega), \quad n \in \mathbb{N}.$$

Lemma

For a given function $f = (f_n)$, and, for every n, the function $h_n(x)$ is harmonic on $V \setminus V_n$ and $h_n(x) = f_n(x), x \in V_n$. Furthermore, $h_n(x)$ is uniquely defined on W_{n-1}.

Theorem

Let $f = (f_n) \geq 0$ be a function on V such that $\overleftarrow{P}_n f_{n+1} = f_n$. Then the sequence $(h_n(x))$ converges pointwise to a harmonic function $H(x)$. Moreover, for every $x \in V$, there exists $n(x)$ such that $h_i(x) = H(x), i \geq n(x)$.