Harmonic functions on Bratteli diagrams

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Outline

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- Harmonic functions through Poisson kernel
- Green’s function, dipoles, and monopoles for transient networks
Network basic settings

Electrical network \((G, c)\): \(G = (V, E)\) is a locally finite connected graph, \(c = c_{xy} = c_{yx} > 0, (xy) \in E\), is a conductance function; \(c(x) := \sum_{y \sim x} c_{xy}\) is called the total conductance at \(x \in V\).

Laplace operator: \((\Delta u)(x) = \sum_{y \sim x} c_{xy}(u(x) - u(y))\)

Monopole: \(\Delta w_{x_0}(x) = \delta_{x_0}(x)\); Dipole: \(\Delta v_{x_1,x_2}(x) = (\delta_{x_1} - \delta_{x_2})(x)\);

Harmonic function: \(\Delta f(x) = 0, \forall x \in V\)

Hilbert space \(\mathcal{H}_E\) of finite energy functions, \((u : V \to \mathbb{R}) \in \mathcal{H}_E\) if

\[
||u||^2_{\mathcal{H}_E} = \frac{1}{2} \sum_{(xy) \in E} c_{xy}(u(x) - u(y))^2 < \infty.
\]

Markov operator: \(P = (p(x, y))_{x, y \in V}\) with transition probabilities \(p(x, y) := \frac{c_{xy}}{c(x)}\). A function \(f\) is harmonic iff \(Pf = f\)

Random walk on \(G = (V, E)\) defined by \(P\) is recurrent if \(\forall x \in V\) it returns to \(x\) infinitely often with probability 1. Otherwise, it is called transient.
Motivational questions

- Are there explicit formulas or algorithms for finding monopoles, dipoles, and harmonic functions for some classes of graphs?
- Under what conditions do these functions have finite (infinite) energy?
- How does the structure of a graph (in particular, a Bratteli diagram) affect the properties of harmonic functions?
- When can a locally finite graph be represented as a Bratteli diagram?
- What are the properties of the random walk defined by the transition matrix $P$ on a Bratteli diagram $B$?
- Are there interesting examples?
Facts about Laplace operators, harmonic functions, monopoles, dipoles

For \((G, c), \Delta,\) and \(P\) as above, the following holds:

(i) \(\Delta\) is an Hermitian, unbounded operator with dense domain in \(\mathcal{H}_E\), but it is not self-adjoint, in general;

(ii) \(\Delta = c(I - P)\) and \(\Delta f = 0 \iff Pf = f;\)

(iii) For a harmonic function \(f,\)

\[
\|f\|_{\mathcal{H}_E}^2 = \frac{1}{2} \sum_{x \in V} c(x)((Pf^2)(x) - f^2(x)),
\]

and

\[
\|f\|_{\mathcal{H}_E}^2 = -\frac{1}{2} \sum_{x \in V}(\Delta f^2)(x).
\]
Facts about Laplace operators, harmonic functions, monopoles, dipoles

(iv) For $x, y \in V$, there exists a vector $v_{xy} \in \mathcal{H}_E$ such that $\langle v_{xy}, u \rangle_{\mathcal{H}_E} = u(x) - u(y)$ (\forall u \in \mathcal{H}_E) is called a dipole.

(v) A monopole at $x \in V$ is an element $w_x \in \mathcal{H}_E$ such that $\langle w_x, u \rangle_{\mathcal{H}_E} = u(x)$, $u \in \mathcal{H}_E$.

(vi) Let $x_0 \in V$ be a fixed vertex. Then $w_{x_0}$ is a monopole if and only if it coincides with a finite energy harmonic function $h$ on $V \setminus \{x_0\}$.

(vii) An electrical network is transient if and only if there exists a monopole $w$ in $\mathcal{H}_E$. 
Definition

A Bratteli diagram is an infinite graph $B = (V, E)$ with the vertex set $V = \bigcup_{i \geq 0} V_i$ and edge set $E = \bigcup_{i \geq 0} E_i$:

1) $V_0 = \{v_0\}$ is a single point;
2) $V_i$ and $E_i$ are finite sets for every $i$;
3) edges $E_i$ connect $V_i$ to $V_{i+1}$: there exist a range map $r$ and a source map $s$ from $E$ to $V$ such that $r(E_i) = V_{i+1}, s(E_i) = V_i$, and $s^{-1}(v) \neq \emptyset; r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

$B$ is stationary if it repeats itself below the first level, and $B$ is of finite rank if $|V_n| \leq d$ (w.l.o.g. one can assume $|V_n| = d$).

The incidence matrix $F_n$ has entries

$$f_{v,w}^{(n)} = \left| \{e \in E_n : s(e) = w, r(e) = v\} \right|, \ v \in V_{n+1}, w \in V_n.$$ 

Every Brattelli diagram is equivalent to a Bratteli diagram with (0,1)-incidence matrices.
Example of a Bratteli diagram
From a graph to a Bratteli diagram

Example ($G$ is not a Bratteli diagram)

Let $G = (V, E)$ be a connected locally finite graph satisfying the property: $\forall x \in V \; \exists y_1, y_2$ such that $y_1 \sim x, y_2 \sim x$ and $(y_1y_2) \in E$. Then $G$ cannot be represented as a Bratteli diagram.

Example ($\mathbb{Z}^d$ is a Bratteli diagram)

Let $d = 2$ for simplicity. Then we take $(0, 0)$ as $V_0 = \{o\}$, and we set $V_n := \{(x, y) \in \mathbb{Z}^d : |x| + |y| = n\}, n \geq 1$. Then $V_n$ is the $n$-th level of $B$. The set of edges $E_n$ between the levels $V_n$ and $V_{n+1}$ is inherited from the lattice. One can take any vertex of $\mathbb{Z}^d$ as the root of the diagram.

Example (Cayley graph)

Let $H$ be a Cayley graph of a group with finitely generating set $S, S = S^{-1}$. Then $H$ can be represented as a Bratteli diagram if and only if $SS \cap S = \emptyset$. 
Example (Infinite “ladder” graph is a Bratteli diagram)

If we add the diagonals in every rectangle, then the “rigid ladder” $G$ is not a Bratteli diagram.
Theorem

A connected locally finite graph \( G(V, E) \) has the structure of a Bratteli diagram if and only if:

(i) for every \( x \in V \), \( \deg(x) \geq 2 \),
(ii) there exists a vertex \( x_0 \in V \) such that, for any \( n \geq 1 \), there are no edges between any vertices from the set \( V_n := \{y \in V : \text{dist}(x_0, y) = n\} \).
(iii) for any vertex \( x \in V_n \) there exists an edge \( e_{(xy)} \) connecting \( x \) with some vertex \( y \in V_{n+1}, n \in \mathbb{N} \).

Theorem

Let \( G = (V, E) \) be a connected locally finite graph that contains at least one path, \( \omega \), without self-intersection. Then \( G \) contains a maximal subgraph \( H \) that is represented as a Bratteli diagram \( B \) such that \( \omega \) belongs to the path space \( X_B \) of \( B \).
Harmonic functions on a Bratteli diagram

Define the matrices \( \left( \overleftarrow{P}_n \right) \) and \( \left( \overrightarrow{P}_{n-1} \right) \) for \( x \in V_n, z \in V_{n+1}, y \in V_{n-1} \):

\[
\left( \overleftarrow{P}_{xz} \right)_{(n)} = \frac{c_{xz}^{(n)}}{c_n(x)}, \quad \left( \overrightarrow{P}_{xy} \right)_{(n-1)} = \frac{c_{yx}^{(n-1)}}{c_n(x)}.
\]

The matrix \( P \) of transition probabilities has the form

\[
P = \begin{pmatrix}
0 & \overleftarrow{P}_0 & 0 & 0 & \cdots & \cdots \\
\overrightarrow{P}_0 & 0 & \overleftarrow{P}_1 & \cdots & \cdots \\
0 & \overrightarrow{P}_1 & 0 & \overleftarrow{P}_2 & \cdots & \cdots \\
0 & 0 & \overrightarrow{P}_2 & 0 & \overleftarrow{P}_3 & \cdots \\
& & & & & \cdots & \cdots \\
& & & & & & \cdots & \cdots 
\end{pmatrix}.
\]
Harmonic functions on a Bratteli diagram
Harmonic functions on a Bratteli diagram

Theorem

(1) Let \((B(V, E), c)\) be a weighted Bratteli diagram with associated sequences of matrices \((\vec{P}_n)\) and \((\vec{P}_n)\). Then a sequence of vectors \((f_n) (f_n \in \mathbb{R}^{V_n})\) represents a harmonic function \(f = (f_n) : V \to \mathbb{R}\) if and only if for any \(n \geq 1\)

\[ f_n - \vec{P}_{n-1}f_{n-1} = \vec{P}_nf_{n+1}. \]

(2) The space of harmonic functions, \(\mathcal{Harm}\), is nontrivial on a weighted Bratteli diagram \((B, c)\) if and only if there exists a sequence of non-zero vectors \(f = (f_n), \) where \(f_n \in \mathbb{R}^{V_n}\), such that

\[ f_n - \vec{P}_{n-1}f_{n-1} \in \text{Col}(\vec{P}_n). \]

(3) Suppose that \(|V_i| \leq |V_{i+1}|, i = 1, \ldots, n - 1, \) and \(|V_{n+1}| < |V_n|\) (a “bottleneck” Bratteli diagram). Then \(\mathcal{Harm}\) is trivial.
Theorem

(4) If a weighted Bratteli diagram \((B, c)\) is not of “bottleneck” type (that is \(|V_n| \leq |V_{n+1}|\) for every \(n\)), and, for infinitely many levels \(n\), the strict inequality holds, then the space \(Harm\) is infinite-dimensional.

(5) There are stationary weighted Bratteli diagrams \((B, c)\) such that the space \(Harm\) is finite-dimensional.

Similar approach can be used to prove the existence of monopoles and dipoles on a weighted Bratteli diagram.
Theorem

Let \((B, c)\) be a stationary weighted Bratteli diagram with incidence matrix \(F\) and \(c_{(xy)} = \lambda^n, e = (xy) \in E_n, \lambda > 1\). Suppose that \(F = F^T\) and \(F\) is invertible. Then any harmonic function \(f = (f_n)\) on \((B, c)\) can be found by the formula:

\[
f_{n+1}(x) = f_1(x) \sum_{i=0}^{n} \lambda^{-i}
\]

where \(x \in V\).

Corollary

Let \((B, c)\) be as in the theorem.

(1) The dimension of the space \(\mathcal{Harm}\) is \(d - 1\) where \(d = |V|\).

(2) If \(\lambda > 1\), then every harmonic function on \((B, c)\) is bounded.
Harmonic functions on trees

Theorem

Let \((T, c)\) be the weighted binary tree. For each positive \(\lambda > 1\) there exists a unique harmonic function \(f = f_\lambda\) satisfying the following conditions:

1. \(f(x_0) = 0;\)
2. \(f(x_1(1)) = -f(x_1(2)) = \lambda\) and
   \[
   f(x_n(1)) = -f(x_n(2^n)) = \frac{1 + \cdots + \lambda^{n-1}}{\lambda^{n-2}}, \quad n \geq 2;
   \]
3. the function \(f\) is constant on each of subtrees \(T_i\) and \(T'_i\) whose all infinite paths start at the roots \(x_i(1)\) and \(x_i(2^i)\), respectively, and go through the vertices \(x_{i+1}(2)\) and \(x_{i+1}(2^{i+1} - 1), \ i \geq 1.\)
Harmonic functions on trees

Diagram showing a tree structure with nodes labeled as $X_0$, $X_1(1)$, $X_2(1)$, $X_1(1)$, $X_{n-1}(1)$, $X_n(1)$, $T_{n-1}$, $T_2$, $T_1$, $T'_1$, $T'_2$, $T'_{n-1}$, and $X_n(2^n)$.
Harmonic functions on the Pascal graph

The incidence matrix of the Pascal graph is

\[ F_n = \begin{pmatrix}
  1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
  0 & 1 & 1 & 0 & \cdots & 0 & 0 \\
  0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
\end{pmatrix} \]

For \( \lambda > 1 \), the matrix of transition probabilities is

\[ \begin{pmatrix}
  \lambda / (1+\lambda) & \lambda / (1+\lambda) & 0 & 0 & \cdots & 0 \\
  0 & \lambda / (2+\lambda) & \lambda / (2+\lambda) & 0 & \cdots & 0 \\
  0 & 0 & \lambda / (2+\lambda) & \lambda / (2+\lambda) & \cdots & 0 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & 0 & 0 & \cdots & \lambda / (1+\lambda) \\
\end{pmatrix} \]
Harmonic function on the Pascal graph

Theorem

Let $c_{xy} = 1$. Define $h(0, 0) = 0$ and set, for every vertex $v = (n, i)$,

$$h(n, i) := \frac{n(n + 1)}{2} - i(n + 1),$$

where $0 \leq i \leq n$ and $n \geq 1$. Then $h : V \rightarrow \mathbb{R}$ is an integer-valued harmonic function on $(B, 1)$ satisfying the symmetry condition $h(n, i) = -h(n, n - i)$. 
Harmonic function on the Pascal graph
Harmonic functions of finite and infinite energy

Let \((B, c)\) be a weighted Bratteli diagram. Denote

\[
\beta_n = \max\{c(x) : x \in V_n\}, \quad I_1 = \sum_{x \in V_1} c_{ox}(f(x) - f(o)).
\]

**Theorem**

(1) Let \(f\) be a harmonic function on a weighted Bratteli diagram \((B, c)\). Then

\[
\sum_{n=0}^{\infty} \frac{I_1^2}{\beta_n |V_n|} \leq \|f\|_{\mathcal{H}_E}^2.
\]

(2) Suppose that a weighted Bratteli diagram \((B, c)\) satisfies the condition

\[
\sum_{n=0}^{\infty} (\beta_n |V_n|)^{-1} = \infty
\]

where \(V = \bigcup_n V_n\). Then any nontrivial harmonic function has infinite energy, i.e., \(\mathcal{H}\text{arm} \cap \mathcal{H}_E = \{\text{const}\}\).
Harmonic functions of finite and infinite energy

Example (Binary tree)
Let the conductance function $c$ be defined by $c(e) = \lambda^n$ for all $e \in E_n, n \in \mathbb{N}_0$, and $f_\lambda = (f_n)$ be the symmetric harmonic function. Then

$$\|f_\lambda\|_{\mathcal{H}_E} < \infty \text{ if and only if } \lambda > 1.$$ 

Example (Pascal graph)
If $c = 1$ (simple random walk), then there is no harmonic function of finite energy on the Pascal graph.

Example (Stationary Bratteli diagram)
For a stationary weighted Bratteli diagram $(B, c)$ with $c_e = \lambda^n, e \in E_n$, $\lambda > 1$, and a harmonic function $f = (f_n)$,

$$\|f\|_{\mathcal{H}_E} < \infty \iff f_1(x) = \text{const}.$$
Integral representation of harmonic functions

\[ \Omega_x = \text{the set of paths that starts at } x. \]
\[ P_x = \text{the Markov measure on } \Omega_x \text{ generated by } P \]
\[ X_i : \Omega_x \rightarrow V = \text{the random variable on } (\Omega_x, P_x) \text{ such that } X_i(\omega) = x_i. \]

\[ \tau(V_n)(\omega) = \min\{i \in \mathbb{N} : X_i(\omega) \in V_n\}, \ \omega \in \Omega_x. \]

**Lemma**

*Let* \((B, c)\) *be a transient network, and* \(W_{n-1} = \bigcup_{i=0}^{n-1} V_i\). *Then for every* \(n \in \mathbb{N}\) *and any* \(x \in W_{n-1}\), *there exists* \(m > n\) *such that for* \(P_x\)-a.e. \(\omega \in \Omega_x\)

\[ \tau(V_{i+1})(\omega) = \tau(V_i)(\omega) + 1, \ i \geq m. \]
Integral representation of harmonic functions

For a vector $f_n \in \mathbb{R}^{|V_n|}$, define the function $h_n : X \to \mathbb{R}$:

$$h_n(x) := \int_{\Omega_x} f_n(X_{\tau(V_n)}(\omega))d\mathbb{P}_x(\omega), \ n \in \mathbb{N}.$$ 

**Lemma**

For a given function $f = (f_n)$, and, for every $n$, the function $h_n(x)$ is harmonic on $V \setminus V_n$ and $h_n(x) = f_n(x)$, $x \in V_n$. Furthermore, $h_n(x)$ is uniquely defined on $W_{n-1}$.

**Theorem**

Let $f = (f_n) \geq 0$ be a function on $V$ such that $\leftarrow P_n f_{n+1} = f_n$. Then the sequence $(h_n(x))$ converges pointwise to a harmonic function $H(x)$. Moreover, for every $x \in V$, there exists $n(x)$ such that $h_i(x) = H(x), i \geq n(x)$. 