Order Bounded Maps and Operator Spaces

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Outline

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- Order Bounded Maps
- Operator Spaces
- An Extension Theorem

Ordered Vector Spaces

Ordered Vector Spaces

An **ordered vector space** is a normed vector space V over $\mathbb{R}$, with a subset V_+ satisfying the following.

1. $a, b \in V_+$ implies that $a + b \in V_+$.
2. $a \in V_+$ and $t \geq 0$ implies that $ta \in V_+$.
3. $V_+ \cap -V_+ = \{0\}$.
4. V_+ is a closed subset of V.

We write $a \geq 0$ to mean that $a \in V_+$. We also write $a \leq b$ whenever $b - a \in V_+$.

An ordered vector space V is called an **order unit space** if there exists an element $e \in V$ (the **order unit**) s.t. for each $a \in V$ there is a $t > 0$ with $-te \leq a \leq te$.

An order unit e is **Archimedean** if $te + a \geq 0$ for all $t > 0$ implies that $a \geq 0$. When an OU space V has an Archimedean order unit, we call V an **Archimedean Order Unit Space** (AOU space).

Setting $\|a\| = \inf\{t > 0 : -te \leq a \leq te\}$ defines a norm on V.

Order Bounds

Definition (Order Bounds)

Let V be an ordered vector space and $a \in V$. We define the **positive order bound** of a, denoted $|a|_+$ via the formula

$$|a|_+ = \inf\{\|a + p\| : p \geq 0\} = dist(a, -V_+).$$

We define the **negative order bound** of a, denoted $|a|_-$ via the formula

$$|a|_- = \inf\{\|a - p\| : p \geq 0\} = dist(a, V_+).$$

Easy observations:

- $a \mapsto |a|_+$ and $a \mapsto |a|_-$ are non-negative sublinear functionals.
- $a \geq 0$ if and only if $|a|_- = 0$.
- If $|a|_+ = |a|_- = 0$, then $a = 0$.

In general, we call a sublinear functional $\omega : V \to \mathbb{R}_+$ **proper** if $\omega(x) = \omega(-x) = 0$ implies $x = 0$.

Example (Trivial Positive Cone)

Suppose that V is an ordered vector space and that $V_+ = \{0\}$. Then for all $a \in V_+$, $|a|_+ = |a|_- = \|a\|$.

Example (AOU Spaces)

Let V be an AOU space. then

$$|a|_+ = \inf\{t > 0 : a \leq te\}$$

and

$$|a|_- = \inf\{t > 0 : a \geq -te\}.$$

Order Bounded Maps

Order Bounded Maps

Definition

Let $S \subset V$, and W be ordered vector spaces, and $\phi : S \to W$ be a linear map. We call ϕ V**-order bounded** if there exists a constant $C > 0$ such that $|\phi(a)|_{+,W} \le C|a|_{+,V}$ and $|\phi(a)|_{-,W} \le C|a|_{-,V}$ for all $a \in S$.

Note:

- Order bounded maps are automatically positive. For, suppose that $x \in S_+$. Then $|\phi(x)|_{-,W} \le C|x|_{-,V} = 0$. So $\phi(x) \in W_+$.
- One inequality suffices: $|\phi(x)|_{+,W} \le C|x|_{+,V}$ for all $x \in S$ implies ϕ is V-order bounded.
- A functional $\phi : S \to \mathbb{R}$ is order bounded if and only if ϕ is positive and bounded.

Order Bounded Functionals

Example

Let S be the span of the element $x = (-2, 1)$ in $V = (\mathbb{R}^2, \|\cdot\|_\infty)$. Then $f : S \to \mathbb{R}$ defined by $f(x) = 2$ is positive (trivially) and contractive. If $\tilde{f}$ is any positive extension of f to $V = \mathbb{R}^2$, observe that $\tilde{f}((1,1)) \geq f(x) = 2$. Thus, $\tilde{f}$ cannot be contractive. So f has no positive contractive extension to V.

The above example shows that the Hahn-Banach theorem fails (in general) in the category of ordered vector spaces. However,

Theorem

Let $S \subset V$ be ordered vector spaces, $f : S \to \mathbb{R}$ a positive bounded functional. Then there exists a positive bounded extension $\tilde{f} : V \to \mathbb{R}$ with $\|\tilde{f}\| = \|f\|$ iff f is V-order bounded.

Operator Spaces

We call a vector space V over $\mathbb{C}$ a $*$-vector space if it posses an involution $* : V \to V$.

For each $B = (b_{i,j}) \in M_{n,m}(V)$, define $B^* = (b^*_{j,i}) \in M_{m,n}(V)$.

Definition

Let V be a $*$-vector space, together with a sequence of proper sublinear functionals $\{\omega_n : M_n(V)_{SA} \to \mathbb{R}_+\}$ satisfying the following conditions.

1. For each $X \in M_{n,k}$ and $A \in M_n(V)_{SA}$, $\omega_k(X^*AX) \leq \|X\|^2\omega_n(A)$.
2. For each $A \in M_n(V)_{SA}$ and $B \in M_k(V)_{SA}$, $\omega_{n+k}(A \oplus B) = \max\{\omega_n(A), \omega_k(B)\}$.

We call $(V, \{\omega_n\})$ an L^∞-**matricially ordered space** (L^∞-MOS). We call $\{\omega_n\}$ the **order bounds** of V.

Every L^∞-MOS $(V, \{\omega_n\})$ posses a sequence of induced norms $\{\|\cdot\|_{n,m}\}$ and positive cones $\{M_n(V)_+\}$.

For each $A \in M_n(V)_{SA}$, define $\|A\|_n = max\{\omega_n(A), \omega_n(-A)\}$, and for each $B \in M_{n,m}(V)$, define

$$\|B\|_{n,m} = \left\| \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix} \right\|_{n+m}.$$

Set $M_n(V)_+ = \ker(\omega_n)$.

$(V, \{\|\cdot\|_{n,m}\})$ is an operator space, and $M_n(V)_+$ is a closed proper cone in $M_n(V)_{SA}$.

We can build L^∞-MOS's out of concrete self-adjoint operator spaces.

Let V be a self-adjoint operator space, and let $S \subseteq V$ be a self-adjoint subspace of V. For each $n \in \mathbb{N}$, define $\omega_n : M_n(S)_{SA} \to \mathbb{R}$ by setting

$$\omega_n(A) = |A|_{-,M_n(V)_{SA}}$$

for each $A \in M_n(S)_{SA}$. Then $(S, \{\omega_n\})$ is an L^∞-matrically ordered space.

Each self-adjoint operator space has potentially many representations as an L^∞-MOS.

Definition

Let $(V, \{\rho_n\})$ and $(W, \{\omega_n\})$ be L^∞-MOS's. We call a linear map $\phi : V \to W$ **completely order bounded** if there exists some $C > 0$ such that $\omega_n(\phi^{(n)}(A)) \leq C\rho_n(A)$ for all $A \in M_n(V)_{SA}$ and $n \in \mathbb{N}$. We call ϕ a **complete order-bound isometry** if $\omega_n(\phi^{(n)}(A)) = \rho_n(A)$ for all $A \in M_n(V)_{SA}$ and $n \in \mathbb{N}$. A bijection which is also a complete order-bound isometry will be called an **complete order-bound isomorphism**.

Unitizations

Definition

Let $(V, \{\omega_n\})$ be an L^∞-MOS. Algebraically, set $M_{n,m}(V_1) = M_{n,m}(V) \oplus M_{n,m}$. For each $A \in M_{n,m}(V)$ and $X \in M_{n,m}$, define $(A, X)^* = (A^*, X^*) \in M_{m,n}(V_1)$. For $X \in M_n$, let $X_t = X + tI_n$. Define $u_n : M_n(V_1)_{SA} \to \mathbb{R}_+$ by

$$u_n(A, X) = \inf\{t > 0 : X_t > 0, \omega_n((X_t)^{-1/2} A (X_t)^{-1/2}) \leq 1\}.$$

Theorem

Let $(V, \{\omega_n\})$ be an L^∞-matricially ordered space. Then $(V_1, \{u_n\})$ described in the above definition is an L^∞-MOS. For each $n \in \mathbb{N}$, $(0, I_n) \in M_n(V_1)$ is a archimedean order unit (so $(V_1, (0, 1))$ is an abstract operator system). The mapping $a \mapsto (a, 0)$ from V to V_1 is completely order-bound isometric.

We now have the main result.

Theorem

Let $(V, \{\omega_n\})$ be an L^∞-matricially ordered space. Then there exists a Hilbert space H and a self-adjoint operator space $V' \subset B(H)$, and a complete order-bound isomorphism $\phi : V \to V'$, where V' is regarded as an L^∞-matricially ordered space $(V', \{\omega'_n\})$ by setting $\omega'_n(A) = |A|_{-,M_n(V')_{SA}}$ for each $A \in M_n(V')_{SA}$.

Proof: Let V_1 be the unitization described on the last slide. By the Choi-Effros characterization of operator systems, we can identify V_1 with a subspace of $B(H)$ for some Hilbert space H. Take $V' = \{(a, 0) : a \in V\} \subset V_1$. The V is completely order-isomorphic to V' by the preceding theorem. $\square$

An Extension Theorem

When (W, e) is an operator system, we will regard it as an L^∞-MOS by setting

$$\rho_n(A) = |A|_- = \inf\{t > 0 : A \geq -t(I_n \otimes e)\}$$

for each n.

Lemma

Let $(V, \{\omega_n\})$ be an L^∞-MOS and W be an operator system with order unit $e \in W$. Suppose that $\phi : V \to W$ is completely order contractive. Define an extension $\phi_1 : V_1 \to W$ by setting $\phi_1((0,1)) = e$. Then ϕ_1 is completely positive.

Second main result:

Theorem

Let S,V be L^∞-MOS's with $S \subset V$, and let H be a Hilbert space. Let $\phi : S \to B(H)$ be a completely order contractive map. Then there exists a completely order contractive map $\tilde{\phi} : V \to B(H)$ extending ϕ.

Proof: Apply the lemma to ϕ and to the inclusion map $S \subset V_1$ to obtain:

$$\begin{array}{cccc} & V_1 & & \\ & \cup\!\mid & & \\ \phi_1 : & S_1 & \to & B(H) \end{array}$$

Apply the Arveson extension theorem, and restrict. $\square$

Definition

Let V, S and W be self-adjoint operator spaces with $S \subset V$. We call a map $\phi : S \to W$ **V-order bounded** if ϕ is self adjoint and if the restricted map $\phi : S_{SA} \to W_{SA}$ is V-order bounded. We call a map $\phi : S \to W$ **completely V-order bounded** if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $A \in M_n(V)_{SA}$, $|\phi^{(n)}(A)|_{-,M_n(W)_{SA}} \leq C|A|_{-,M_n(V)}$.

Theorem

Let V and W be self-adjoint operator spaces and $\phi : V \to W$ be a linear map. Then ϕ is completely order bounded if and only if ϕ is completely positive and completely bounded.

Corollary

Let S,V be self-adjoint operator spaces with $S \subset V$, and let H be a Hilbert space. Let $\phi : S \to B(H)$ be a completely contractive and completely positive linear map. Then there exists a completely contractive completely positive extension $\tilde{\phi} : V \to B(H)$ if and only if ϕ is completely V-order contractive.

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The End - thanks for listening!