Positivity in Function Algebras

Jason Ekstrand

Intel Corporation

INFAS, March 2015
What is a functional analyst doing at Intel?

Not functional analysis.

- I work on the Open-source 3-D graphics driver team
- Modern graphics cards are specialized processors that perform moderate calculations millions of times per second.
- My work has focused on the compiler for Intel GPUs
- My work so far has been:
  - 20% Graph Theory
  - 15% Algebraic Identities/Reductions
  - 65% Problem Solving and writing C Code
Overview

- Introduction
  - Problem Statement
  - Notation
- Positivity
  - Positivity in the Disc
  - Positivity in the Annulus
  - Positivity in more General Domains
- Connections with Representation Theory
- Future Work
- References
Problem Statement

Let $\mathcal{A}(\mathbb{D})$ be the disc algebra and give $\mathcal{A}(\mathbb{D})$ the involution

$$f \mapsto f^*; \quad f^*(z) = \overline{f(\bar{z})}$$

This yields a Banach $*$-algebra that is not a $C^*$-algebra.

Properties of $\mathcal{A}(\mathbb{D}, \cdot)$

- $\mathcal{A}(\mathbb{D})$ (without the involution) is a norm-closed subalgebra of $C(\mathbb{T})$ so it is an operator algebra
- $\mathcal{A}(\mathbb{D}, \cdot)$ is a $*$-subalgebra of $C[-1, 1]$
- For every $f \in \mathcal{A}(\mathbb{D})$, $\sigma(f) = f(\mathbb{D}^-)$
We wish to study the positive elements of $\mathcal{A}(\mathbb{D}, \ast)$.

**Definition**

Let $\mathcal{A}$ be a general $\ast$-algebra (no assumptions of norm). Then the set of *positive* elements of $\mathcal{A}$, denoted $\mathcal{A}_+$, is given by

$$\mathcal{A}_+ = \left\{ \sum_k a_k^* a_k : a_k \in \mathcal{A} \right\}.$$

**Definition**

Let $\mathcal{A}$ be a unital $C^*$-algebra. Then an element $a \in \mathcal{A}$ is said to be *positive* if $a^* = a$ and $\sigma(a) \subseteq \mathbb{R}_+$. 

---

**Jason Ekstrand**

**Positivity in Function Algebras**
What is a good definition of positivity in $\mathcal{A}(\mathbb{D},*)$?

**Definition**

Let $f \in \mathcal{A}(\mathbb{D},*)$. Then $f$ is said to be *positive* if

$$f([-1,1]) \subseteq \mathbb{R}^+.$$

Is this the right definition?

**Theorem (Ekstrand & Peters, 2013)**

Let $f \in \mathcal{A}(\mathbb{D},*)$. Then $f$ is positive (as defined above) if and only if $f = g^*g$ for some $g \in \mathcal{A}(\mathbb{D})$. 

---

**Jason Ekstrand**  
**Positivity in Function Algebras**
For a domain $G \subseteq \mathbb{C}$, we have the following algebras:

- $\mathcal{H}(G)$ of holomorphic functions on $G$
- $H^\infty(G)$ of bounded holomorphic functions on $G$
- $\mathcal{A}(G)$ of bounded holomorphic functions on $G$ which have continuous extension to $G^-$

If $f : G \to \mathbb{C}$ and $r \mathbb{T} \subseteq G$ and, we define the function

$$f_r : [-\pi, \pi] \to \mathbb{C}; \quad f_r(t) = f(re^{it}).$$

When it makes sense, we define the $p^{th}$ Hardy space

$$H^p(G) = \{ f \in \mathcal{H}(G) : \|f_r\|_p \text{ is bounded in } r \}$$
We begin with the case of non-vanishing functions.

Let $f \in A(\mathbb{D})$ be non-vanishing. Since $\mathbb{D}$ is simply connected,

$$f(z) = e^{h(z)} \text{ for some } h \in \mathcal{H}(\mathbb{D}).$$

However, $h$ need be neither bounded nor continuous on $\mathbb{D}^-$.

**Lemma**

*Suppose $h : \mathbb{D} \to \mathbb{C}$ is continuous and that there is a continuous function $F : \mathbb{D}^- \to \mathbb{C}$ with $F = e^h$ on $\mathbb{D}$. If $K$ is the set of zeros of $F$ on $\mathbb{T}$ then $h$ can be continuously extended to $\mathbb{D}^- \setminus K$.***
Theorem

Let $f \in \mathcal{H}(\mathbb{D})$ be positive with no roots in $\mathbb{D}$. Then, for every integer $n > 0$ there is a unique positive function $g \in \mathcal{H}(\mathbb{D})$ such that $f = g^n$. If $f \in H^p(\mathbb{D})$ for some $1 \leq p \leq \infty$, then $g \in H^{np}(\mathbb{D})$. If $f \in \mathcal{A}(\mathbb{D})$, then $g \in \mathcal{A}(\mathbb{D})$.

Sketch of proof.

- $f = e^h$ for some $h \in \mathcal{H}(\mathbb{D})$; let $g = e^{h/n}$ on $\mathbb{D}$
- Define $x : \mathbb{T} \to \mathbb{C}$ as $x = e^{h/n}$ on $\mathbb{T} \setminus K$ and $x = 0$ on $K$
- Then $x$ is continuous on $\mathbb{T}$ and $x$ is a.e. the boundary values of $g$ so $g \in \mathcal{A}(\mathbb{D})$. 

Jason Ekstrand  
Positivity in Function Algebras
BSF Factorization

For any function $f \in H^1(\mathbb{D})$, we can write $f = BSF$ where

$$F(z) = \lambda \exp \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f(e^{i\theta})| \, d\theta \right],$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$B(z) = z^{\rho_0} \prod_{n=1}^{\infty} \left[ \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \right]^{p_n}$$

where $\{\alpha_n\}$ are the roots of $f$ with multiplicities $p_n$ and

$$S(z) = \exp \left[ - \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \right]$$

for some singular positive measure $\mu$ on $[-\pi, \pi]$. 
If we are going to use the BSF factorization, we need to handle the positivity and continuity of the different pieces.

**Theorem**

Let $f \in \mathcal{A}(\mathbb{D})$ and decompose $f$ as $f = gB$ where $g \in H^\infty(\mathbb{D})$ and $B$ is a Blaschke product. Then $g \in \mathcal{A}(\mathbb{D})$ and $g$ has the same zeros on $\mathbb{T}$ as $f$.

**Theorem**

Let $f \in \mathcal{A}(\mathbb{D})$ and let $B$ be a Blaschke product such that $f(z) = 0$ whenever $z$ is a limit point of the roots of $B$. Then $fB \in \mathcal{A}(\mathbb{D})$. 
Theorem

Let $B$ be the Blaschke product. If $B$ has the same roots as some positive $f \in \mathcal{H}(\mathbb{D})$, then there is another Blaschke product $B_+$ with $B = B_+^* B_+$. 

Theorem

Let $f \in H^p(\mathbb{D})$ for some $1 \leq p \leq \infty$. Then $f$ is positive if and only if there exists $g \in H^{2p}(\mathbb{D})$ so that $f = g^* g$. If $f \in A(\mathbb{D})$ then $g$ may also be chosen to be in $A(\mathbb{D})$. 
Definition

Fix $0 < r_0 < 1$ and define the annulus

$$A = \{ z \in \mathbb{C} : r_0 < |z| < 1 \}.$$ 

We define the following algebras:

- $\mathcal{H}(A)$ of all holomorphic functions on $A$,
- $H^p(A)$ of all holomorphic functions on $A$ with $\|f_r\|_p$ bounded for $r_0 < r < 1$,
- $\mathcal{A}(A)$ of all holomorphic functions on $A$ with continuous extension to $A^-$.
Properties of $\mathcal{H}(A)$

Given a function $f \in \mathcal{H}(A)$, we have the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \frac{1}{z^n}$$

so $f(z) = g(z) + h(r_0/z)$ where $g, h \in \mathcal{H}(\mathbb{D})$.

**Observation**

- $f \in H^p(A)$ if and only if $g, h \in H^p(\mathbb{D})$
- $f \in A(A)$ if and only if $g, h \in A(\mathbb{D})$
- $f \in H^p(A)$ can be recovered from its boundary values
How do we study positive functions on $A$?

For $f \in \mathcal{H}(A)$, $f(z) = g(z) + h(r_0/z)$ where $g, h \in \mathcal{H}(D)$. However, $f$ positive does not imply that $g$ or $h$ is positive.

$f \in \mathcal{H}(A)$ non-vanishing does not imply $f = e^g$.

How do we replace our use of the BSF factorization?
The problem here is that $A$ is not simply connected.

**Theorem**

*Let $G$ be a domain and $f$ be holomorphic on $G$. Suppose $f$ is non-vanishing and*

$$
\oint_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0
$$

*for every simple closed curve $\gamma$. Then there exists a holomorphic function $g$ on $G$ so that $f = e^g$.**
Definition

For $f \in \mathcal{H}(A)$ non-vanishing, define the winding number of $f$ by

$$\text{wn}(f) = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f'(z)}{f(z)} \, dz$$

where $\gamma_r(t) = re^{it}$ for $t \in [-\pi, \pi]$ and $r_0 < r < 1$.

Theorem (Ekstrand, 2014)

Let $f \in \mathcal{H}(A)$ be positive and non-vanishing. Then $\text{wn}(f)$ is an even number.

For any positive $f \in \mathcal{H}(A)$, the function $g(z) = f(z)z^{-\text{wn}(f)}$ is positive with $\text{wn}(g) = 0$. 
Theorem (Ekstrand, 2014)

Let \( f \in \mathcal{H}(A) \) be positive and non-vanishing. Then there exists a function \( g \in \mathcal{H}(A) \) so that \( f = g^*g \). Furthermore, if \( f \in H^p(A) \), then \( g \in H^{2p}(A) \) for \( 1 \leq p \leq \infty \) and, if \( f \in \mathcal{A}(A) \), then \( g \in \mathcal{A}(A) \).

Sketch of proof.

- Let \( f_0(z) = f(z)z^{-\text{wn}(f)}; \text{wn}(f_0) = 0 \).
- \( f_0 = e^h \) for some \( h \in \mathcal{H}(A) \).
- Define \( g \) by \( g(z) = e^{h(z)/2}z^{\text{wn}(f)/2} \).
- Continuity is similar to the disc case.
In his 1965 work, Sarason studies holomorphic functions on $A$ and tries to recover a BSF factorization for the annulus.

- Sarason’s work focuses on the universal covering surface

$$\hat{A} = \{(r, t) \in \mathbb{R}^2 : r_0 < r < 1\}$$

with the covering map

$$\varphi : \hat{A} \rightarrow A; \quad \varphi(r, t) = re^{it}.$$ 

- Sarason develops a BSF factorization for \textit{modulus automorphic} functions $\hat{A}$

- Unfortunately, these result don’t translate easily to $H^p(A)$
Sarason’s construction is enough to get us the following:

**Theorem (Sarason, 1965; Ekstrands, 2014)**

Let \( f \in H^\infty(A) \) that is not identically zero and let \( \{ a_n \}_{n=1}^\infty \) be the set of zeros of \( f \) repeated according to multiplicity. Then

\[
\sum_{n=1}^{\infty} \min \left( 1 - |a_n|, 1 - \frac{r_0}{|a_n|} \right) < \infty.
\]
Theorem (Ekstrands, 2014)

Let \( f \in H^\infty(A) \) that is not identically zero and let \( \{a_n\} \) be the roots of \( f \) repeated according to multiplicity. Then the Blaschke products

\[
B_1(z) = \prod_{|a_n| \geq \sqrt{r_0}} \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z} \quad \text{and} \quad B_2(z) = \prod_{|a_n| < \sqrt{r_0}} \frac{a_n}{|a_n|} \frac{r_0/a_n - z}{1 - (r_0/\bar{a}_n)z}
\]

converge and we may decompose \( f \) as \( f(z) = g(z)B_1(z)B_2(r_0/z) \) where \( g \) is bounded, holomorphic, and non-vanishing on \( A \). If \( f \) has a continuous extension to \( A^- \) then so does \( g \).

Theorem (Ekstrands, 2014)

An element \( f \in H^p(A) \) is positive if and only if \( f = g^*g \) for some \( g \in H^{2p}(A) \). Furthermore, if \( f \) is continuous on \( A^- \), then \( g \) may be chosen continuous on \( A^- \).
Generalizations to other domains

Definition

Let $G$ be a domain. We say that $G$ is symmetric if

$$G = G^* = \{ \bar{z} : z \in G \}.$$

Theorem (Ekstrand, 2014)

Let $G$ be a symmetric domain where $\partial G$ is the union of finitely many disjoint Jordan curves and let $f \in H^\infty(G)$. Then $f$ is positive if and only if there is some $g \in H^\infty(G)$ so that $f = g^* g$. Furthermore, if $f \in \mathcal{A}(G)$ then $g$ may be chosen in $\mathcal{A}(G)$. 
Definition

Let $\mathcal{A}$ be a $\ast$-algebra. Then a $\ast$-representation of $\mathcal{A}$ is a pair $(\mathcal{H}, \varphi)$ where $\mathcal{H}$ is a Hilbert space and $\varphi : \mathcal{A} \to B(\mathcal{H})$ is a $\ast$-homomorphism.

What about $\mathcal{A}(G, \ast)$?

- If $(\mathcal{H}, \varphi)$ is a $\ast$-representation of $\mathcal{A}$ then, for all $g \in \mathcal{A}$, $\varphi(g^* g) = \varphi(g)^* \varphi(g)$ is positive in $\mathcal{H}$.
- The one-dimensional $\ast$-representations of $\mathcal{A}(G, \ast)$ are exactly the point-evaluations on $G \cap \mathbb{R}$.
- $f \in \mathcal{A}(G, \ast)$ is positive if and only if $\varphi(f) \geq 0$ for every one-dimensional $\ast$-representation $\varphi$ of $\mathcal{A}(G, \ast)$. 
Theorem (Ekstrand, 2014)

Let $G$ be a region so that $\partial G$ is the union of finitely many disjoint Jordan curves in $\mathbb{C}_\infty$. For each $f \in \mathcal{A}(G)$, TFAE:

1. $f$ is positive, i.e., $f(G \cap \mathbb{R}) \geq 0$,
2. $f = g^* g$ for some $g \in \mathcal{A}$,
3. $f = \sum_{i=1}^{n} g_i^* g_i$ for some $g_1, \ldots, g_n \in \mathcal{A}$,
4. $f = \lim_{n \to \infty} f_n$ where each $f_n$ is of the form given in 3.
5. $\varphi(f) \geq 0$ for every one-dimensional $\ast$-rep. $(\mathbb{C}, \varphi)$ of $\mathcal{A}(G)$

5. is equivalent to $\sigma(a) \geq 0$ in abelian $C^*$-algebras
Future Work

1. Extend the results to even more general domains
   - While the restriction that $\partial G$ is the union of finitely many disjoint Jordan curves is sufficient, I have no proof that it is necessary.
   - Unfortunately, such an extension would probably need a new technique.

2. Try and extend these results to a non-abelian case
   - These definitions extend fairly easily to $\mathcal{M}_{n \times n}(\mathcal{A}(G))$

3. Consider domains not in $\mathbb{C}$ such as Riemann surfaces
   - There is a 1965 paper by Voichick and Zalcman that gives a BSF factorization for a certain class of Riemann surfaces
References I


References II


Thank You!
**Theorem (Riemann Mapping Theorem)**

Let $G \subseteq \mathbb{C}$ be a simply connected region that is not the whole plane and let $a \in G$. Then there is a unique holomorphic bijection $\phi : G \to \mathbb{D}$ so that $\phi(a) = 0$ and $\phi'(a) > 0$.

**Theorem (Carathéodory)**

Let $G \subseteq \mathbb{C}$ be a simply connected region whose boundary is a Jordan curve. Then the Riemann map $\phi : G \to \mathbb{D}$ extends to a homeomorphism $\Phi : G^- \to \mathbb{D}^-$. 
Sketch of Proof

Start with some symmetric region $G$ and $f \in H^\infty(G)$
Sketch of Proof

Pick a single hole $H$ in $G$
Define a Carathéodory map \( \phi : \mathbb{C} \setminus H \rightarrow \mathbb{D}^- \)
Pick $r_0$ so that $\{ z \in \mathbb{C} : r_0 \leq |z| < 1 \} \subseteq \phi(G)$
This gives us an annulus $A = \{z \in \mathbb{C} : r_0 \leq |z| < 1\}$.

We can factor $f \circ \phi^{-1}$ as $f \circ \phi^{-1} = gB$ where $g \in H^\infty(A)$ and $B$ is a Blaschke product.

Translating back to $G$, $f = (g \circ \phi)(B \circ \phi)$.

A similar trick can be used to ensure $\text{wn}(f \circ \varphi^{-1}) = 0$.

Decompose $f$, square root the non-vanishing part and put it back together as we did before.

Thanks to the Carathéodory theorem, $\phi$ is a homeomorphism of $\mathbb{C} \setminus H$ and $\mathbb{D}^-$ so continuity follows from results in the annulus.