On Spectra of a Cantor Measure

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Overview

Consider the Cantor set obtained from the interval $[0, 1]$, dividing it into four equal intervals and keeping the first and the third, $[0, 1/4]$ and $[1/2, 3/4]$, and repeating the procedure. This can be described in terms of iterated function systems: let

$$\tau_0(x) = x/4 \text{ and } \tau_2(x) = (x + 2)/4, \quad (x \in \mathbb{R}).$$

The Cantor set $X_4$ is the unique compact set that satisfies the invariance condition

$$X_4 = \tau_0(X_4) \cup \tau_2(X_4).$$

The set $X_4$ is described also in terms of the base 4 decomposition of real numbers:

$$X_4 = \left\{ \sum_{k=1}^{n} 4^{-k}b_k : b_k \in \{0, 2\}, n \in \mathbb{N} \right\}.$$
Overview

On the set $X_4$ one considers the Hausdorff measure $\mu$ of dimension $\log_4 2 = \frac{1}{2}$. In terms of iterated function systems, the measure $\mu$ is the invariant measure for the iterated function system, that is, the unique Borel probability measure that satisfies the invariance equation

$$\mu(E) = \frac{1}{2} \left( \mu(\tau_0^{-1} E) + \mu(\tau_2^{-1} E) \right), \text{ for all Borel sets } E \subset \mathbb{R}. \quad (0.1)$$

Equivalently, for all continuous compactly supported functions $f$,

$$\int f \, d\mu = \frac{1}{2} \left( \int f \circ \tau_0 \, d\mu + \int f \circ \tau_2 \, d\mu \right). \quad (0.2)$$
Overview

We denote, for $\lambda \in \mathbb{R}$:

$$e^\lambda(x) = e^{2\pi i \lambda \cdot x}, \quad (x \in \mathbb{R}).$$

The Hilbert space $L^2(\mu)$ has an orthonormal basis formed with exponential functions, i.e., a Fourier basis, $E(\Gamma_0) := \{e^\lambda : \lambda \in \Gamma_0\}$ where

$$\Gamma_0 := \left\{ \sum_{k=0}^{n} 4^k l_k : l_k \in \{0, 1\}, n \in \mathbb{N} \right\}. \quad (0.3)$$

Definition

We say that the subset $\Gamma$ of $\mathbb{R}$ is a spectrum for the measure $\mu$ if the corresponding family of exponential $E(\Gamma) := \{e^\lambda : \lambda \in \Gamma\}$ is an orthonormal basis for $L^2(\mu)$. We say that $\Gamma$ is complete/incomplete if the set $E(\Gamma)$ is as such in $L^2(\mu)$. 
Overview

Question

For what digits \( \{0, m\} \) with \( m \in \mathbb{N} \) odd is the set

\[
\Gamma(m) := m\Gamma_0 \left\{ \sum_{k=0}^{n} 4^k l_k : l_k \in \{0, m\}, n \in \mathbb{N} \right\}
\]

a spectrum for \( L^2(\mu) \)?
Extreme Cycles

Definition

Let \( m \in \mathbb{N} \) be an odd number. We say that a finite set \( \{x_0, x_1, \ldots, x_{r-1}\} \) is an extreme cycle (for the digits \( \{0, m\} \)) if there exist \( l_0, \ldots, l_{r-1} \in \{0, m\} \) such that

\[
x_1 = \frac{x_0 + l_0}{4}, \quad x_2 = \frac{x_1 + l_1}{4}, \quad \ldots, \quad x_{r-1} = \frac{x_{r-2} + l_{r-2}}{4}, \quad x_0 = \frac{x_{r-1} + l_{r-1}}{4},
\]

and

\[
\left| 1 + e^{2\pi i 2x_k} \right| = 1, \quad (k \in \{0, \ldots, r - 1\}). \quad (0.4)
\]

The points \( x_i \) are called extreme cycle points.
Theorem

Let $m \in \mathbb{N}$ be odd. The set $\Gamma(m)$ is a spectrum for the measure $\mu$ if and only if the only extreme cycle for the digit set $\{0, m\}$ is the trivial one $\{0\}$. 
Extreme Cycles: Examples

Recall

\[ x_1 = \frac{x_0 + l_0}{4}, \quad x_2 = \frac{x_1 + l_1}{4}, \quad \ldots, \]
\[ x_{r-1} = \frac{x_{r-2} + l_{r-2}}{4}, \quad x_0 = \frac{x_{r-1} + l_{r-1}}{4}, \]

where \( l_j \in \{0, m\} \).

Let \( m = 3 \).

\[ \frac{1 + 3}{4} = 1, \]

so \( \{1\} \) is an extreme cycle for the digit set \( \{0, 3\} \).

Let \( m = 85 \).

\[ \frac{7 + 85}{4} = 23, \quad \frac{23 + 85}{27} = 1, \quad \frac{27 + 85}{4} = 28, \quad \frac{28 + 0}{4} = 7, \]

so \( \{7, 23, 27, 28\} \) is an extreme cycle for the digit set \( \{0, 85\} \).
Lemma

If $x_0$ is an extreme cycle point then $x_0 \in \mathbb{Z}$, $x_0$ has a periodic base 4 expansion

$$x_0 = \frac{a_0}{4} + \frac{a_1}{4^2} + \cdots + \frac{a_{r-1}}{4^r} + \frac{a_0}{4^{r+1}} + \cdots + \frac{a_{r-1}}{4^{2r}} + \cdots,$$

(0.5)

with $a_k \in \{0, m\}$, and $0 \leq x_0 \leq \frac{m}{3}$.

Proposition

If $\Gamma(m)$ is incomplete then $\Gamma(km)$ is incomplete for all $k \in \mathbb{Z}$, $k \geq 1$. 
Let $m > 3$ be an odd number not divisible by 3. Let $G = \{4^j(\text{mod } m)| j \in \mathbb{N}\}$. If any of the numbers $−1(\text{mod } m)$, $−2(\text{mod } m)$, or $2(\text{mod } m)$, then $\Gamma(m)$ is complete.
Prime Powers

Assume for contradiction's sake that $\Gamma(m)$ is not spectral. Then there is a non-trivial extreme cycle $X = \{x_0, \ldots, x_{r-1}\}$ for the digit set $\{0, m\}$. From the relation between the cycle points,

$$x_{j+1} = \frac{x_j + b_j}{4}, \quad (0.6)$$

where $b_j \in \{0, m\}$, we have that $4x_{j+1} \equiv x_j (\text{mod } m)$. Thus,

$$4^{r-k}x_0 \equiv x_0 (\text{mod } m, k \in \{0, \ldots, r\}), \quad (0.7)$$

so, for all $k \in \mathbb{N}$, the number $4^k x_0$ is congruent modulo $m$ with an element of the extreme cycle $X$. But then, the hypothesis implies that there is a number $c \in \{-1, 2, -2\}$, so the number $cx_0$ is congruent modulo $m$ with an element in $X$, and since $x_0$ is arbitrary in the cycle, we get that $cx_j$ is congruent to an element in $X$ for any $j$. 
In the following arguments we use the fact that since $m$ is not divisible by 3, the condition on cycle points $0 \leq x_j \leq \frac{m}{3}$ implies $0 \leq x_j < \frac{m}{3}$.

If $c = -1$, then $-x_0 \pmod{m} \in X$. Since $x_0 < \frac{m}{3}$, $-x_0 \pmod{m} > \frac{m}{3}$, a contradiction.

If $c = -2$, then $-2x_0 \pmod{m} \in X$. Since $x_0 < \frac{m}{3}$, $-2x_0 \pmod{m} > \frac{m}{3}$, a contradiction.

If $c = 2$, then $2x_j \pmod{m} \in X$ for all $j$. Let $x_N$ be the largest element of the extreme cycle. Since $x_N < \frac{m}{3}$, $2x_N \pmod{m} = 2x_N$. This number is in $X$, a contradiction to the maximality of $x_N$. 
Prime Powers

**Theorem**

If $p$ is a prime number, $p > 3$ and $n \in \mathbb{N}$, then $\Gamma(p^n)$ is complete.

It is well known that the equation $x^2 \equiv b \pmod{p^n}$ has zero or two solutions.

Let $a$ be the smallest positive integer such that $4^a \equiv 1 \pmod{p^n}$.  
If $a$ is even, then we have $(4^{a/2})^2 \equiv 1 \pmod{p^n}$ so $4^{a/2} \equiv \pm 1 \pmod{p^n}$.  
Since $4^{a/2} \not\equiv 1 \pmod{p^n}$ we get $4^{a/2} \equiv -1 \pmod{p^n}$.  
If $a$ is odd, then $(4^{a+1}/2)^2 \equiv 4 \pmod{p^n}$.  Therefore $4^{a+1}/2 \equiv \pm 2 \pmod{p^n}$.  
In both cases, the result follows from the previous Theorem.
Composite Numbers

Definition

We say that an odd number $m$ is primitive if $\Gamma(m)$ is incomplete and, for all proper divisors $d$ of $m$, $\Gamma(d)$ is complete. In other words, there exist extreme cycles for the digits $\{0, m\}$ and there are no extreme cycles for the digits $\{0, d\}$ for any proper divisor $d$ of $m$.

For an integer $m$, the order of 4 in the group $U(\mathbb{Z}_m)$ is the smallest positive integer $a$ such that $4^a \equiv 1 \pmod{m}$. We denote $a$ by $o_4(m)$, and the set of powers of 4 in $U(\mathbb{Z}_m)$ by $G$. 
Proposition

Let $m$ be a primitive number and let $C = \{x_0, \ldots, x_{p-1}\}$ be an extreme cycle. Then:

1. The length $p$ of the cycle is equal to $o_4(m)$.
2. Every element of the cycle $x_j$ is mutually prime with $m$.
3. The extreme cycle $C$ is a coset of the group $G$: $C = x_0G$. 
Composite Numbers

Theorem

There are infinitely many primitive numbers.

Proposition

Let $m$ and $n$ be mutually prime odd integers. Then

\[ o_4(mn) = \text{lcm}(o_4(m), o_4(n)). \]

Definition

For a prime number $p \geq 3$, we say that $p$ is simple if $o_4(p) < o_4(p^2)$. 
Composite Numbers

Proposition

Let $m$ be an odd number. If

$$o_4(m) > \sqrt[3]{\frac{4m}{3}}$$

then $m$ cannot be primitive.

Lemma

Let $a, b \geq 1$ be some odd numbers. Assume that $o_4(ab) > \frac{a}{3}o_4(b)$. Then $ab$ cannot be primitive.
Corollary

Let $p_1, \ldots, p_r$ be distinct prime numbers strictly larger than 5. Assume the following conditions are satisfied:

1. The numbers $o_4(p_1), \ldots, o_4(p_r), p_1, \ldots, p_r$ are mutually prime.
2. $o_4(p_g) = \frac{p_g-1}{2}$ for some $g$, and $p_g$ is simple.

Then the set $\Gamma(p_1^{k_1} \ldots p_r^{k_r})$ is not primitive for any $k_1 \geq 0, \ldots, k_r \geq 0$ provided that $k_g \geq 1$. 
Corollary

Let \( p_1, \ldots, p_r \) be distinct simple prime numbers strictly larger than 3. Assume the following conditions are satisfied:

1. The numbers \( o_4(p_1), \ldots, o_4(p_r), p_1, \ldots, p_r \) are mutually prime.

2. \( o_4(p_j) > \sqrt[4]{\sqrt[3]{4}} p_j \) for all \( j \).

Then the set \( \Gamma(p_1^{k_1} \ldots p_r^{k_r}) \) is complete for any \( k_1 \geq 0, \ldots, k_r \geq 0 \).
Thank you!