Cartan MASAs and Exact Sequences of Inverse Semigroups

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Let $\mathcal{M}$ be a von Neumann algebra. A maximal abelian subalgebra (MASA) $\mathcal{D}$ in $\mathcal{M}$ is a Cartan MASA if

1. the unitaries $U \in \mathcal{M}$ such that $UDU^* = U^*DU = \mathcal{D}$ span a weak-$^*$ dense subset in $\mathcal{M}$;
2. there is a normal, faithful conditional expectation $E : \mathcal{M} \to \mathcal{D}$.

We will call the pair $(\mathcal{M}, \mathcal{D})$ a Cartan pair. We call the normalizing partial isometries groupoid normalizers, written $G^\mathcal{M}(\mathcal{D})$. 
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Alternatively

1. the partial isometries $V \in \mathcal{M}$ such that $V\mathcal{D}V^*, V^*\mathcal{D}V \subseteq \mathcal{D}$ span a weak-$^*$ dense subset in $\mathcal{M}$;
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### Examples of Cartan Pairs

#### Example

Let $M_n$ be the $n \times n$ complex matrices, and let $D_n$ be the diagonal $n \times n$ matrices. Then $(M_n, D_n)$ is a Cartan pair:

1. the matrix units normalize $D_n$ and generate $M_n$;
2. The map

   $$E: [a_{ij}] \mapsto \text{diag}[a_{11}, \ldots, a_{nn}]$$

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Example

Let $\mathcal{D} = L^\infty(\mathbb{T})$ and let $\alpha$ be an action of $\mathbb{Z}$ on $\mathbb{T}$ by irrational rotation. Then $L^\infty(\mathbb{T})$ is a Cartan MASA in $L^\infty(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$. 
Example

Let
\[ G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, \ a \neq 0 \right\}, \]
and let
\[ H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}. \]

Then \( H \) is a normal subgroup of \( G \) and \( L(H) \) is Cartan MASA in \( L(G) \).
Feldman and Moore (1977) explored Cartan pairs $(\mathcal{M}, \mathcal{D})$ where $\mathcal{M}_*$ is separable and $\mathcal{D} = L^\infty(X, \mu)$. They showed:

1. there is a measurable equivalence relation $R$ on $X$ with countable equivalence classes and a 2-cocycle $\sigma$ on $R$ s.t.

   $$\mathcal{M} \simeq \mathcal{M}(R, \sigma) \text{ and } \mathcal{D} \simeq \mathcal{A}(R, \sigma),$$

   where $\mathcal{M}(R, \sigma)$ are “functions on $R$” and $\mathcal{A}(R, \sigma)$ are the “functions” supported on diag. $\{(x, x) : x \in X\}$;

2. every sep. acting pair $(\mathcal{M}, \mathcal{D})$ arises this way.
Consider the Cartan pair \((M_3, D_3)\). Let \(G = G_{M_3}(D_3)\). E.g., an element of \(G\) could look like

\[
V = \begin{bmatrix}
0 & \lambda & 0 \\
\mu & 0 & 0 \\
0 & 0 & \gamma
\end{bmatrix},
\]

with \(\lambda, \mu, \gamma \in \mathbb{T}\).

Let \(\mathcal{P} = G \cap D_n\). And let \(S = G/\mathcal{P}\). So elements of \(S\) are of the form

\[
S = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

From \((M_n, D_n)\) we have 3 semigroups: \(\mathcal{P}\), \(G\) and \(S\).
Conversely, starting with $S$, we can construct $P$: $P$ is all the continuous functions from the idempotents of $S$ into $\mathbb{T}$. From $S$ and $P$ we can construct $G$, since every element of $G$ is the product of an element in $S$ and an element in $P$. From $G$ we can construct $(M_n, D_n)$ as the span of $G$. 
Our Objective: Give an alternative approach using algebraic rather than measure theoretic tools which
• conceptually simpler;
• applies to the non-separably acting case.
A semigroup $S$ is an *inverse semigroup* if for each $s \in S$ there is a unique “inverse” element $s^\dagger$ such that

$$ss^\dagger s = s \text{ and } s^\dagger ss^\dagger = s^\dagger.$$

We denote the idempotents in an inverse semigroup $S$ by $E(S)$. The idempotents form an abelian semigroup. For any element $s \in S$, $ss^\dagger \in E(S)$. 
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An inverse semigroup $S$ has a natural partial order defined by

$$s \leq t \text{ if and only if } s = te$$

for some idempotent $e \in \mathcal{E}(S)$. 
Example

Consider the Cartan pair \((M_n, D_n)\) again. Again, let

\[
G = G_{M_n}(D_n)
\]

\[
= \{ \text{partial isometries } V \in M_n : VD_nV^* \subseteq D_n, \ V^*D_nV \subseteq D_n \}.
\]

Then \(G\) is an inverse semigroup:

- If \(V, W \in G\) then

\[
(VW)D_n(VW)^* = V(WD_nW^*)V^* \subseteq D_n,
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so \(VW \in G\);
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  so \( VW \in G; \)
- the “inverse” of \( V \) is \( V^*; \)
- the idempotents are the projections in \( D_n; \)
- \( V \leq W \) if \( V = WP \) for some projection \( P \in D_n. \)
More generally...

**Example**

Let \((\mathcal{M}, \mathcal{D})\) be a Cartan pair. Then the groupoid normalizers \(G_M(\mathcal{D})\) form an inverse semigroup.

- if \(V, W \in G_M(\mathcal{D})\) then
  \[
  (VW)D(VW)^* = V(WDW^*)V^* \subseteq \mathcal{D},
  \]
  so \(VW \in G_M(\mathcal{D})\);
- the “inverse” of \(V\) is \(V^*\);
- the idempotents are the projections in \(\mathcal{D}\);
- \(V \leq W\) if \(V = WP\) for some projection \(P \in \mathcal{D}\).
Let $S$ and $P$ be inverse semigroups. And let

$$\pi : P \to S,$$

be a surjective homomorphism such that $\pi|_{\mathcal{E}(P)}$ is an isomorphism from $\mathcal{E}(P)$ to $\mathcal{E}(S)$. An *idempotent separating extension of $S$ by $P$* is an inverse semigroup $G$ with

$$P \xleftarrow{\iota} G \xrightarrow{q} S$$

and

- $\iota$ is an injective homomorphism;
- $q$ is a surjective homomorphism;
- $q(g) \in \mathcal{E}(S)$ if and only if $g = \iota(p)$ for some $p \in P$;
- $q \circ \iota = \pi$.

Note that $\mathcal{E}(P) \cong \mathcal{E}(G) \cong \mathcal{E}(S)$. 

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**Extensions of Inverse Semigroups**
Let $G$ be an inverse semigroup. Define an equivalence relation \((the Munn congruence)\) $\sim$ on $G$ by

$$s \sim t \text{ if } ses^\dagger = tet^\dagger \text{ for all } e \in \mathcal{E}(G).$$
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If $s \sim t$ and $u \sim v$ then 

$$su \sim tv.$$

Thus $S = G/\sim$ is an inverse semigroup.
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Thus $S = G/\sim$ is an inverse semigroup.

Let $P = \{v \in G : v \sim e \text{ for some } e \in \mathcal{E}(G)\}$. Then $P$ is an inverse semigroup.

And $G$ is an extension of $S$ by $P$:

$$P \hookrightarrow G \twoheadrightarrow S.$$
Let \((\mathcal{M}, \mathcal{D})\) be a Cartan pair. Let

\[ G = G_\mathcal{M}(\mathcal{D}) = \{ v \in \mathcal{M} \text{ a partial isometry: } v\mathcal{D}v^* \subseteq \mathcal{D} \text{ and } v^*\mathcal{D}v \subseteq \mathcal{D} \}. \]
Let \((M, D)\) be a Cartan pair. Let

\[
G = G_M(D) = \{ v \in M \text{ a partial isometry: } vDv^* \subseteq D \text{ and } v^*Dv \subseteq D \}.
\]

Let \(S = G/\sim\), where \(\sim\) is the Munn congruence on \(G\) and let

\[
P = \{ V \in G : V \sim P, \ P \in \text{Proj}(D) \}.
\]

**Definition**

We call the extension \( \overset{}{P} \rightarrow G \rightarrow S \),

the *extension associated to the Cartan pair* \((M, D)\).
Properties of associated extensions

Let $(\mathcal{M}, \mathcal{D})$ be a Cartan pair, and let

$$P \hookrightarrow G \rightarrow S,$$

be the associated extension.

Then $P = \mathcal{G}_\mathcal{M}(\mathcal{D}) \cap \mathcal{D}$, i.e. $P$ is simply the partial isometries in $\mathcal{D}$. 
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The inverse semigroup \( S \) has the following properties

1. \( S \) is fundamental: \( \mathcal{E}(S) \) is maximal abelian in \( S \);
2. \( \mathcal{E}(S) \) is a hyperstonean boolean algebra, i.e. the idempotents are the projection lattice of an abelian \( W^* \)-algebra;
3. \( S \) is a meet semilattice under the natural partial order on \( S \);
4. for every pairwise orthogonal family \( \mathcal{F} \subseteq S \), \( \bigvee \mathcal{F} \) exists in \( S \);
5. \( S \) contains 1 and 0.
Properties of associated extensions

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Definition

An inverse semigroup \(S\), satisfying the conditions above is called a Cartan inverse monoid.
Example

In the matrix example \((M_n, D_n)\), the semigroups \(P\), \(G\) and \(S\) are the semigroups discussed earlier:

1. \(G\) is the partial isometries \(V\) such that \(VD_nV^*, \ V^*D_nV \subseteq D_n\);
2. \(P\) is the partial isometries in \(D_n\);
3. \(S\) is the matrices in \(G\) with only 0 and 1 entries.
Let $\alpha: S_1 \to S_2$ be an isomorphism of Cartan inverse monoids. Then $\mathcal{E}(S_i)$ is the lattice of projections for a $\mathcal{W}^*$-algebra, $\mathcal{D}_i = \mathcal{C}(\mathcal{E}(S_i))$. The isomorphism $\alpha$ induces an isomorphism $\tilde{\alpha}$ from $\mathcal{D}_1$ to $\mathcal{D}_2$. 

\[ 
\begin{array}{c}
\alpha: S_1 \to S_2 \\
\tilde{\alpha}: \mathcal{D}_1 \to \mathcal{D}_2
\end{array} 
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Let $\alpha: S_1 \rightarrow S_2$ be an isomorphism of Cartan inverse monoids. Then $\mathcal{E}(S_i)$ is the lattice of projections for a $W^*$-algebra, $\mathcal{D}_i = C(\mathcal{E}(S_i))$. The isomorphism $\alpha$ induces an isomorphism $\tilde{\alpha}$ from $\mathcal{D}_1$ to $\mathcal{D}_2$.

**Definition**

Let $S_1$ and $S_2$ be isomorphic Cartan inverse monoids. Let $P_i$ be the partial isometries in $\mathcal{D}_i$. Extensions $G_i$ of $S_i$ by $P_i$ are *equivalent* if there is an isomorphism $\alpha: G_1 \rightarrow G_2$ such that

$$
\begin{array}{c}
P_1 \xrightarrow{\iota_1} G_1 \xrightarrow{q_1} S_1 \\
\tilde{\alpha} \downarrow \quad \alpha \downarrow \quad \alpha \downarrow \\
P_2 \xrightarrow{\iota_2} G_2 \xrightarrow{q_2} S_2.
\end{array}
$$

commutes.
It was shown by Laush (1975) that there is one-to-one correspondence between extensions of $S$ by $P$ and the second cohomology group $H^2(S, P)$.

It is also shown that every extension of $S$ by $P$ is determined by cocycle function $\sigma: S \times S \to P$. 
Theorem

Let \((\mathcal{M}_1, \mathcal{D}_1)\) and \((\mathcal{M}_2, \mathcal{D}_2)\) be two Cartan pairs with associated extensions

\[ P_i \leftrightarrow G_i \rightarrow S_i \]

for \(i = 1, 2\).

There is a normal isomorphism \(\theta: \mathcal{M}_1 \rightarrow \mathcal{M}_2\) such that \(\theta(\mathcal{D}_1) = \mathcal{D}_2\) if and only if the two associated extensions are equivalent.
Let $S$ be a Cartan inverse monoid. Let $\mathcal{D} = C(\widehat{\mathcal{E}(S)})$, and let $P$ be the partial isometries in $\mathcal{D}$. Given an extension

$$P \hookrightarrow G \twoheadrightarrow S$$

we want to construct a Cartan pair $(\mathcal{M}, \mathcal{D})$ with associated extension (equivalent to) $P \hookrightarrow G \twoheadrightarrow S$. 
A $\mathcal{D}$-valued Reproducing kernel space

Let $j$ be an order-preserving map, $j: S \to G$ such that $j \circ q = \text{id}$. That is $j(s) \leq j(t)$ when $s \leq t$ and $j: \mathcal{E}(S) \to \mathcal{E}(G)$ is an isomorphism.
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Define a map

$$K: S \times S \rightarrow D$$

by $K(s, t) = j(s^\dagger t \wedge 1)$. 


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$$K: S \times S \to \mathcal{D}$$

by $K(s, t) = j(s^\dagger t \wedge 1)$. The idempotent $s^\dagger t \wedge 1$ is the minimal idempotent $e$ such that

$$se = te = s \wedge t.$$

Thus $K(s, t)$ is the idempotent in $G$ defining $j(s) \wedge j(t)$.
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The idempotent \( s^\dagger t \wedge 1 \) is the minimal idempotent \( e \) such that

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se = te = s \wedge t.
\]

Thus \( K(s, t) \) is the idempotent in \( G \) defining \( j(s) \wedge j(t) \).
The map \( K \) is positive: that is for \( c_1, \ldots, c_k \in \mathbb{C} \) and \( s_1, \ldots, s_k \in S \)

\[
\sum_{i,j} \overline{c_i} c_j K(s_i, s_j) \geq 0.
\]
For each $s \in S$ define a “kernel-map” $k_s : S \to D$ by

$$k_s(t) = K(t, s).$$

Let $\mathcal{A}_0 = \text{span}\{k_s : s \in S\}$. The positivity of $K$ shows that the

$$\langle \sum c_i k_{s_i}, \sum d_j k_{t_j} \rangle = \sum_{i,j} \overline{c_i}d_j K(s_i, t_j)$$

defines a $D$-valued inner product on $\mathcal{A}_0$. Let $\mathcal{A}$ be completion of $\mathcal{A}_0$. Thus $\mathcal{A}$ is a reproducing kernel Hilbert $D$-module of functions from $S$ into $D$. 
For $g \in G$ define an adjointable operator $\lambda(g)$ on $\mathcal{A}$ by

$$\lambda(g)k_s = k_{q(g)s}\sigma(g, s),$$

where $\sigma : G \times S \to P$ is a “cocycle-like” function (related to the cocycles of Lausch). This is determined by the equation

$$gj(s) = j(q(g)s)\sigma(g, s),$$

i.e. elements of the form $gj(s)$ can be factored into the product of an element in $j(S)$ by an element in $P$. 
A left representation of $G$

For $g \in G$ define an adjointable operator $\lambda(g)$ on $\mathcal{A}$ by

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i.e. elements of the form $gj(s)$ can be factored into the product of an element in $j(S)$ by an element in $P$. The mapping

$$\lambda: G \to L(\mathcal{A})$$

is a representation of $G$ by partial isometries.
Let $\pi$ be a faithful representation of $\mathcal{D}$ on a Hilbert space $\mathcal{H}$. We can form a Hilbert space $\mathcal{A} \otimes \pi \mathcal{H}$ by completing $\mathcal{A} \otimes \mathcal{H}$ with respect to the inner product

$$\langle a \otimes h, b \otimes k \rangle := \langle h, \pi(\langle a, b \rangle)k \rangle.$$
Let $\pi$ be a faithful representation of $D$ on a Hilbert space $\mathcal{H}$. We can form a Hilbert space $\mathfrak{A} \otimes_\pi \mathcal{H}$ by completing $\mathfrak{A} \otimes \mathcal{H}$ with respect to the inner product

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Then $\pi$ determines a faithful representation $\hat{\pi}$ of $L(\mathfrak{A})$ on the Hilbert space $\mathfrak{A} \otimes_\pi \mathcal{H}$ by

$$\hat{\pi}(T)(a \otimes h) = (Ta) \otimes h.$$
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Thus, we have a faithful representation of $G$ on the hilbert space $\mathcal{A} \otimes_{\pi} \mathcal{H}$ by

$$\lambda_{\pi}: g \mapsto \hat{\pi}(\lambda(g)).$$
Let $M_q = \lambda(G)''$, and $D_q = \lambda(E(S))''$. Then $(M_q, D_q)$ is a Cartan pair such that

1. The pair $(M_q, D_q)$ is independent of choice of $j$ and $\pi$;
Creating Cartan pairs

Let \( \mathcal{M}_q = \lambda(G)'' \), and \( \mathcal{D}_q = \lambda(\mathcal{E}(S))'' \). Then \((\mathcal{M}_q, \mathcal{D}_q)\) is a Cartan pair such that

1. The pair \((\mathcal{M}_q, \mathcal{D}_q)\) is independent of choice of \(j\) and \(\pi\);
2. \(\mathcal{D}_q\) is isomorphic to \(\mathcal{D} = C(\widehat{\mathcal{E}(S)})\);
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2. $\mathcal{D}_q$ is isomorphic to $\mathcal{D} = C(\widehat{\mathcal{E}(S)})$;
3. The conditional expectation $E: \mathcal{M}_q \rightarrow \mathcal{D}_q$ is induced from the map

$$S \rightarrow \mathcal{E}(S)$$

$$s \mapsto s \wedge 1.$$
Creating Cartan pairs

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\]

4. The extension associated to \((\mathcal{M}_q, \mathcal{D}_q)\) is equivalent to

\[
P \hookrightarrow G \overset{q}{\to} S
\]

(the extension we started with).
Main Theorem

Theorem (Feldman-Moore; Donsig-F-Pitts)

If $S$ is a Cartan inverse monoid and $P \hookrightarrow G \xrightarrow{q} S$ is an extension of $S$ by $P := p.i.(C^*(\mathcal{E}(S)))$, then the extension determines a Cartan pair $(\mathcal{M}, \mathcal{D})$ which is unique up to isomorphism. Equivalent extensions determine isomorphic Cartan pairs.

Every Cartan pair $(\mathcal{M}, \mathcal{D})$ determines uniquely an extension of a Cartan inverse semigroup $S$ by $P$, $P \hookrightarrow G \xrightarrow{q} S$. 