Corners of C*-algebras

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Introduction

The Peirce[1] decomposition of an algebra A with respect to an idempotent $e \in A$ is

$$A = eAe \oplus eA(1-e) \oplus (1-e)Ae \oplus (1-e)A(1-e);$$

M is a bimodule over $S := eAe$.

The corresponding projection from A onto S,

$$\mathcal{E} \colon A \to A \qquad \mathcal{E}(x) = exe \quad (x \in A),$$

is an S-bimodule map (an algebraic conditional expectation).

[1] Benjamin Peirce, *Linear Associative Algebra*. Amer. J. Math. 4 (1881)

Introduction

The Peirce decomposition leads to the following definition.

Definition

Let A be an algebra.

A *corner* of A is a subalgebra $S \subseteq A$ for which there is a subspace $M \subseteq A$ such that $A = S \oplus M$ and $sm, ms \in M$ for all $s \in S$, $m \in M$.

Examples include:
Peirce corners eAe.
Injective C^*-algebras.
The graph of a homomorphism between C^*-algebras.
The graph of a derivation from a C^*-algebra A to a Banach A-bimodule.

Non-example: A non-unital C^*-subalgebra of a von Neumann algebra.

(The above definition makes no reference to idempotents.)

Retracts

A *retraction* on a topological space X is a continuous function (homeomorphism) $\tau\colon X \to X$ with $\tau \circ \tau = \tau$; $\tau(X)$ is called the *retract* of X.

Proposition

Let K be a compact Hausdorff space and let $\tau\colon K \to K$ be a retraction. Define

$$\mathcal{E}_\tau\colon C(K) \to C(K) \quad \text{by} \quad \mathcal{E}_\tau(f) = f \circ \tau.$$

Then:

(1) $\mathcal{E}_\tau$ is a conditional expectation (in particular $\|\mathcal{E}_\tau\| = 1$);

(2) $\mathcal{E}_\tau$ is multiplicative;

(3) $\ker \mathcal{E}_\tau = \{f \in C(K) \,:\, f_{|\tau(K)} = 0\}$.

Thus, every retraction τ on K gives rise to a corner in $C(K)$.

But there are corners in $C(K)$ which do not come from retractions on K.

Retracts

Proposition.

Let $K = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and define

$$\mathcal{E}\colon C(K) \to C(K) \quad \text{by} \quad \mathcal{E}(f)(\zeta) = \frac{f(\zeta) + f(-\zeta)}{2}.$$

Then:

(1) $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$;

(2) $S = \mathcal{E}(C(K)) = \{f \in C(K) : \mathcal{E}(f) = f\}$ is a closed *-subalgebra of $C(K)$ (C^*-subalgebra);

(3) $\mathcal{E}$ is an S-module map (not multiplicative) and S is a corner in $C(K)$.

There is no retraction $\tau\colon K \to K$ with $S = \{f \in C(K) : f \circ \tau = f\}$.

Corollary

The functor $\tau \mapsto \mathcal{E}_\tau$ is not surjective.

Retracts

Theorem

Let K be a compact Hausdorff space and let $\mathcal{E}\colon C(K) \to C(K)$ be a unital algebraic conditional expectation which is also an algebra homomorphism. Then there is a retraction $\tau\colon K \to K$ such that $\mathcal{E} = \mathcal{E}_\tau$.

Sketch of Proof:

1. $\mathcal{E}$ is a contraction ($\|\mathcal{E}\| \leq 1$).
2. $\ker \mathcal{E}$ is a closed ideal in $C(K)$.
3. There exists a closed $K_1 \subseteq K$ such that $\ker \mathcal{E} = \{f \in C(K) \ : \ f|K_1 = 0\}$.
4. There exists a retraction $\tau\colon K \to K$ with $\tau(K) = K_1$ such that $\mathcal{E} = \mathcal{E}_\tau$.

Retracts

Theorem

Let K be a compact Hausdorff space and let $\mathcal{E}\colon C(K) \to C(K)$ be a unital algebraic conditional expectation which is also an algebra homomorphism.

Then there is a clopen set $L \subseteq K$ and a retraction $\tau\colon L \to L$ such that $\mathcal{E}$ is given by

$$\mathcal{E}(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{for } t \in K \setminus L \end{cases}$$

(for $f \in C(K)$, $t \in K$). Moreover $\mathcal{E}(\bar{f}) = \overline{\mathcal{E}f}$ for $f \in C(K)$.

Construction of L:
Let 1_K be the constant function in $C(K)$. Then
$\mathcal{E}(1_K) = \mathcal{E}(1_K \cdot 1_K) = \mathcal{E}(1_K) \cdot \mathcal{E}(1_K) = \mathcal{E}(1_K)^2$,
$L = \{t \in K \quad : \quad \mathcal{E}(1_K)(t) = 1\}$,
$\mathcal{E}(1_K) = 1_L$ (the characteristic function of L).

Retracts

We have shown:

$$\begin{pmatrix}\text{Retracts of compact} \\ \text{Hausdorff spaces } K\end{pmatrix} \equiv \begin{pmatrix}\text{Multiplicative conditional} \\ \text{expectations on } C(K)\end{pmatrix}$$

$$|||$$

$$\begin{pmatrix}\text{Corner } C^*\text{-subalgebras of} \\ C(K) \text{ complemented by ideals,} \\ \text{`supported' on clopen sets}\end{pmatrix}$$

Use the the Gelfand Duality (categorical equivalence):

$$\Big(\text{Compact Hausdorff spaces } K\Big) \equiv \Big(\text{Unital commutative } C^*\text{-algebras } C(K)\Big)$$

Retracts

Let X be a locally compact, non-compact Hausdorff space and let $X^* = X \cup \{\omega\}$ be the space X with one point adjoined according to the recipe for the one point compactification.
We consider $C_0(X)$ as embedded in $C(X^*)$ via

$$f \mapsto \tilde{f} \colon C_0(X) \to C(X^*)$$

where

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \in X \\ 0 & \text{for } t = \omega. \end{cases}$$

Proposition

If $\tau \colon X^* \to X^*$ is a retraction such that $\tau(\omega) = \omega$ then the map defined by

$$\mathcal{E}_{\tau,*} \colon C_0(X) \to C_0(X), \quad \mathcal{E}_{\tau,*}(f) = (\tilde{f} \circ \tau)|_X$$

is a conditional expectation.

In fact, every multiplicative (algebraic) conditional expectation if of this type.

Retracts

Theorem

If X is a locally compact, non-compact Hausdorff space and $\mathcal{E}\colon C_0(X) \to C_0(X)$ is an algebraic conditional expectation which is also an algebra homomorphism, then there is a retraction $\tau\colon X^* \to X^*$ with $\tau(\omega) = \omega$ such that $\mathcal{E} = \mathcal{E}_{\tau,*}$.

This gives the following equivalence:

$$\begin{pmatrix}\text{Noncommutative}\\ \text{retracts}\end{pmatrix} \equiv \begin{pmatrix}\text{Corners of } C^*\text{-algebras } A\\ \text{given by}\\ \text{multiplicative conditional}\\ \text{expectations } \mathcal{E}\colon A \to A\\ \text{(describe this in terms of } \|\mathcal{E}\|\text{)}\end{pmatrix}$$

$$\Big(\text{Noncommutative topology}\Big) \equiv \Big(C^*\text{-algebras}\Big)$$

Retracts

Theorem

Let A be a unital C^*-algebra and let $\mathcal{E}\colon A \to A$ be an algebraic conditional expectation.
Then $\mathcal{E}$ is a conditional expectation (in the category of C^*-algebras) if and only if

$$\|\mathcal{E}(x)\|^2 \leq \|\mathcal{E}(x^*x)\|$$

for all $x \in A$.

Proof of this theorem relies on the **Russo-Dye** theorem (1966) which asserts that in a unital C^*-algebra, the closure of the convex hull of the unitary elements is the closed unit ball.

So if $x \in A$ has $\|x\| < 1$, then $\|x\| \leq 1 - \frac{2}{n}$ for some integer n, and (by the Russo-Dye theorem) there are unitary elements $u_1, \ldots, u_n$ in A with $x = \frac{1}{n}(u_1 + \cdots + u_n)$.

Retracts

Proposition

Let A be a unital C^*-algebra and let $\mathcal{E}\colon A \to A$ be an algebraic conditional expectation. If

$$\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|$$

for all $x \in A$, then $\ker \mathcal{E}$ is a closed Jordan ideal in A.

Theorem

Let A be a unital C^*-algebra and let $\mathcal{E}\colon A \to A$ be an algebraic conditional expectation. Then $\mathcal{E}$ is a multiplicative $*$-homomorphism if and only if

$$\|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|$$

for all $x \in A$.

Proof Use the Proposition and the fact that closed Jordan $*$-ideals in C^*-algebras are automatically 2-sided ideals.

Retracts

$$\begin{pmatrix}\text{Noncommutative}\\ \text{Retracts}\end{pmatrix} \equiv \begin{pmatrix}\text{Multiplicative conditional}\\ \text{expectations on } C^*\text{-algebras}\end{pmatrix}$$

$$|||$$

$$\begin{pmatrix}\text{Corner } C^*\text{-subalgebras}\\ \text{complemented by ideals}\end{pmatrix}$$

$$|||$$

$$\begin{pmatrix}\text{Conditional expectations } \mathcal{E}\\ \text{on } C^*\text{-algebras satisfying}\\ \|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\|\\ \text{(in general } \|\mathcal{E}(x)\|^2 \leq \|\mathcal{E}(x^*x)\| \text{)}\end{pmatrix}$$

Closure question

Does every $*$-corner of a commutative C^*-algebra is closed?

Theorem

Let A be a unital commutative C^*-algebras. Then there are no dense proper corner subalgebras of A.
That is: if K is a compact Hausdorff space and $S \subseteq A = C(K)$ a dense corner in A, then $S = A$.

Theorem

Every $*$-corner of a unital commutative C^*-algebra is closed.

Closure question

Theorem

If $S \subseteq A$ is a dense corner in a commutative C^*-algebra A then $S = A$.

Theorem

If S is a $*$-corner in a commutative C^*-algebra A then S is closed.

Thus, every $*$-corner of a commutative C^*-algebra is closed.

Closure question

Unlike conditional expectations in the category of C^*-algebras, algebraic conditional expectations need not be bounded; however, the next theorem shows that, in the commutative case, bounded conditional expectations can always be found.

Theorem

Let K be a compact Hausdorff space, and let $\mathcal{E}\colon C(K) \to C(K)$ be a unital algebraic $*$-conditional expectation.
Then there exists a bounded conditional expectation operator $\bar{\mathcal{E}}\colon C(K) \to C(K)$ with

$$\operatorname{range} \bar{\mathcal{E}} = \operatorname{range} \mathcal{E}.$$

Corners containing diagonals

Let H be a Hilbert space with an orthonormal basis $(e_i)_{i \in I}$
(which may be countable or uncountable).
Let $\mathcal{E}\colon B(H) \to B(H)$ be a linear map with range S a subalgebra such that:
$\mathcal{E} \circ \mathcal{E} = \mathcal{E}$,
$\mathcal{E}$ is an S-bimodule map,
$\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for $x \in B(H)$.

Theorem

If $e_i \otimes e_i^* \in S$ for $i \in I$, there is an equivalence relation on I such that
$\mathcal{E}(x) = \sum_{j \in J} p_j x p_j$ for $x \in B(x)$, where J is the set of equivalence classes,
$p_j = \sum_{i \in j} e_i \otimes e_i^*$ for $j \in J$, and $e_i \otimes e_i^*$ is the operator that sends an element
$h \in H$ to $< h, e_i > e_i \in H$.

This result can be generalized to expectations in purely atomic von Neumann algebras.

A method of constructing discontinuous expectations

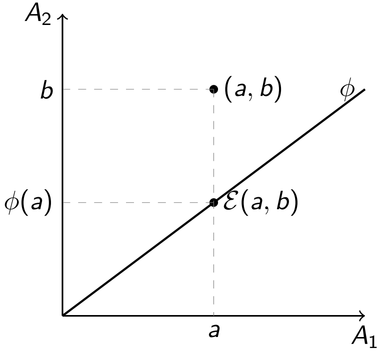
Statement

Assuming the Continuum Hypothesis **(CH)**, there exist discontinuous algebra homomorphisms between C^*-algebras.
For example, assuming **(CH)**, there is a discontinuous algebra homomorphism $\phi\colon C([0,1]) \to B(H)$.

This is used to construct a discontinuous expectation (next slide).

A method of constructing discontinuous expectations

- Suppose $\phi\colon A_1 \to A_2$ is a discontinuous homomorphism between C^*-algebras A_1 and A_2.
- Let $A = A_1 \oplus A_2$, a C^*-algebra with the summands as ideals (where we take the max norm).
- Let S be the graph of ϕ, i.e., $S = \{(x, \phi(x)) : x \in A_1\}$.
  Define $\mathcal{E}\colon A \to A$ by $\mathcal{E}(a, b) = (a, \phi(a))$.



Then $\mathcal{E}$ is a discontinuous expectation from A onto S.

Thank you

References:

Robert Pluta, *Ranges of bimodule projections and conditional expectations*, Cambridge Scholar Publishing, 2013.