Corners of C*-algebras

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The Peirce\(^1\) decomposition of an algebra \(A\) with respect to an idempotent \(e \in A\) is

\[
A = eAe \oplus eA(1 - e) \oplus (1 - e)Ae \oplus (1 - e)A(1 - e);
\]

\(M\) is a bimodule over \(S := eAe\).

The corresponding projection from \(A\) onto \(S\),

\[\mathcal{E}: A \to A \quad \mathcal{E}(x) = exe \quad (x \in A),\]

is an \(S\)-bimodule map (an algebraic conditional expectation).

\(^1\)Benjamin Peirce, *Linear Associative Algebra*. Amer. J. Math. 4 (1881)
Introduction

The Peirce decomposition leads to the following definition.

**Definition**

Let \( A \) be an algebra.

A *corner* of \( A \) is a subalgebra \( S \subseteq A \) for which there is a subspace \( M \subseteq A \) such that \( A = S \oplus M \) and \( sm, ms \in M \) for all \( s \in S, m \in M \).

**Examples include:**

Peirce corners \( eAe \).

Injective \( C^* \)-algebras.

The graph of a homomorphism between \( C^* \)-algebras.

The graph of a derivation from a \( C^* \)-algebra \( A \) to a Banach \( A \)-bimodule.

**Non-example:** A non-unital \( C^* \)-subalgebra of a von Neumann algebra.

(The above definition makes no reference to idempotents.)
Retracts

A retraction on a topological space $X$ is a continuous function (homeomorphism) $\tau: X \to X$ with $\tau \circ \tau = \tau$; $\tau(X)$ is called the retract of $X$.

**Proposition**

Let $K$ be a compact Hausdorff space and let $\tau: K \to K$ be a retraction. Define

$$E_\tau: C(K) \to C(K) \quad \text{by} \quad E_\tau(f) = f \circ \tau.$$

Then:

1. $E_\tau$ is a conditional expectation (in particular $\|E_\tau\| = 1$);
2. $E_\tau$ is multiplicative;
3. $\ker E_\tau = \{ f \in C(K) : f|_{\tau(K)} = 0 \}$.

Thus, every retraction $\tau$ on $K$ gives rise to a corner in $C(K)$.
But there are corners in $C(K)$ which do not come from retractions on $K$. 
Proposition.

Let $K = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and define

$$ \mathcal{E} : C(K) \to C(K) \quad \text{by} \quad \mathcal{E}(f)(\zeta) = \frac{f(\zeta) + f(-\zeta)}{2}. $$

Then:

1. $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$;
2. $S = \mathcal{E}(C(K)) = \{f \in C(K) : \mathcal{E}(f) = f\}$ is a closed *-subalgebra of $C(K)$ ($C^*$-subalgebra);
3. $\mathcal{E}$ is an $S$-module map (not multiplicative) and $S$ is a corner in $C(K)$.

There is no retraction $\tau : K \to K$ with $S = \{f \in C(K) : f \circ \tau = f\}$.

Corollary

The functor $\tau \mapsto \mathcal{E}_\tau$ is not surjective.
Retracts

**Theorem**

Let $K$ be a compact Hausdorff space and let $\mathcal{E} : C(K) \to C(K)$ be a unital algebraic conditional expectation which is also an algebra homomorphism. Then there is a retraction $\tau : K \to K$ such that $\mathcal{E} = \mathcal{E}_\tau$.

**Sketch of Proof:**

1. $\mathcal{E}$ is a contraction ($\|\mathcal{E}\| \leq 1$).

2. $\ker \mathcal{E}$ is a closed ideal in $C(K)$.

3. There exists a closed $K_1 \subseteq K$ such that $\ker \mathcal{E} = \{f \in C(K) : f|_{K_1} = 0\}$.

4. There exists a retraction $\tau : K \to K$ with $\tau(K) = K_1$ such that $\mathcal{E} = \mathcal{E}_\tau$. 
Theorem

Let $K$ be a compact Hausdorff space and let $\mathcal{E}: C(K) \to C(K)$ be a unital algebraic conditional expectation which is also an algebra homomorphism.

Then there is a clopen set $L \subseteq K$ and a retraction $\tau: L \to L$ such that $\mathcal{E}$ is given by

$$\mathcal{E}(f)(t) = \begin{cases} 
 f(\tau(t)) & \text{if } t \in L \\
 0 & \text{for } t \in K \setminus L
\end{cases}$$

(for $f \in C(K)$, $t \in K$). Moreover $\mathcal{E}(\bar{f}) = \overline{\mathcal{E}f}$ for $f \in C(K)$.

Construction of $L$:

Let $1_K$ be the constant function in $C(K)$. Then

$$\mathcal{E}(1_K) = \mathcal{E}(1_K \cdot 1_K) = \mathcal{E}(1_K) \cdot \mathcal{E}(1_K) = \mathcal{E}(1_K)^2,$$

$L = \{ t \in K : \mathcal{E}(1_K)(t) = 1 \}$,

$\mathcal{E}(1_K) = 1_L$ (the characteristic function of $L$).
Retracts

We have shown:

\[
\left( \text{Retracts of compact Hausdorff spaces } K \right) \equiv \left( \text{Multiplicative conditional expectations on } C(K) \right)
\]

\[
\equiv \begin{bmatrix}
\text{Corner } C^*-\text{subalgebras of } C(K) \text{ complemented by ideals,}
\text{‘supported’ on clopen sets}
\end{bmatrix}
\]

Use the Gelfand Duality (categorical equivalence):

\[
\left( \text{Compact Hausdorff spaces } K \right) \equiv \left( \text{Unital commutative } C^*-\text{algebras } C(K) \right)
\]
Retracts

Let $X$ be a locally compact, non-compact Hausdorff space and let $X^* = X \cup \{\omega\}$ be the space $X$ with one point adjoined according to the recipe for the one point compactification.

We consider $\mathcal{C}_0(X)$ as embedded in $C(X^*)$ via

$$f \mapsto \tilde{f} : \mathcal{C}_0(X) \to C(X^*)$$

where

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \in X \\ 0 & \text{for } t = \omega. \end{cases}$$

Proposition

If $\tau : X^* \to X^*$ is a retraction such that $\tau(\omega) = \omega$ then the map defined by

$$\mathcal{E}_{\tau,*} : \mathcal{C}_0(X) \to \mathcal{C}_0(X), \quad \mathcal{E}_{\tau,*}(f) = (\tilde{f} \circ \tau)|_X$$

is a conditional expectation.

In fact, every multiplicative (algebraic) conditional expectation is of this type.
Retracts

**Theorem**

If $X$ is a locally compact, non-compact Hausdorff space and $\mathcal{E}: C_0(X) \to C_0(X)$ is an algebraic conditional expectation which is also an algebra homomorphism, then there is a retraction $\tau: X^* \to X^*$ with $\tau(\omega) = \omega$ such that $\mathcal{E} = \mathcal{E}_{\tau,*}$.

This gives the following equivalence:

\[
\begin{pmatrix}
\text{(Noncommutative retracts)} \\
\end{pmatrix}
\equiv
\begin{pmatrix}
\text{Corners of C*-algebras } A \\
given \text{by multiplicative conditional expectations } \mathcal{E}: A \to A \\
\text{(describe this in terms of } ||\mathcal{E}||) 
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{(Noncommutative topology)} \\
\end{pmatrix}
\equiv
\begin{pmatrix}
\text{C*-algebras} 
\end{pmatrix}
\]
Retracts

**Theorem**

Let $A$ be a unital $C^*$-algebra and let $\mathcal{E}: A \to A$ be an algebraic conditional expectation. Then $\mathcal{E}$ is a conditional expectation (in the category of $C^*$-algebras) if and only if

$$\|\mathcal{E}(x)\|^2 \leq \|\mathcal{E}(x^* x)\|$$

for all $x \in A$.

**Proof** of this theorem relies on the Russo-Dye theorem (1966) which asserts that in a unital $C^*$-algebra, the closure of the convex hull of the unitary elements is the closed unit ball.

So if $x \in A$ has $\|x\| < 1$, then $\|x\| \leq 1 - \frac{2}{n}$ for some integer $n$, and (by the Russo-Dye theorem) there are unitary elements $u_1, \ldots, u_n$ in $A$ with $x = \frac{1}{n}(u_1 + \cdots + u_n)$. 
Retracts

**Proposition**
Let $A$ be a unital $C^*$-algebra and let $\mathcal{E}: A \to A$ be an algebraic conditional expectation. If
\[ \|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\| \]
for all $x \in A$, then $\ker \mathcal{E}$ is a closed Jordan ideal in $A$.

**Theorem**
Let $A$ be a unital $C^*$-algebra and let $\mathcal{E}: A \to A$ be an algebraic conditional expectation. Then $\mathcal{E}$ is a multiplicative $*$-homomorphism if and only if
\[ \|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\| \]
for all $x \in A$.

**Proof** Use the Proposition and the fact that closed Jordan $*$-ideals in $C^*$-algebras are automatically 2-sided ideals.
Retracts

\[
(\text{Noncommutative Retracts}) \equiv (\text{Multiplicative conditional expectations on } C^*-\text{algebras})
\]

\[
\equiv (\text{Corner } C^*-\text{subalgebras complemented by ideals})
\]

\[
\equiv \left( \text{Conditional expectations } \mathcal{E} \text{ on } C^*-\text{algebras satisfying} \right.
\]
\[
\left. \|\mathcal{E}(x)\|^2 = \|\mathcal{E}(x^*x)\| \right.
\]
\[
\left. \text{(in general } \|\mathcal{E}(x)\|^2 \leq \|\mathcal{E}(x^*x)\| \text{) } \right)
\]
Closure question

Does every \(*\)-corner of a commutative \(C^*\)-algebra is closed?

**Theorem**

Let \(A\) be a unital commutative \(C^*\)-algebras. Then there are no dense proper corner subalgebras of \(A\). That is: if \(K\) is a compact Hausdorff space and \(S \subseteq A = C(K)\) a dense corner in \(A\), then \(S = A\).

**Theorem**

Every \(*\)-corner of a unital commutative \(C^*\)-algebra is closed.
Theorem
If $S \subseteq A$ is a dense corner in a commutative $C^*$-algebra $A$ then $S = A$.

Theorem
If $S$ is a $\ast$-corner in a commutative $C^*$-algebra $A$ then $S$ is closed.

Thus, every $\ast$-corner of a commutative $C^*$-algebra is closed.
Closure question

Unlike conditional expectations in the category of $C^*$-algebras, algebraic conditional expectations need not be bounded; however, the next theorem shows that, in the commutative case, bounded conditional expectations can always be found.

**Theorem**

Let $K$ be a compact Hausdorff space, and let $\mathcal{E} : C(K) \rightarrow C(K)$ be a unital algebraic $*$-conditional expectation. Then there exists a bounded conditional expectation operator $\bar{\mathcal{E}} : C(K) \rightarrow C(K)$ with

$$\text{range } \bar{\mathcal{E}} = \text{range } \mathcal{E}.$$
Corners containing diagonals

Let $H$ be a Hilbert space with an orthonormal basis $(e_i)_{i \in I}$ (which may be countable or uncountable).
Let $\mathcal{E} : B(H) \to B(H)$ be a linear map with range $S$ a subalgebra such that:

- $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$,
- $\mathcal{E}$ is an $S$-bimodule map,
- $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for $x \in B(H)$.

**Theorem**

If $e_i \otimes e_i^* \in S$ for $i \in I$, there is an equivalence relation on $I$ such that

$\mathcal{E}(x) = \sum_{j \in J} p_j x p_j$ for $x \in B(x)$, where $J$ is the set of equivalence classes,

$p_j = \sum_{i \in j} e_i \otimes e_i^*$ for $j \in J$, and $e_i \otimes e_i^*$ is the operator that sends an element $h \in H$ to $< h, e_i > e_i \in H$.

This result can be generalized to expectations in purely atomic von Neumann algebras.
A method of constructing discontinuous expectations

Statement

Assuming the Continuum Hypothesis \((\text{CH})\), there exist discontinuous algebra homomorphisms between \(C^*\)-algebras. For example, assuming \((\text{CH})\), there is a discontinuous algebra homomorphism \(\phi: C([0, 1]) \rightarrow B(H)\).

This is used to construct a discontinuous expectation (next slide).
A method of constructing discontinuous expectations

- Suppose $\phi: A_1 \rightarrow A_2$ is a discontinuous homomorphism between $C^*$-algebras $A_1$ and $A_2$.
- Let $A = A_1 \oplus A_2$, a $C^*$-algebra with the summands as ideals (where we take the max norm).
- Let $S$ be the graph of $\phi$, i.e., $S = \{(x, \phi(x)) : x \in A_1\}$. Define $\mathcal{E}: A \rightarrow A$ by $\mathcal{E}(a, b) = (a, \phi(a))$.

Then $\mathcal{E}$ is a discontinuous expectation from $A$ onto $S$. 
Thank you

References: