The Invariant Basis Number Property for C^*-Algebras

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NIFAS April 19th, 2014.

All rings R will be unital and all modules X will be right modules.

- a **basis** for an R-module X is a R-linearly independent generating set.
- An R-module X **has dimension** if
  - it admits a finite basis, and
  - all finite bases of X have the same cardinality.
- A ring R has **Invariant Basis Number** if the free R-modules R^n all have dimension.
- A ring R is **dimensional** if all R-modules with finite basis have dimension.

Examples: Commutative, right-Noetherian, and division rings are dimensional.The ring

$$\langle a, b, c, d : ca = db = 1,\ cb = da = 0,\ ac+bd = 1\rangle\langle v_1, v_2, v_1^*, v_2^* : v_1^* v_1 = v_2^*$$

is not dimensional. Viewed as a module over itself it contains the bases $\{1\}$ and $\{a, b\}\{v_1, v_2\}$.

Leavitt's Work

Theorem (Leavitt, 1962)

A ring R is dimensional if and only if there exists a dimensional ring R' and a unital homomorphism $\psi : R \to R'$.

Theorem (Leavitt, 1962)

If R is not dimensional then there exist unique positive integers N and K such that:

1. *if X is an R-module with finite basis of size m then $m < N$ iff X has dimension, and*
2. *if X is an R-module with finite bases of distinct sizes n and m then $m \equiv n \mod K$.*

The pair (N, K) is termed the **module type** of the ring.

Order and Lattice Structure of Module Types

The Module Types form a distributive lattice under the ordering

$$(N_1, K_1) \leq (N_2, K_2) \Leftrightarrow N_1 \leq N_2,\ \ K_2 \equiv 0 \mod K_1$$

and operations

$$(N_1, K_1) \wedge (N_2, K_2) := (\min(N_1, N_2), gcd(K_1, K_2))$$

$$(N_1, K_2) \vee (N_2, K_2) := (\max(N_1, N_2), lcm(K_1, K_2))$$

Proposition (Leavitt, 1962)

For unital non-dimensional rings A and B

$$type(A \oplus B) = type(A) \vee type(B)$$

while

$$type(A \otimes_{\mathbb{Z}} B) \leq type(A) \wedge type(B).$$

Existence of all Types

Theorem (Leavitt)

Given a basis type (N, K) there is a unital ring R with that basis type.

The Leavitt Path algebra

$$L_F(1,k) = alg_F\langle v_i, v_i^* : i = 1, ..., k, \sum_{i=1}^{k} v_i v_i^* = 1,\ v_i^* v_j = \delta_{ij}\rangle$$

is of module type $(1, k-1)$.

Leavitt also constructs algebras of types $(n, 1)$ for arbitrary $n \geq 1$: e.g.

$$alg_F\left\langle v_{ij}, v_{ij}^* : i = 1, ..., n,\ j = 1, ..., n+1,\ \sum_{k=1}^{n} v_{ki}^* v_{kj} = \delta_{ij},\ \sum_{k=1}^{n+1} v_{ik} v_{jk}^* = \delta_{ij}\right\rangle$$

These have not been extensively studied.

Definitions

For a C^*-algebra A, a (right) A-module X is a complex vector space with

1. a right action of A, and
2. an "A-valued inner product" i.e. a mapping $\langle \cdot, \cdot \rangle : X \times X \to A$ satisfying:
    - $\langle x, ya \rangle = \langle x, y \rangle a$
    - $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
    - $\langle x, y \rangle = \langle y, x \rangle^*$
    - $\langle x, x \rangle > 0$ if $x \neq 0$.

The assignment $||x|| := ||\langle x, x \rangle||_A^{\frac{1}{2}}$ defines a norm on X.

If X is complete with respect to this norm it is a **Hilbert A-module**.

Examples

- If $A = \mathbb{C}$ then Hilbert A-modules are Hilbert spaces.
- $C(0,1)$ is a Hilbert $C[0,1]$-module with $\langle f, g \rangle = \overline{f} g$.
- Any C^*-algebra A with $\langle a, b \rangle = a^* b$ is a Hilbert A-module.
- If X and Y are Hilbert A-modules then $X \oplus Y$ is a Hilbert A-module with the inner-product $\langle (x,y), (z,w) \rangle = \langle x, z \rangle_X + \langle y, w \rangle_Y$.
- The **standard A-modules** are $A^n := \bigoplus_{i=1}^n A$ with inner products $\langle (a_i), (b_i) \rangle = \sum_{i=1}^n a_i^* b_i$.

Homomorphisms and Unitaries

An A-module homomorphism $\phi : X \to Y$ is an A-linear map, i.e. $\phi(xa + y) = \phi(x)a + \phi(y)$. A homomorphism is:

- **bounded** if $\sup_{x \in X} \frac{||\phi(x)||_Y}{||x||_X} < \infty$.
- **adjointable** if there is a homomorphism $\phi^* : Y \to X$ satisfying $\langle \phi(x), y \rangle_Y = \langle x, \phi^*(y) \rangle_X$.
- **unitary** if it is adjointable and $\phi\phi^* = I_Y$, $\phi^*\phi = I_X$.

Bounded homomorphisms need not be adjointable, e.g. the inclusion $i : C(0,1) \hookrightarrow C[0,1]$.

$L(X, Y)$

For Hilbert A-modules X and Y we set

$$L(X, Y) := \{\phi : X \to Y : \ \phi \textit{ an adjointable homomorphism}\},$$

$$L(X) := L(X, X).$$

Examples:

- H, K Hilbert $\mathbb{C}$-modules; then $L(H, K) = B(H, K)$ and $L(H) = B(H)$.
- Viewing (unital) A as a Hilbert module over itself; then $L(A) = A$. (If A non-unital then $L(A)$ is the multiplier algebra.)
- The standard A-module A^n; then $L(A^n) = M_n(A)$ and $L(A^n, A^m) = M_{m,n}(A)$.

Two Hilbert A-modules are **unitarily equivalent**, denoted $X \simeq Y$, if $L(X, Y)$ has a unitary element. This is an equivalence relation.

Bases of Hilbert Modules

Assumption

Henceforth all C^*-algebras will be assumed unital.

Let X be Hilbert A-module.

A set $\{x_\alpha\} \subset X$ is **orthogonal** if $\langle x_\alpha, x_\beta \rangle = 0$ when $\alpha \neq \beta$, and **orthonormal** if in addition $\langle x_\alpha, x_\alpha \rangle = 1_A$.

A **basis** for a Hilbert A-module is an orthonormal set whose A-linear span is norm-dense.

Remark: Orthonormality guarantees the "A-linear independence" of the basis.

Examples

- If X is a Hilbert $\mathbb{C}$-module (i.e. a Hilbert space) then its Hilbert space basis is a $\mathbb{C}$-module basis.
- The singleton set $\{1_A\}$ is a basis for A considered as a module over itself.
- The standard modules have the "standard basis" $\{e_1, ..., e_n\}$ with $e_i := (..., 0, 1, 0, ...)$.
- $C(0,1)$ is a Hilbert $C[0,1]$-module with no bases.

Finite Bases

Proposition

If X is a Hilbert A-module with finite basis $x_1, \ldots, x_n$ then for each $x \in X$ we have the *Fourier decomposition* $x = \sum_{i=1}^{n} x_i \langle x_i, x \rangle$.

Short Proof. Use same proof as for finite Hilbert space bases. Completeness of A is essential.

Proposition

If X is a Hilbert A-module with finite basis $x_1, \ldots, x_n$ then $X \simeq A^n$. Further, a unitary $u \in L(X, A^n)$ may be found such that $ux_i = e_i$ (e_i the standard basis element of A^n) for all $i = 1, \ldots, n$.

Short Proof. Map each element of X to the tuple whose elements are its "Fourier coefficients."

Uniqueness of Basis Size?

Natural Question: Is the cardinality of a basis unique to the module?

Answer: Yes... in some cases.

Example: Hilbert $\mathbb{C}$-modules have unique basis sizes. This is because they are just Hilbert spaces.

Answer: But not in general.

Example: The Cuntz algebra $\mathcal{O}_2$ has a singleton basis (the identity) and a basis of size two (the generating isometries).

Invariant Basis Number

Definition

A C^*-algebra A has **Invariant Basis Number (IBN)** if every Hilbert A-module X with finite basis has a unique finite basis size.

Proposition

A has IBN if and only if whenever $A^j \simeq A^k$ then $j = k$.

Short Proof. If X is a Hilbert A-module with bases of sizes j and k then $A^j \simeq X \simeq A^k$.

Corollary

A does not have IBN if and only if $A^j \simeq A^k$ for some $j \neq k$.

Examples

- Commutative C^*-algebras have IBN.
- No Cuntz algebra $\mathcal{O}_n$ ($n \geq 2$) has IBN.
- Stably finite C^*-algebras (ones with no proper matrix isometries) have IBN.

Proposition

A has IBN if and only if every unitary matrix over A is square.

Short Proof. Since $L(A^j, A^k) = M_{j,k}(A)$ we have that $A^j \simeq A^k$ if and only if there is a unitary $j \times k$ matrix over A.

K_0

Shamelessly brief review of C^*-algebraic K-theory.

- $K_0(A)$ is an abelian group.
- $K_0(A)$ is generated by elements $[p]$ for $p \in P_n(A)$, $n \geq 1$.
- $[p] = [q]$ if $\begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix} \sim \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}$. Here " $\sim$ " is Murray-von Neumann equivalence: $x \sim y$ if $vv^* = x$ and $v^*v = y$ for some v.
- The map $K_0 : A \mapsto K_0(A)$ is a covariant, half exact functor.

The K-theory of many classes of C^*-algebras is well known and, in some cases, provides a classification invariant.

Fact: $K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}$.

Characterization of IBN

Theorem (G. '14)

A C^*-algebra A has IBN if and only if the element $[1_A] \in K_0(A)$ has infinite order.

Short Proof. Let $|\cdot|$ denote the order of a group element.
If $|[1_a]| = k < \infty$ then $k[1_A] = [I_k] = 0$.
By definition this means there is some matric projection p for which
$\begin{bmatrix} I_k & 0 \\ 0 & p \end{bmatrix} \sim [p]$.
We can in fact choose $p = I_n$ for some $n > 0$, hence $I_{k+n} \sim I_n$.
Thus there is $u \in M_{n,n+k}(A)$ for which $uu^* = I_n,\ u^*u = I_{n+k}$, i.e. there is a unitary in $M_{n,n+k}(A) = L(A^n, A^{n+k})$.

Consequences

We may now recover the C^*-algebraic version of Leavitt's characterization.

Corollary

If $\phi : B \to A$ is a unital $*$-homomorphism and A has IBN then B has IBN as well.

Short Proof. The functoriality of K_0 gives a group homomorphism $K_0(\phi) : K_0(B) \to K_0(A)$ with $K_0(\phi)[1_B] = [1_A]$.
Thus $|[1_B]| \equiv 0 \mod |[1_A]|$.

Corollary

If B is an extension of A, i.e. for some C we have the short exact sequence,

$$0 \to C \to B \to A \to 0$$

and A has IBN then B also has IBN.

Examples

- The irrational rotation algebras $\mathcal{A}_\theta$ have IBN since $K_0(\mathcal{A}_\theta) = \mathbb{Z}^2$.
- The Toeplitz algebra has IBN since it is an extension of $C(\mathbb{T})$ by the compacts.
- Neither the Calkin algebra nor $B(H)$ has IBN since both have trivial K_0.
- If A is non-unital then its unitization $\widetilde{A} \cong \mathbb{C} \oplus A$ has IBN since $K_0(\widetilde{A}) = K_0(A) \oplus \mathbb{Z}$.

Algebras without IBN

Theorem (G. '14)

If A is a unital C^*-algebra without IBN then there are unique positive integers N and K such that

1. *if $n < N$ and $A^n \simeq A^j$ for some j then $j = n$, and*
2. *if $A^j \simeq A^k$ then $j \equiv k \mod K$.*

Short Proof. Since A doesn't have IBN there are at least two distinct positive integers for which $A^j \simeq A^k$. Let N be the smallest of all such integers. Let K be the smallest positive integer for which $A^N \simeq A^{N+K}$.

The pair (N, K) will be termed the **basis type** of the C^*-algebra A.

Examples

- The Cuntz algebra $\mathcal{O}_2$ has basis type $(1,1)$ since $\mathcal{O}_2 \simeq \mathcal{O}_2^2$.
- In general, $\mathcal{O}_n$ has type $(1, n-1)$.
- $B(H)$ has basis type $(1,1)$.

Theorem (G. '14)

If A has basis type (N, K) then $K = |[1_A]|_{K_0}$.

Corollary

If $K_0(A) = 0$ then A does not have IBN and $K = 1$.

The C^*-algebraic **basis types** have the same lattice structure as the purely algebraic *module types*:

$$(N_1, K_1) \leq (N_2, K_2) \Leftrightarrow N_1 \leq N_2,\ \ K_2 \equiv 0 \mod K_1$$
$$(N_1, K_1) \wedge (N_2, K_2) := (\min(N_1, N_2), gcd(K_1, K_2))$$
$$(N_1, K_2) \vee (N_2, K_2) := (\max(N_1, N_2), lcm(K_1, K_2))$$

Theorem (G. '14)

If A and B are C^*-algebras of basis types (N_1, K_1) and (N_2, K_2) respectively then $A \oplus B$ is of basis type $(N_1, K_1) \vee (N_2, K_2)$.

For example, $\mathcal{O}_3$ is of type $(1, 2)$, $\mathcal{O}_4$ is of type $(1, 3)$ and $\mathcal{O}_3 \oplus \mathcal{O}_4$ is of type $(1, 6)$.
See this either because $\mathcal{O}_7 \subset \mathcal{O}_3 \oplus \mathcal{O}_4$ or

$$K_0(\mathcal{O}_3 \oplus \mathcal{O}_4) = K_0(\mathcal{O}_3) \oplus K_0(\mathcal{O}_4) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}.$$

Theorem (G. '14)

If A has basis type (N_1, K_2) and $\pi : A \to B$ is a unital $*$-homomorphism then B has basis type $(N_2, K_2) \leq (N_1, K_1)$.

Theorem (G. '14)

If A and B have basis types (N_1, K_1) and (N_2, K_2) respectively then $A \otimes B$ has basis type $\leq (N_1, K_1) \wedge (N_2, K_2)$.

The proof is an application of the first Theorem and, as such, applies to $A \otimes_{max} B$ as well.

Equality can occur. For example,

$$type(\mathcal{O}_3 \otimes \mathcal{O}_4) = (1,1) = (1,2) \wedge (1,3) = type(\mathcal{O}_3) \wedge type(O_4).$$

Existence of all Basis Types

Theorem (G. '14)

For each pair of positive integers (N, K) there is a C^*-algebra A with that basis type.

Sketch of Proof. By the previous result, if $type(A) = (N, 1)$ and $type(B) = (1, K)$ then $type(A \oplus B) = (N, K)$. Thus it is enough to exhibit C^*-algebras with the types $(N, 1)$ and $(1, K)$ for each $N, K \geq 1$.

We have already seen that $type(\mathcal{O}_{K+1}) = (1, K)$.

A series of papers by Rørdam contains such an algebra, which is additionally simple and nuclear.

Theorem (Rørdam, 1998)

Let A be a simple, σ-unital C^*-algebra with stable rank one. Then $\mathcal{M}(A)$ is finite if A is non-stable and $\mathcal{M}(A)$ is properly infinite if A is stable.

Finite-ness (or lack thereof) is important because the existence of isometries is necessary to have a module basis.

Theorem (Rørdam, 1997)

For each integer $n \geq 2$ there exists a C^*-algebra B such that $M_n(B)$ is stable and $M_k(B)$ is non-stable for $1 \leq k < n$. Moreover, B may be chosen to be σ-unital and with stable rank one.

Recall that if A is of basis type (N, K) then the standard modules are "nice" for indices below N and "interesting" above N.

Theorem (Rørdam, 1998)

For each $n \geq 2$ there is a C^*-algebra A such that $M_k(A)$ is finite for $1 \leq k < n$ and $M_n(A)$ is properly infinite.

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For each $n \geq 2$ there is a C^*-algebra A such that $M_k(A)$ is finite for $1 \leq k < n$ and $M_n(A)$ is properly infinite.

Fix $n \geq 2$ and take A from the third Theorem. Then A is the multiplier algebra of a stable C^*-algebra and hence $K_0(A) = 0$ and so A does not have IBN and is of basis type $(N, 1)$ for some N. We also have $K_0(M_n(A)) = K_0(A) = 0$.

Since $M_n(A)$ is properly infinite and has trivial K_0 there exists a unital embedding $\mathcal{O}_2 \hookrightarrow M_n(A)$. We can use these isometries to show $M_n(A) \simeq M_n(A)^2$, i.e. there is a unitary in $L(M_n(A), M_n(A)^2) = M_{1,2}(M_n(A)) = M_{n,2n}(A)$.

This gives us the equivalence $A^n \simeq A^{2n}$ and so $N \leq n$. A more technical argument, using the finite-ness of the algebras $M_k(A)$ for $1 \leq k < n$, gives that $N \geq n$.

Summary

Definition

A C^*-algebra A has IBN if $A^n \simeq A^m \Leftrightarrow n = m$.

Theorem

A C^*-algebras has IBN if and only if the element $[1_A]$ has infinite order in $K_0(A)$.

Theorem

C^*-algebras without IBN have a unique basis type (N, K).

Theorem

All basis types are realized by C^*-algebras.

Thank you.

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