The Invariant Basis Number Property for $C^*$-Algebras

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All rings $R$ will be unital and all modules $X$ will be right modules.

- a **basis** for an $R$-module $X$ is a $R$-linearly independent generating set.
- An $R$-module $X$ **has dimension** if
  - it admits a finite basis, and
  - all finite bases of $X$ have the same cardinality.
- A ring $R$ has **Invariant Basis Number** if the free $R$-modules $R^n$ all have dimension.
- A ring $R$ is **dimensional** if all $R$-modules with finite basis have dimension.

Examples: Commutative, right-Noetherian, and division rings are dimensional. The ring

$$\langle a, b, c, d : ca = db = 1, \ cb = da = 0, \ ac + bd = 1 \rangle \langle v_1, v_2, v_1^*, v_2^* : v_1^* v_1 = v_2^* \rangle$$

is not dimensional. Viewed as a module over itself it contains the bases

$\{1\}$ and $\{a, b\}\{v_1, v_2\}$. 
Leavitt’s Work

Theorem (Leavitt, 1962)

A ring \( R \) is dimensional if and only if there exists a dimensional ring \( R' \) and a unital homomorphism \( \psi : R \rightarrow R' \).

Theorem (Leavitt, 1962)

If \( R \) is not dimensional then there exist unique positive integers \( N \) and \( K \) such that:

1. if \( X \) is an \( R \)-module with finite basis of size \( m \) then \( m < N \) iff \( X \) has dimension, and
2. if \( X \) is an \( R \)-module with finite bases of distinct sizes \( n \) and \( m \) then \( m \equiv n \mod K \).

The pair \((N, K)\) is termed the module type of the ring.
Order and Lattice Structure of Module Types

The Module Types form a distributive lattice under the ordering

\[(N_1, K_1) \leq (N_2, K_2) \iff N_1 \leq N_2, \ K_2 \equiv 0 \mod K_1\]

and operations

\[(N_1, K_1) \land (N_2, K_2) : = (\min(N_1, N_2), \gcd(K_1, K_2))\]
\[(N_1, K_2) \lor (N_2, K_2) : = (\max(N_1, N_2), \text{lcm}(K_1, K_2))\]

Proposition (Leavitt, 1962)

For unital non-dimensional rings \(A\) and \(B\)

\[\text{type}(A \oplus B) = \text{type}(A) \lor \text{type}(B)\]

while

\[\text{type}(A \otimes \mathbb{Z} B) \leq \text{type}(A) \land \text{type}(B).\]
Existence of all Types

**Theorem (Leavitt)**

*Given a basis type \((N, K)\) there is a unital ring \(R\) with that basis type.*

The Leavitt Path algebra

\[
L_F(1, k) = \text{alg}_F\langle v_i, v_i^* : i = 1, \ldots, k, \sum_{i=1}^{k} v_i v_i^* = 1, \ v_i^* v_j = \delta_{ij}\rangle
\]

is of module type \((1, k - 1)\).

Leavitt also constructs algebras of types \((n, 1)\) for arbitrary \(n \geq 1\): e.g.

\[
\text{alg}_F\left\langle v_{ij}, v_{ij}^* : i = 1, \ldots, n, \ j = 1, \ldots, n + 1, \ \sum_{k=1}^{n} v_{ki}^* v_{kj} = \delta_{ij}, \ \sum_{k=1}^{n+1} v_{ik} v_{jk}^* = \delta_{ij}\right\rangle
\]

These have not been extensively studied.
Definitions

For a $C^*$-algebra $A$, a (right) $A$-module $X$ is a complex vector space with

1. a right action of $A$, and
2. an “$A$-valued inner product” i.e. a mapping $\langle \cdot, \cdot \rangle : X \times X \to A$ satisfying:
   - $\langle x, ya \rangle = \langle x, y \rangle a$
   - $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
   - $\langle x, y \rangle = \langle y, x \rangle^*$
   - $\langle x, x \rangle > 0$ if $x \neq 0$.

The assignment $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}_A$ defines a norm on $X$.

If $X$ is complete with respect to this norm it is a Hilbert $A$-module.
Examples

- If $A = \mathbb{C}$ then Hilbert $A$-modules are Hilbert spaces.
- $C(0, 1)$ is a Hilbert $C[0, 1]$-module with $\langle f, g \rangle = \overline{f}g$.
- Any $C^*$-algebra $A$ with $\langle a, b \rangle = a^*b$ is a Hilbert $A$-module.
- If $X$ and $Y$ are Hilbert $A$-modules then $X \oplus Y$ is a Hilbert $A$-module with the inner-product $\langle (x, y), (z, w) \rangle = \langle x, z \rangle_X + \langle y, w \rangle_Y$.
- The standard $A$-modules are $A^n := \bigoplus_{i=1}^n A$ with inner products $\langle (a_i), (b_i) \rangle = \sum_{i=1}^n a_i^* b_i$. 
An $A$-module homomorphism $\phi : X \rightarrow Y$ is an $A$-linear map, i.e. $\phi(xa + y) = \phi(x)a + \phi(y)$. A homomorphism is:

- **bounded** if $\sup_{x \in X} \frac{||\phi(x)||_Y}{||x||_X} < \infty$.
- **adjointable** if there is a homomorphism $\phi^* : Y \rightarrow X$ satisfying $\langle \phi(x), y \rangle_Y = \langle x, \phi^*(y) \rangle_X$.
- **unitary** if it is adjointable and $\phi \phi^* = I_Y$, $\phi^* \phi = I_X$.

Bounded homomorphisms need not be adjointable, e.g. the inclusion $i : C(0,1) \hookrightarrow C[0,1]$. 
For Hilbert $A$-modules $X$ and $Y$ we set

$$L(X, Y) := \{ \phi : X \to Y : \phi \text{ an adjointable homomorphism} \},$$

$$L(X) := L(X, X).$$

Examples:

- $H, K$ Hilbert $\mathbb{C}$-modules; then $L(H, K) = B(H, K)$ and $L(H) = B(H)$.
- Viewing (unital) $A$ as a Hilbert module over itself; then $L(A) = A$. (If $A$ non-unital then $L(A)$ is the multiplier algebra.)
- The standard $A$-module $A^n$; then $L(A^n) = M_n(A)$ and $L(A^n, A^m) = M_{m,n}(A)$.

Two Hilbert $A$-modules are **unitarily equivalent**, denoted $X \simeq Y$, if $L(X, Y)$ has a unitary element. This is an equivalence relation.
Bases of Hilbert Modules

**Assumption**

Henceforth all $C^*$-algebras will be assumed unital.

Let $X$ be Hilbert $A$-module.

A set $\{x_\alpha\} \subset X$ is **orthogonal** if $\langle x_\alpha, x_\beta \rangle = 0$ when $\alpha \neq \beta$, and **orthonormal** if in addition $\langle x_\alpha, x_\alpha \rangle = 1_A$.

A **basis** for a Hilbert $A$-module is an orthonormal set whose $A$-linear span is norm-dense.

Remark: Orthonormality guarantees the “$A$-linear independence” of the basis.
Examples

- If $X$ is a Hilbert $\mathbb{C}$-module (i.e. a Hilbert space) then its Hilbert space basis is a $\mathbb{C}$-module basis.
- The singleton set $\{1_A\}$ is a basis for $A$ considered as a module over itself.
- The standard modules have the “standard basis” $\{e_1, \ldots, e_n\}$ with $e_i := (\ldots, 0, 1, 0, \ldots)$.
- $C(0, 1)$ is a Hilbert $C[0, 1]$-module with no bases.
Finite Bases

Proposition
If $X$ is a Hilbert $A$-module with finite basis $x_1, \ldots, x_n$ then for each $x \in X$ we have the *Fourier decomposition* $x = \sum_{i=1}^{n} x_i \langle x_i, x \rangle$.

*Short Proof.* Use same proof as for finite Hilbert space bases. Completeness of $A$ is essential.

Proposition
If $X$ is a Hilbert $A$-module with finite basis $x_1, \ldots, x_n$ then $X \simeq A^n$. Further, a unitary $u \in L(X, A^n)$ may be found such that $ux_i = e_i$ ($e_i$ the standard basis element of $A^n$) for all $i = 1, \ldots, n$.

*Short Proof.* Map each element of $X$ to the tuple whose elements are its "Fourier coefficients."
Uniqueness of Basis Size?

Natural Question: Is the cardinality of a basis unique to the module?

Answer: Yes... in some cases.

Example: Hilbert $\mathbb{C}$-modules have unique basis sizes. This is because they are just Hilbert spaces.

Answer: But not in general.

Example: The Cuntz algebra $\mathcal{O}_2$ has a singleton basis (the identity) and a basis of size two (the generating isometries).
Invariant Basis Number

Definition
A $C^*$-algebra $A$ has **Invariant Basis Number (IBN)** if every Hilbert $A$-module $X$ with finite basis has a unique finite basis size.

Proposition
$A$ has IBN if and only if whenever $A^j \simeq A^k$ then $j = k$.

*Short Proof.* If $X$ is a Hilbert $A$-module with bases of sizes $j$ and $k$ then $A^j \simeq X \simeq A^k$.

Corollary
$A$ does not have IBN if and only if $A^j \simeq A^k$ for some $j \neq k$. 

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Examples

- Commutative $C^*$-algebras have IBN.
- No Cuntz algebra $\mathcal{O}_n$ ($n \geq 2$) has IBN.
- Stably finite $C^*$-algebras (ones with no proper matrix isometries) have IBN.

**Proposition**

$A$ has IBN if and only if every unitary matrix over $A$ is square.

**Short Proof.** Since $L(A^j, A^k) = M_{j,k}(A)$ we have that $A^j \simeq A^k$ if and only if there is a unitary $j \times k$ matrix over $A$. 
Shamelessly brief review of $C^*$-algebraic $K$-theory.

- $K_0(A)$ is an abelian group.
- $K_0(A)$ is generated by elements $[p]$ for $p \in P_n(A)$, $n \geq 1$.
- $[p] = [q]$ if $\begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix} \sim \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}$. Here "\sim" is Murray-von Neumann equivalence: $x \sim y$ if $vv^* = x$ and $v^*v = y$ for some $v$.
- The map $K_0 : A \mapsto K_0(A)$ is a covariant, half exact functor.

The $K$-theory of many classes of $C^*$-algebras is well known and, in some cases, provides a classification invariant.

Fact: $K_0(O_n) = \mathbb{Z}/(n-1)\mathbb{Z}$.
Theorem (G. '14)

A $C^*$-algebra $A$ has IBN if and only if the element $[1_A] \in K_0(A)$ has infinite order.

**Short Proof.** Let $|\cdot|$ denote the order of a group element. If $|[1_a]| = k < \infty$ then $k[1_A] = [l_k] = 0$. By definition this means there is some matric projection $p$ for which

$$\begin{bmatrix} l_k & 0 \\ 0 & p \end{bmatrix} \sim [p].$$

We can in fact choose $p = I_n$ for some $n > 0$, hence $l_{k+n} \sim I_n$. Thus there is $u \in M_{n,n+k}(A)$ for which $uu^* = I_n$, $u^*u = I_{n+k}$, i.e. there is a unitary in $M_{n,n+k}(A) = L(A^n, A^{n+k})$. 
Consequences

We may now recover the $C^*$-algebraic version of Leavitt’s characterization.

**Corollary**

If $\phi : B \to A$ is a unital $\ast$-homomorphism and $A$ has IBN then $B$ has IBN as well.

**Short Proof.** The functoriality of $K_0$ gives a group homomorphism $K_0(\phi) : K_0(B) \to K_0(A)$ with $K_0(\phi)[1_B] = [1_A]$.

Thus $|[1_B]| \equiv 0 \pmod{|[1_A]|}$.

**Corollary**

If $B$ is an extension of $A$, i.e. for some $C$ we have the short exact sequence,

$$0 \to C \to B \to A \to 0$$

and $A$ has IBN then $B$ also has IBN.
Examples

- The irrational rotation algebras $A_\theta$ have IBN since $K_0(A_\theta) = \mathbb{Z}^2$.
- The Toeplitz algebra has IBN since it is an extension of $C(\mathbb{T})$ by the compacts.
- Neither the Calkin algebra nor $B(H)$ has IBN since both have trivial $K_0$.
- If $A$ is non-unital then its unitization $\tilde{A} \cong \mathbb{C} \oplus A$ has IBN since $K_0(\tilde{A}) = K_0(A) \oplus \mathbb{Z}$.
Algebras without IBN

Theorem (G. ’14)

If $A$ is a unital $C^*$-algebra without IBN then there are unique positive integers $N$ and $K$ such that

1. if $n < N$ and $A^n \simeq A^j$ for some $j$ then $j = n$, and
2. if $A^j \simeq A^k$ then $j \equiv k \mod K$.

Short Proof. Since $A$ doesn’t have IBN there are at least two distinct positive integers for which $A^j \simeq A^k$. Let $N$ be the smallest of all such integers. Let $K$ be the smallest positive integer for which $A^N \simeq A^{N+K}$.

The pair $(N, K)$ will be termed the basis type of the $C^*$-algebra $A$. 
Examples

- The Cuntz algebra $\mathcal{O}_2$ has basis type $(1, 1)$ since $\mathcal{O}_2 \simeq \mathcal{O}_2^2$.
- In general, $\mathcal{O}_n$ has type $(1, n-1)$.
- $B(H)$ has basis type $(1, 1)$.

**Theorem (G. ’14)**

*If $A$ has basis type $(N, K)$ then $K = |[1_A]|_{K_0}$.***

**Corollary**

*If $K_0(A) = 0$ then $A$ does not have IBN and $K = 1$.***
The $C^*$-algebraic **basis types** have the same lattice structure as the purely algebraic **module types**:

$$(N_1, K_1) \leq (N_2, K_2) \iff N_1 \leq N_2, K_2 \equiv 0 \mod K_1$$

$$(N_1, K_1) \wedge (N_2, K_2) := (\min(N_1, N_2), \gcd(K_1, K_2))$$

$$(N_1, K_2) \vee (N_2, K_2) := (\max(N_1, N_2), \lcm(K_1, K_2))$$

**Theorem (G. '14)**

*If $A$ and $B$ are $C^*$-algebras of basis types $(N_1, K_1)$ and $(N_2, K_2)$ respectively then $A \oplus B$ is of basis type $(N_1, K_1) \vee (N_2, K_2)$.***

For example, $O_3$ is of type $(1, 2)$, $O_4$ is of type $(1, 3)$ and $O_3 \oplus O_4$ is of type $(1, 6)$.

See this either because $O_7 \subset O_3 \oplus O_4$ or

$$K_0(O_3 \oplus O_4) = K_0(O_3) \oplus K_0(O_4) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}.$$
Theorem (G. ’14)

If $A$ has basis type $(N_1, K_2)$ and $\pi : A \to B$ is a unital $\ast$-homomorphism then $B$ has basis type $(N_2, K_2) \leq (N_1, K_1)$.

Theorem (G. ’14)

If $A$ and $B$ have basis types $(N_1, K_1)$ and $(N_2, K_2)$ respectively then $A \otimes B$ has basis type $\leq (N_1, K_1) \wedge (N_2, K_2)$.

The proof is an application of the first Theorem and, as such, applies to $A \otimes_{\text{max}} B$ as well.

Equality can occur. For example,

$$\text{type}(\mathcal{O}_3 \otimes \mathcal{O}_4) = (1, 1) = (1, 2) \wedge (1, 3) = \text{type}(\mathcal{O}_3) \wedge \text{type}(\mathcal{O}_4).$$
Existence of all Basis Types

**Theorem (G. ’14)**

For each pair of positive integers \((N, K)\) there is a \(C^*\)-algebra \(A\) with that basis type.

**Sketch of Proof.** By the previous result, if \(\text{type}(A) = (N, 1)\) and \(\text{type}(B) = (1, K)\) then \(\text{type}(A \oplus B) = (N, K)\). Thus it is enough to exhibit \(C^*\)-algebras with the types \((N, 1)\) and \((1, K)\) for each \(N, K \geq 1\).

We have already seen that \(\text{type}(O_{K+1}) = (1, K)\).

A series of papers by Rørdam contains such an algebra, which is additionally simple and nuclear.
Theorem (Rørdam, 1998)

Let $A$ be a simple, $\sigma$-unital $C^*$-algebra with stable rank one. Then $\mathcal{M}(A)$ is finite if $A$ is non-stable and $\mathcal{M}(A)$ is properly infinite if $A$ is stable.

Finite-ness (or lack thereof) is important because the existence of isometries is necessary to have a module basis.

Theorem (Rørdam, 1997)

For each integer $n \geq 2$ there exists a $C^*$-algebra $B$ such that $M_n(B)$ is stable and $M_k(B)$ is non-stable for $1 \leq k < n$. Moreover, $B$ may be chosen to be $\sigma$-unital and with stable rank one.

Recall that if $A$ is of basis type $(N, K)$ then the standard modules are “nice” for indices below $N$ and “interesting” above $N$.

Theorem (Rørdam, 1998)

For each $n \geq 2$ there is a $C^*$-algebra $A$ such that $M_k(A)$ is finite for $1 \leq k < n$ and $M_n(A)$ is properly infinite.
Theorem (Rørdam, 1998)

For each \( n \geq 2 \) there is a \( C^* \)-algebra \( A \) such that \( M_k(A) \) is finite for \( 1 \leq k < n \) and \( M_n(A) \) is properly infinite.

Fix \( n \geq 2 \) and take \( A \) from the third Theorem. Then \( A \) is the multiplier algebra of a stable \( C^* \)-algebra and hence \( K_0(A) = 0 \) and so \( A \) does not have IBN and is of basis type \((N, 1)\) for some \( N \). We also have \( K_0(M_n(A)) = K_0(A) = 0 \).

Since \( M_n(A) \) is properly infinite and has trivial \( K_0 \) there exists a unital embedding \( O_2 \hookrightarrow M_n(A) \). We can use these isometries to show \( M_n(A) \cong M_n(A)^2 \), i.e. there is a unitary in \( L(M_n(A), M_n(A)^2) = M_{1,2}(M_n(A)) = M_{n,2n}(A) \).

This gives us the equivalence \( A^n \cong A^{2n} \) and so \( N \leq n \). A more technical argument, using the finite-ness of the algebras \( M_k(A) \) for \( 1 \leq k < n \), gives that \( N \geq n \).
Summary

Definition
A $C^*$-algebra $A$ has IBN if $A^n \simeq A^m \iff n = m$.

Theorem
A $C^*$-algebras has IBN if and only if the element $[1_A]$ has infinite order in $K_0(A)$.

Theorem
$C^*$-algebras without IBN have a unique basis type $(N, K)$.

Theorem
All basis types are realized by $C^*$-algebras.
Thank you.
References


