Crossed product results for inverse semigroup algebras

David Milan   Benjamin Steinberg

U. Texas at Tyler   Carleton University

April 14, 2012

C*-algebras generated by partial isometries

1. $C^*(T)$, T is the unilateral shift $\quad T^*T = I$
2. **Toeplitz-Cuntz algebras**: $\mathcal{TO}_n$ generated by isometries $T_1, \ldots, T_n$ with mutually orthogonal ranges.
3. **Cuntz algebras**: $\mathcal{O}_n$ generated by isometries $T_1, \ldots, T_n$ such that $\sum T_i T_i^* = I$
4. **Graph algebras**: $C^*(\Gamma)$, Γ a directed graph
   S_e, e an edge $\quad P_v$, v a vertex
   Cuntz-Krieger relations:
   - $S_e^* S_e = P_{s(e)}$
   - $P_v = \sum S_e S_e^*$ (over all directed edges with range v)
5. **Tiling C*-algebras**: Kellendonk's algebra of an aperiodic tiling

The generating set in each case is an inverse semigroup.

Definition

A semigroup S is an **inverse semigroup** if for each s there exists unique s^* such that $s = ss^*s$ and $s^* = s^*ss^*$.

Structure of inverse semigroups:

idempotents: $E = E(S) = \{s : s^2 = s\}$ a commutative subsemigroup.

partial order: $s \leq t$ if and only if $s = te$ for some $e \in E$.

minimal group congruence σ**:** $s\sigma t$ iff $se = te$ for some $e \in E$.

group homomorphic image: $G(S) = S/\sigma$

The Bicyclic Monoid

$B = < t : t^*t = 1 >$

Every word in t, t^* reduces to $t^i t^{*j}$ (e.g. $t^2t^*t^4t^{*3} = t^5t^{*3}$).

$B \cong \mathbb{N} \times \mathbb{N} \qquad (i,j)(m,n) = \begin{cases} (i+m-j, n) & \text{if } m \geq j \\ (i, n+j-m) & \text{otherwise} \end{cases}$

idempotents: $E(B) = \{(m,m) : m \in \mathbb{N}\}$

partial order: $(i,j)(m,m) = \begin{cases} (i+m-j, m) & \text{if } m \geq j \\ (i,j) & \text{otherwise} \end{cases}$

min. group congruence: $(i,j)\sigma(m,n)$ iff $i - j = m - n$

group image: $G(S) = \mathbb{Z}$

Other Examples

1. **polycyclic (Cuntz)** $P_n = < a_1, \dots a_n : a_i^* a_i = 1,\ a_i^* a_j = 0\ i \neq j >$
2. **McAlister** $M_n = < a_1, \dots a_n : a_i a_j^* = a_i^* a_j = 0\ i \neq j >$
3. **graph inv. semigroups** Γ - directed graph Γ^* - path category

   $$S_\Gamma = \{(\alpha, \beta) : s(\alpha) = s(\beta)\} \cup \{0\}$$
   $$(\alpha, \beta)(\mu, \nu) = \begin{cases} (\alpha\overline{\mu}, \nu) & \beta\overline{\mu} = \mu \\ (\alpha, \nu\overline{\beta}) & \mu\overline{\nu} = \beta \\ 0 & \text{otherwise} \end{cases}$$
4. **tiling semigroups**

E-unitary inverse semigroups

Each of the previous examples is E- or 0-E-unitary.

- S is *E-unitary* if: $\quad e \leq s,\ e^2 = e$ implies $s^2 = s$.
- **Theorem:** S is E-unitary iff $S \to G(S)$ is idempotent pure.
- S is *0-E-unitary* if: $\quad e \leq s,\ e^2 = e \neq 0$ implies $s^2 = s$.
- S is *strongly 0-E-unitary* iff $\exists S \to G^0$ that is idempotent pure.

McAlister's P-theorem

If S is E-unitary then the map $s \mapsto (ss^{-1}, \sigma(s))$ from $S \to E \times G$ is injective.

P-theorem (McAlister): If S is E-unitary then G acts partially on E and $S = E \times_\alpha G$

Question: What is the correct structure theorem for the C^*-algebras of E-unitary inverse semigroups?

C^*-algebras of inverse semigroups

left regular representation: Define $\Lambda : S \to \mathcal{B}(\ell^2(S))$ by

$$\Lambda(a)\delta_b = \begin{cases} \delta_{ab} & \text{if } a^*ab = b \\ 0 & \text{otherwise} \end{cases}$$

Definition

$C^*(S) := \overline{\mathbb{C}S}^{\|\cdot\|_u}$ where $\| f \|_u := \sup\{\| \pi(f) \|\}$ over all $\pi : S \to \mathcal{B}(\mathcal{H})$.

$C^*_r(S) := \Lambda(C^*(S))$

G "acting" on $C^*(E)$

1. $C^*(E) = C_0(\widehat{E})$
2. $\widehat{E} = \{x : E \to \{0,1\} : x(0) = 0\} = \{$ filters $x \subseteq E : 0 \notin x$ $\}$
3. Let $D(e) = \{x \in \widehat{E} : x(e) = 1\}$. Then S acts on $\widehat{E}$ by homeomorphisms $\beta_s : D(s^*s) \to D(ss^*)$ where

$$\beta_s(x)(e) = x(s^*es)$$

4. For S E-unitary, G acts by $\alpha_g = \cup\beta_s$ where $s \in \sigma^{-1}(g)$.

Note: Not an action in the usual sense, so we shouldn't nec. expect a crossed product theorem.

Paterson's groupoid

Paterson defined a groupoid $\mathcal{G}(\mathcal{S})$ that is a groupoid of germs for the above action of S.

$\mathcal{G}(\mathcal{S})$ is the set of equiv. classes of pairs (s, x) $(s \in S, x \in D(s^*s))$ where $(s, x) \sim (t, y)$ if $x = y$ and $\exists u \leq s, t$ with $x \in D(u^*u)$.

$[s, x][t, y] = [st, y]$ provided $x = \beta_t(y)$

Theorem: (Paterson) $C^*(S) = C^*(\mathcal{G}(S))$ and $C_r^*(S) = C_r^*(\mathcal{G}(S))$.

Khoshkam and Skandalis

Gave conditions on a cocycle $\rho : \mathcal{G} \to G$ so that $C^*(\mathcal{G})$ is Morita equivalent to $C_0(X) \times_\alpha G$.

Let $X = G^{(0)}$. The conditions are:

1. (faithful) $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is injective from $\mathcal{G}$ into $X \times G \times X$,
2. (closed) $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$ is closed, and
3. (transverse) $(g, \gamma) \mapsto (g\rho(\gamma), s(\gamma))$ from $G \times \mathcal{G} \to G \times X$ is open.

Theorem (K. and S.) If $\rho : \mathcal{G} \to G$ satisfies (1) - (3) above, then $C^*(\mathcal{G})$ is Morita equivalent to $C_0(Y) \times_\alpha G$ and $C_r^*(\mathcal{G})$ is Morita equivalent to $C_0(Y) \times_{\alpha,r} G$.

Applications to inverse semigroups

K. and S. applied their results to Paterson's groupoid $\mathcal{G}(S)$.

$\sigma : S \to G$ induces a cocycle $\rho : \mathcal{G}(S) \to G$. The conditions (1)-(3) are satisfied provided

1. S is E-unitary.
2. $\sigma : S \to G$ satisfies the **KS condition**: $eS_g f$ is finitely generated as an order ideal.

Cor: (K. and S.) If S is E-unitary and satisfies the KS condition then $C^*(S)$ is Morita equivalent to $C_0(Y) \times_\alpha G$ and $C_r^*(S)$ is Morita equivalent to $C_0(Y) \times_{\alpha,r} G$.

The KS condition does not include all inverse semigroups, but it does include a large class called the F-inverse semigroups.

Graph inverse semigroups S_Γ, for example, are 0-F-inverse.

We would still like a crossed product theorem that applies in the same generality as the P-theorem.

Partial Crossed Products

Partial action α **of** G **on a C*-algebra A**:
closed ideals $\{A_g\}_{g\in G}$ of A isomorphisms $\alpha_g : A_{g^{-1}} \to A_g$ such that

1. $A_e = A$
2. α_{gh} extends $\alpha_g \alpha_h$

Covariant representation (π, u) **of** (A, G, α):
$\pi : A \to \mathcal{B}(\mathcal{H})$ a rep. of A
$u : G \to \mathcal{B}(\mathcal{H})$ a partial rep. of G:

1. u_g is a partial isometry for all g in G
2. u_{gh} extends $u_g u_h$

Partial Crossed Products

The **partial crossed product** $A \times_\alpha G$ is built from summable $f : G \to A$ and is universal for covariant representations (π, u) of (A, G, α).

History:

1. (Nica, 1992) Studied C*-algebras of quasi-lattice ordered groups G. Such an algebra has a large abelian subalgebra $\mathcal{D}$ and an expectation $\epsilon : A \to \mathcal{D}$. Nica remarks that there is a "crossed product-like structure" of $\mathcal{D}$ by G.
2. (Exel, 1994) Studied $A \times_\alpha \mathbb{Z}$, a crossed product by a single partial automorphism.
3. (McClanahan, 1995) Partial crossed products by arbitrary discrete groups.
4. (Quigg and Raeburn, 1997) Identified Cuntz algebras and Nica's algebras as partial crossed products.

Isomorphism Theorem

Suppose S is strongly 0-E-unitary with group image G. The partial action of G on E extends to $C^*(E)$

$$C_{g^{-1}} = \overline{\text{span}}\left(\bigcup_{s\in\varphi^{-1}(g)} Es^*s\right)$$

For an idempotent x in $C_{g^{-1}}$, $\alpha_g(x) := sxs^*$, where s in S is any element such that $x \leq s^*s$ and $\varphi(s) = g$

Theorem (M., Steinberg) Let S be strongly 0-E-unitary. Then $C^*(S) \cong C^*(E) \times_\alpha G$ and $C^*_r(S) \cong C^*(E) \times_{r,\alpha} G$.

$C^*(S) \cong C^*(E) \times_\alpha G$

- The crossed product is the closed span of $F_s : G \to C^*(E)$ where

$$F_s(g) = \begin{cases} ss^* & \text{if } \sigma(s) = g \\ 0 & \text{otherwise} \end{cases}$$

- $s \mapsto F_s$ extends to a surjection $C^*(S) \to C^*(E) \times_\alpha G$

- For injectivity, we need to know a representation $\pi : S \to \mathcal{B}(\mathcal{H})$ induces a covariant representation of (π_E, π_G) of $(C^*(E), G, \alpha)$.

Defining (π_E, π_G)

- **Lemma:** If X is a set of compatible partial isometries then there exists a partial isometry $\bigvee_{T \in X} A$ that extends every operator in X.
- If $\sigma(s) = \sigma(t)$ in G then $st^*, s^*t \in E$. Thus $\pi(s), \pi(t)$ are compatible partial isometries.
- $\pi_G(g) := \bigvee_{\sigma(s)=g} \pi(s)$ is a partial representation of G and (π_E, π_G) is a covariant representation.

Limitations of C*(S)

The crossed product theorem applies to all semigroups mentioned so far, including polycyclic (Cuntz), graph inverse, and tiling semigroups. However, in each case $C^*(S) \cong C^*(E) \times_\alpha G$ is lacking the Cuntz-Krieger type relations.

To fix this, one must restrict the representations of S considered in some way.

Recall, $C^*(E) = C_0(\widehat{E})$, where

$$\widehat{E} = \{x : E \to \{0,1\} : x(0) = 0\} = \{ \text{ filters } x \subseteq E : 0 \notin x \}$$

Enforcing Cuntz-Krieger Relations

- In order to enforce Cuntz-Krieger type relations on general inverse semigroups, Exel introduced the notion of the tight algebra of S.
- Exel defined $\widehat{E}_\infty$ to be the ultrafilters in $\widehat{E}$ and

$$\widehat{E}_{\text{tight}} = \overline{\widehat{E}_\infty}.$$

- Then S acts on $\widehat{E}_{\text{tight}}$ partially and the tight algebra of S is defined as a crossed product of $C_0(\widehat{E}_{\text{tight}})$ by S. (Defined as a groupoid algebra for the groupoid of germs of the action.)

Tight C*-algebra $C^*_{\text{tight}}(S)$

The tight algebra of S gives the correct C^*-algebra in many cases.

P_n	$\mathcal{O}_n$
S_Γ	$C^*(\Gamma)$
tiling semigroups	Kellendonk's C*-algebra

Theorem (M., Steinberg) Let S be strongly 0-E-unitary. Then $\widehat{E}_{\text{tight}}$ is invariant for the partial action α of G on $\widehat{E}$ and
$C^*_{\text{tight}}(S) \cong C_0(\widehat{E}_{\text{tight}}) \times_\alpha G$.

The crossed product result encompasses other results in the literature:

- (Quigg, Raeburn)
  1. $\mathcal{TO}_n = D \times_\alpha \mathbb{F}$
  2. $C^*(G, P) = D \times_\alpha G$ (Nica's quasi-lattice ordered group (G,P).
- (Crip, Laca) $C(\partial\Omega) \times G$

Structure of partial crossed products

Having a partial crossed product by a group has some advantages.

Results of Exel, Laca, Quigg (2002):

- If α is topologically free then a representation of $C_0(X) \times_{r,\alpha} G$ is faithful if and only if it is faithful on $C_0(X)$.
- If α is topologically free and minimal then $C_0(X) \times_{r,\alpha} G$ is simple.
- If α is topologically free on closed invariant subsets of X and α has the approximation property then $U \mapsto \langle C_0(U) \rangle$ is a lattice isomorphism between open invariant subset of U and ideals in $C_0(X) \times_\alpha G$

Partial dynamical properties for inverse semigroups

Let S be strongly 0-E-unitary and α the partial action of G on $C_0(\widehat{E})$

- If S is combinatorial then α is topologically free.
- P_n, S_Γ, and one-dimensional tiling semigroups have the approximation property.

Question: Is there a connection between partial crossed products and the results of Khoshkam and Skandalis?

Theorem: (Abadie) Let α be a partial action of G on the locally compact Hausdorff space X. Then $C^*(G \times_\alpha X) = C_0(X) \times_\alpha G$.

From α, G, and X, Abadie constructs the **enveloping action** $\widehat{\alpha}$ of G on the space $Y = (G \times X)/\sim$.

Theorem: (Abadie) Let $\widehat{\alpha}$ be the enveloping action of α on the space Y. If Y is Hausdorff, then $C_0(X) \times_\alpha G$ is Morita equivalent to $C_0(Y) \times_{\widehat{\alpha}} G$.

Moreover, the map $\rho : G \times_\alpha X \to G$ is a faithful, transverse cocycle. The results of Abadie imply that ρ is closed if and only if the enveloping space Y is Hausdorff.

To summarize:

1. $\mathcal{G}(S) = \widehat{E} \times_\beta S$.
2. If S is strongly 0-E-unitary then G partially acts on $\widehat{E}$ and $\mathcal{G}(S) = \widehat{E} \times_\alpha G$.
3. If S also satisfies the KS condition, then $C^*(S)$ is Morita equivalent to a full crossed product $C_0(Y) \times_\alpha G$.