Extensions of Hilbert Modules over Tensor Algebras

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Spring 2012
Outline of topics

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Setup

$C^*$-correspondences

- $A$ - a unital $C^*$ algebra
- $X$ a $C^*$-correspondence. Recall that this means $X$ is a certain kind of bimodule over $A$. Specifically,
  - $X$ is a right Hilbert $C^*$-module over $A$.
  - Its left $A$-action is given by a $C^*$-homomorphism $\phi : A \to \mathcal{L}(X)$. 

Tensor Powers

\[ X^{\otimes 2} = X \otimes_A X \] is a \( C^* \)-correspondence satisfying

- \( a \cdot (x \otimes y) := \phi(a)x \otimes y \).
- \((x \otimes y) \cdot b := x \otimes yb \).
- \(xa \otimes y := x \otimes \phi(a)y \).
- \( \langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_2, \phi(\langle x_1, y_1 \rangle)y_2 \rangle \).

Similarly, define \( X^{\otimes 3} \), \( X^{\otimes 4} \), ...
Constructing the Tensor Algebra

Form the Fock space:

\[ \mathcal{F}(X) := A \oplus X \oplus X \otimes 2 \oplus X \otimes 3 \oplus \cdots. \]

Define \( \phi_{\infty} : A \to \mathcal{L}(\mathcal{F}(X)) \) by

\[
\phi_{\infty}(a) = \begin{bmatrix}
    a \\
    \phi(a) \\
    \phi_2(a) \\
    \phi_3(a) \\
    \vdots
\end{bmatrix}
\]

where \( \phi_n(a)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = (\phi(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n. \)
Fock space $= \mathcal{F}(X) := A \oplus X \oplus X \otimes 2 \oplus X \otimes 3 \oplus \cdots$

For each $x \in X$, we define the creation operator $T_x \in \mathcal{L}(\mathcal{F}(X))$ by

$$T_x = \begin{bmatrix} 0 & & & \\ T_x^{(1)} & 0 & & \\ & T_x^{(2)} & 0 & \\ & & T_x^{(3)} & 0 \\ & & & \ldots \ldots \end{bmatrix}$$

where $T_x^{(k)} : X \otimes k \to X \otimes (k+1)$ is

$T_x^{(k)}(x_1 \otimes \cdots \otimes x_k) = x \otimes x_1 \otimes \cdots \otimes x_k$. 
Constructing the Tensor Algebra

Definition

The \textit{tensor algebra} of $X$, denoted $\mathcal{T}_+(X)$, is the norm closed subalgebra of $\mathcal{L}(\mathcal{F}(X))$ generated by $\phi_\infty(A)$ and $\{ T_x | x \in X \}$.
**Examples**

1. \( A = X = \mathbb{C} \), \( \mathcal{T}_+(X) = A(\mathbb{D}) \) - classical disc algebra

2. \( A = \mathbb{C}, X = \mathbb{C}^d \), \( \mathcal{T}_+(X) = \mathcal{A}_d \) - Popescu’s noncommutative disc algebra

3. Let \( \alpha \) be an automorphism of a unital \( C^* \)-algebra \( A \). Let \( X = \alpha A \) by defining

   1. \( x \cdot a := xa \).
   2. \( \phi(a)x = \alpha(a)x \).
   3. \( \langle x, y \rangle := x^*y \).

   - \( \phi : A \rightarrow \mathcal{L}(A) \) equals \( \alpha \) since \( \mathcal{L}(A) = M(A) = A \).
   - \( \mathcal{F}(X) = l^2(\mathbb{Z}^+; A) \)
   - \( \mathcal{T}_+(X) \) is generated by \( \phi_\infty(A) \) and \( S = T_1 \), a shift.
   - \( \mathcal{T}_+(X) = A \times_\alpha \mathbb{Z}^+ \) is the analytic crossed product of \( A \) by \( \mathbb{Z}^+ \) determined by \( \alpha \).
Definition

1. A Hilbert space $H$ is a (c.b.) Hilbert module over an operator algebra $B$ if the action of $B$ on $H$ is given by a completely bounded homomorphism $\pi : B \to B(H)$.

2. $\varphi : H \to H'$ is a Hilbert module map if it is a $B$-module map between Hilbert modules that is bounded as a Hilbert space operator.

Note: We will assume $A \subset B$ is a $C^*$-algebra, although $B$ need not be self-adjoint. Furthermore, the representation $(\pi|_A) : A \to B(H)$ is a $C^*$-representation.
Extensions

Definition

An extension $\xi$ is a short exact sequence

$$\xi : 0 \rightarrow H \xrightarrow{\varphi} J \xrightarrow{\psi} K \rightarrow 0$$

where $H$, $J$, and $K$ are Hilbert modules over an operator algebra $B$ and $\varphi$ and $\psi$ are Hilbert-module maps.

Note: In particular, the range of $\varphi$ equals the kernel of $\psi$. So $\varphi$ is bounded below and $\psi$ is bounded below on its initial space.
Equivalent of Extensions

Two extensions $\xi$ and $\xi'$ are equivalent if and only if there exist a Hilbert-module map $\theta : J \rightarrow J'$ making the following diagram commute:

$$
\begin{array}{ccccccccc}
\xi : 0 & \longrightarrow & H & \overset{\varphi}{\longrightarrow} & J & \overset{\psi}{\longrightarrow} & K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\xi' : 0 & \longrightarrow & H' & \overset{\varphi'}{\longrightarrow} & J' & \overset{\psi'}{\longrightarrow} & K' & \longrightarrow & 0
\end{array}
$$

The collection (in fact group) of equivalence classes of extensions is denoted $\text{Ext}^1(K, H)$. 

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Hilbert Space Decomposition

\[ \xi : 0 \to H \xrightarrow{\varphi} J \xrightarrow{\psi} K \to 0 \]

As Hilbert spaces, \( J \cong H \oplus K \) (but not necessarily as \( B \)-modules.)
Let $\pi : B \to B(H)$ and $\rho : B \to B(K)$ be the representations of $B$ on $H$ and $K$, respectively.
The $B$-module action on $H \oplus K$, is given by

\[
\begin{pmatrix}
\pi(\cdot) & \delta(\cdot) \\
0 & \rho(\cdot)
\end{pmatrix} : B \to B(H \oplus K)
\]

where $\delta : B \to B(K, H)$ is a completely bounded $A$-derivation

1. $\delta(fg) = \delta(f)\rho(g)k + \pi(f)\delta(g) \quad \forall f, g \in B$
2. $\delta(a) = 0$ for all $a \in A$.

Note: $\delta$ is, technically, a $\phi_\infty(A)$-derivation).
If the derivations $\delta$ and $\delta'$ correspond, respectively, to extensions $\xi$ and $\xi'$, then $\xi \approx \xi'$ if and only if $\delta - \delta'$ is an inner derivation: there exists $L \in B(K, H)$ such that

$$(\delta - \delta')(f) = \pi(f)L - L\rho(f) \forall f \in B.$$ 

An inner derivation is $A$-linear iff $\pi(a)L = L\rho(a) \forall a \in A$. 
Alternatively, we can describe extensions in terms of cocycles:

**Definition**

A *cocycle* is a bilinear map \( \sigma : B \times K \to H \) satisfying

\[
\sigma(fg, k) = \pi(f)\sigma(g, k) + \sigma(f, \rho(g)k).
\]

which is completely bounded when \( H \) and \( K \) are given their column Hilbert space structure.

Derivations and cocycles are related via the equation

\[
\sigma(f, k) = \delta(f)k.
\]
Extension Equivalence

\[ \xi \approx \xi' \text{ if and only if } \]

\[ \sigma(f, k) - \sigma'(f, k) = \pi(f) Lk - L\rho(f)k. \]
Proposition

Suppose $H$ and $K$ are Hilbert modules over $B$ with representations

$\pi : \mathcal{T}_+(\alpha A) \to B(H)$ and $\rho : \mathcal{T}_+(\alpha A) \to B(K)$, respectively. If

$\sigma : \mathcal{T}_+(\alpha A) \times K \to H$ is a cocycle, then

$$\sigma(S^{n+1}, k) = \sum_{j=0}^{n} \pi(S^{n-j}) \sigma(S, \rho(S^j)k)$$

for every $n \geq 0, S \in B, k \in K$. 
Induced Representation

- Let $\psi : A \to B(E)$ be a representation and let $\{e_m\}_{m \geq 0}$ be an orthonormal basis for $E$.
- From now on, we only consider $B = \mathcal{T}_+(\alpha A)$ and $H = \ell^2(\mathbb{Z}^+; A) \otimes_\psi E$.
- $\{\delta_n \otimes e_m\}_{n,m \geq 0}$ is an orthonormal basis for $\ell^2(\mathbb{Z}^+; A) \otimes_\psi E$., where $\delta_n(k) = \delta_{nk} 1_A$.
- $\pi : \mathcal{T}_+(\alpha A) \to B(\ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$ is given by $\pi|_A = \phi_\infty \otimes id_E$ and $\pi(T_1) = U_+ \otimes id_E$. 
Cocycles Defined by Vectors

**Definition**

We say a sequence of vectors in $K$, $\{k_m\}$ define a cocycle $\sigma$ if

$$\sigma(S, k) = \sum_m \langle k, k_m \rangle \delta_0 \otimes e_m.$$
Theorem (Carlson & Clark, 1995)

Let $K$ be a Hilbert $A(\mathbb{D})$-module. Then a vector $k_0 \in K$ defines a cocycle $\sigma : A(\mathbb{D}) \times K \to H^2$ if and only if

$$\sum_{n=0}^{\infty} |\langle \rho(S^n)k, k_0 \rangle|^2 < \infty$$

for all $k \in K$.

Note: $H^2$ is the classical Hardy space and $\sigma(S, k) = \langle k, k_0 \rangle \in H^2$. 

Motivation
Boundedness Criterion

Theorem (Greene, 2011)

Let $K$ be a Hilbert $\mathcal{T}_+ (\alpha A)$-module. Then a sequence in $K, \{k_m\}_{m=0}^{\infty}$ defines a cocycle $\sigma : \mathcal{T}_+ (\alpha A) \times K \to \ell^2 (\mathbb{Z}^+; A) \otimes_{\psi} E$ if and only if

1. \[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\langle \rho (S^n) k, k_m \rangle|^2 < \infty \quad \forall k \in K
\]

2. \[
\pi (\alpha (a)) k_m = \sum_{m'} \langle \psi (a) e_m, e_{m'} \rangle k_{m'}
\]
Corollary

If $N = \dim(E) < \infty$ and $\text{sp}(\rho(S)) \subset \mathbb{D}$, then any $\{k_m\}_{1 \leq m \leq N}$ satisfying (2) defines a cocycle $\sigma$.

Proof.

Define the functions $h_m(z) = \langle \sum_n (z\rho(S))^n k, k_m \rangle$. By hypothesis $h_m(z) = \langle (id_K - z\rho(S))^{-1} k, k_m \rangle$ for $|z| < \|\rho(S)\|^{-1}$ and $h_m(z)$ are analytic across the unit circle.
Continuation of proof.

\[
\sum_{m=1}^{N} \sum_{n=0}^{\infty} |\langle z \rho(S^n)k, k_m \rangle|^2 = \| \sum_{n,m} \langle z \rho(S)^n k, k_m \rangle \delta_n \otimes e_m \| \\
\leq \sum_{m} \| \langle (id_K - z \rho(S))^{-1}k, k_m \rangle \| \\
\leq \sum_{m=1}^{N} \| h_m(z) \| \\
< \infty.
\]
Corollary

If $\rho(S) = id_K$, then $\{k_m\}$ defines a cocycle $\sigma$ only if $k_m = 0$ for every $m$. It follows that $\text{Ext}(K, \ell^2(\mathbb{Z}^+; A) \otimes \psi E) = 0$.

Proof.

$$\sum_{n,m} |\langle \rho(S^n)k, k_m \rangle|^2 = \sum_{n,m} |\langle k, k_m \rangle|^2 < \infty \iff k_m = 0 \forall m.$$
Characterization of Cocycles

Theorem (Greene, 2011)

Every cocycle $\sigma$ is equivalent to a cocycle defined by some $\{k_m\}$.

Proof.

1. Let $\sigma$ be a cocycle.
2. By the Riesz Representation theorem, there exist $K_{n,m} \in K$ with

$$\sigma(S, k) = \sum_{n,m} \langle k, K_{n,m} \rangle \delta_n \otimes e_m.$$
Characterization of Cocycles

Proof.

3 By the product formula,

\[
\sigma(S^{N+1}, k) = \sum_{j=0}^{N} \pi(S^{N-j}) \sigma(S, \rho(S^j)k) = \sum_{j=0}^{N} \pi(S^{N-j}) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle \rho(S^j)k, K_{n,m} \rangle \delta_n \otimes e_m \\
= \sum_{j=0}^{N} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle k, \rho(S)^j K_{n,m} \rangle \delta_{N+n-j} \otimes e_m \\
= \sum_{n=0}^{\infty} \sum_{j=0}^{N} \sum_{m=0}^{\infty} \langle k, \rho(S)^j K_{n,m} \rangle \delta_{N+n-j} \otimes e_m
\]
Characterization of Cocycles

Proof.

4 The coefficient of the $\delta_\nu \otimes e_m$ term of $\sigma(S^{N+1}, k)$ is

$$\begin{cases} 
\sum_{j=0}^{N} \langle k, \rho(S)^j K_{\nu+j-N,m} \rangle & \text{for } \nu \geq N \\
\sum_{j=0}^{\nu} \langle k, \rho(S)^{N-\nu+j} K_j, m \rangle & \text{for } \nu < N.
\end{cases}$$

5 Therefore, $\left\{ \langle k, \sum_{j=1}^{N} \rho(S)^j K_{j+p,m} \rangle \right\}_{N=1}^\infty$ is a bounded sequence in $N$.

6 Letting $\text{Lim}$ be a Banach limit on $\ell^\infty$, we define $k_{p,m} \in K$ by

$$\langle k, k_{p,m} \rangle = \text{Lim}_{N \to \infty} \left\langle k, \sum_{j=0}^{N} \rho(S)^j K_{j+p,m} \right\rangle.$$
Characterization of Cocycles

Proof.

7 Define $\sigma_0$ by $\sigma_0(S, k) = \sum_m \langle k, k_{0,m} \rangle \delta_0 \otimes e_m$.
   Note: $\sigma_0$ is $A$-linear iff $\pi(\alpha(a)) k_{0,m} = \sum_p \langle \psi(a)e_m, e_p \rangle k_{0,p}$.

8 Define $L : K \rightarrow \ell^2(\mathbb{Z}^+; A) \otimes \psi E$ by
   $Lk = \sum_{j,m} \langle k, k_{j+1,m} \rangle \delta_j \otimes e_m$.

9 $\sigma(S, k) - \sigma_0(S, k) = (\pi(S)L - L\rho(S))k$. 


Ongoing and Future Work

1. Characterize the coboundaries.
2. Calculate $\text{Ext}^1(K, \ell^2(\mathbb{Z}^+; A) \otimes_\psi E)$.
3. Study the more general setting with $\alpha \in \text{End}(A)$.
4. Generalize to $\mathcal{T}_+(X)$.
5. Study projectivity and injectivity in terms of $\text{Ext}$.
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