Labeled graphs C^*-algebras with group actions

Teresa Bates, David Pask, Paulette N. Willis*

University of Iowa

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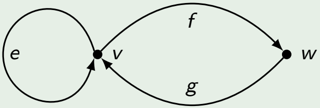
Definition

A directed graph $E = (E^0, E^1, r, s)$ consists of a vertex set E^0, an edge set E^1, and range and source maps $r, s : E^1 \to E^0$.

Example



Definition (Cuntz-Krieger algebra)

One can construct a C^*-algebra, $C^*(E)$ with generators $\{P_v\}_{v\in E^0}$ and $\{S_e\}_{e\in E^1}$ subject to the following relations:

(CK1) *The P_v are projections in $C^*(E)$.*

(CK2) *The S_e are partial isometries in $C^*(E)$.*

(CK3) $S_e^* S_e = P_{s(e)}$ *for all* $e \in E^1$; *and*

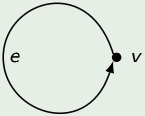
(CK4) $P_v = \sum_{r(e)=v} S_e S_e^*$ *provided the sum is finite and* $r^{-1}(v) \neq \emptyset$.

The algebra $C^*(E)$ is called a Cuntz-Krieger algebra or a graph C^*-algebra.

The C^*-algebra $C^*(E)$ encodes the properties of E, however non-isomorphic graphs can give rise to isomorphic $C^*(E)$. Graph C^*-algebras are of interest because they are easy to construct and provide an environment in which to test questions about arbitrary C^*-algebras.

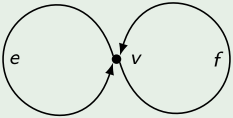
Example

Consider the following directed graph



The CK relations say that $S_e^* S_e = P_v = S_e S_e^*$. Therefore P_v is the identity and S_e is a unitary operator, hence $C^*(S_e)$ is isomorphic to the continuous functions on the circle $C(\mathbb{T})$.

Example



The CK relations say that

$$S_e^* S_e = P_v = S_f^* S_f \qquad \text{and} \qquad P_v = S_e S_e^* + S_f S_f^*.$$

Notice that P_v is the identity for $C^*(E)$. This C^*-algebra is simple (i.e. no non-zero ideals) and is called the Cuntz algebra. It is denoted $\mathcal{O}_2$.

Relationship to dynamical systems

If $E = (E^0, E^1, r, s)$ is a finite graph then the infinite path space

$$E^\infty := \{(e_1, e_2, \ldots) \mid s(e_i) = r(e_{i+1})\}$$

is a compact subset of $(E^1)^{\mathbb{N}}$ with the product topology that is invariant under the map σ defined by the equation

$$\sigma(e_1, e_2, \ldots) := (e_2, e_3, \ldots).$$

The map σ (restricted to E^∞) is called a shift of finite type. Much of the theory of shifts of finite type has direct analogues in the structure theory of the associated graph C^*-algebras.

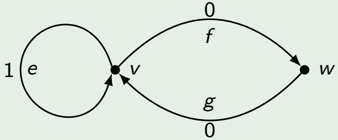
Definition

A labeled graph $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E together with a labeling map $\mathcal{L} : E^1 \to \mathcal{A}$.

We assume that $\mathcal{L} : E^1 \to \mathcal{A}$ is surjective, that is $\mathcal{L}(E^1) = \mathcal{A}$.

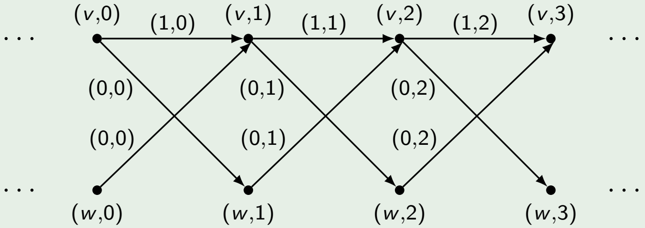
Examples

1. A directed graph may be considered a labeled graph where $\mathcal{A} = E^1$ and the labeling is the identity map.
2. The graph $(E, \mathcal{L})$ over the alphabet $\{0, 1\}$ is an example of a labeled graph.



Example

The graph $(E, \mathcal{L})$ over the alphabet $\{(\{0,1\}, \mathbb{Z})\}$ is an example of a labeled graph, where $E^0 = \{(\{v, w\}, \mathbb{Z})\}$.



Definition

Let $e, f \in E^1$ and $s(e) = r(f)$, then ef is a path of length 2. E^2 is the collection of all paths of length 2. Let $\mathcal{L}(e) = a$ and $\mathcal{L}(f) = b$, then $ab \in \mathcal{L}^2(E)$, where $\mathcal{L}^2(E)$ denotes the collection of labels for paths of length 2. More generally, E^n is the collection of paths of length n and $\mathcal{L}^n(E)$ is the collection of labeled paths of length n. $E^* = \cup_{n\in\mathbb{N}} E^n$, and $\mathcal{L}^*(E) = \cup_{n\in\mathbb{N}} \mathcal{L}^n(E)$.

Definition

A vertex v is a sink if it emits no edges, i.e. there does not exist an edge such that $s(e) = v$

Definition

For $B \subseteq E^0$, let $L_B := \{\beta \in \mathcal{A} : B \cap s(\beta) \neq \emptyset\}$ denote the labeled edges whose source intersects B nontrivially.

C^*-algebra of labeled graph

Definition

A Cuntz-Krieger $(E, \mathcal{L})$-family consists of projections $\{p_{r(\beta)} : \beta \in \mathcal{L}^*(E)\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that:

(CK1a) $p_{r(\beta)} p_{r(\omega)} = 0$ *if and only if* $r(\beta) \cap r(\omega) = \emptyset$

(CK1b) *For all* $\beta, \omega, \kappa \in \mathcal{L}^*(E)$, *if* $r(\beta) \cap r(\omega) = r(\kappa)$ *then* $p_{r(\beta)} p_{r(\omega)} = p_{r(\kappa)}$ *and if* $r(\beta) \cup r(\omega) = r(\kappa)$ *then* $p_{r(\beta)} + p_{r(\omega)} - p_{r(\beta)} p_{r(\omega)} = p_{r(\kappa)}$.

Remark

Using relations (CK1a) and (CK1b) we may now unambiguously define $p_{r(\beta) \cap r(\omega)} = p_{r(\beta)} p_{r(\omega)}$ *and* $p_{r(\beta) \cup r(\omega)} = p_{r(\beta)} + p_{r(\omega)} - p_{r(\beta)} p_{r(\omega)}$. *If* $r(\beta) \cap r(\omega) \neq \emptyset$, *then we write* $p_{r(\beta)} p_{r(\omega)} = p_{r(\beta) \cap r(\omega)}$, *so (CK1) implies that* $p_\emptyset = 0$.

C^*-algebra of labeled graph

Definition (continued)

(CK2) *If* $a \in \mathcal{A}$ *and* $\beta \in \mathcal{L}^*(E)$ *then* $p_{r(\beta)} s_a = s_a p_{r(\beta a)}$

(CK3) *If* $a, b \in \mathcal{A}$ *then* $s_a^* s_a = p_{r(a)}$ *and* $s_a^* s_b = 0$ *unless* $a = b$

(CK4) *For* $\beta \in \mathcal{L}^*(E)$, *if* $L_{r(\beta)}$ *is finite, non-empty, and* $r(\beta)$ *contains no sinks we have*

$$p_{r(\beta)} = \sum_{a \in L_{r(\beta)}} s_a p_{r(\beta a)} s_a^*.$$

Relationship to dynamical systems

Recall that the theory of shifts of finite type has direct analogues in the theory of graph C^*-algebras.
Labeled graph C^*-algebras are similarly connected to sofic shifts.

Definition

Let $(E,\mathcal{L})$ and $(F,\mathcal{M})$ be labeled graphs over alphabets $\mathcal{A}_E$ and $\mathcal{A}_F$ respectively. A labeled graph isomorphism is a triple $\phi := (\phi^0, \phi^1, \phi^{\mathcal{A}})$ where $\phi^0 : E^0 \to F^0$, $\phi^1 : E^1 \to F^1$, and $\phi^{\mathcal{A}_E} : \mathcal{A}_E \to \mathcal{A}_F$ such that

1. $\phi^0, \phi^1, \phi^{\mathcal{A}_E}$ *are bijective*
2. *For all $e \in E^1$ we have $\phi^0(r(e)) = r(\phi^1(e))$ and $\phi^0(s(e)) = s(\phi^1(e))$;*
3. $\phi^{\mathcal{A}_E} : \mathcal{A}_E \to \mathcal{A}_F$ *is a map such that $\mathcal{M} \circ \phi^1 = \phi^{\mathcal{A}_E} \circ \mathcal{L}$.*

If $F = E$ and $\mathcal{A}_E = \mathcal{A}_F$, then ϕ is called a labeled graph automorphism.

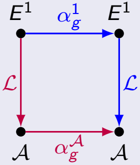
Let $(E,\mathcal{L})$ be a labeled graph over the alphabet $\mathcal{A}$, then the set

$$\mathrm{Aut}(E,\mathcal{L}) := \{\phi : (E,\mathcal{L}) \to (E,\mathcal{L}) : \phi \text{ is a labeled graph automorphism}\}$$

forms a group under composition.

Definition (group action)

Let $(E, \mathcal{L})$ *be a labeled graph over the alphabet* $\mathcal{A}$ *and* G *be a group. A labeled graph action of* G *on* $(E, \mathcal{L})$ *is a triple* $((E, \mathcal{L}), G, \alpha)$ *where* $\alpha : G \to \operatorname{Aut}(E, \mathcal{L})$ *is a group homomorphism. So for all* $e \in E^1$ *and* $g \in G$ *we have* $\mathcal{L}(\alpha_g^1(e)) = \alpha_g^{\mathcal{A}}(\mathcal{L}(e))$.



We say that the label graph action $((E, \mathcal{L}), G, \alpha)$ *is free if* $\alpha_g^0(v) = v$ *for all* $v \in E^0$ *implies* $g = 1_G$ *and if* $\alpha_g^{\mathcal{A}}(a) = a$ *for all* $a \in \mathcal{A}$ *then* $g = 1_G$.

Definition (quotient labeled graph)

Let $((E,\mathcal{L}),G,\alpha)$ be a labeled graph action. For $i=0,1$, $x\in E^i$, and $a\in\mathcal{A}$ let

$$Gx := \{\alpha^i_g(x) : g\in G\} \qquad Ga = \{\alpha^{\mathcal{A}}_g(a) : g\in G\}$$

$$(E/G)^i = \{Gx : x\in E^i\} \qquad \mathcal{A}/G = \{Ga : a\in\mathcal{A}\}$$

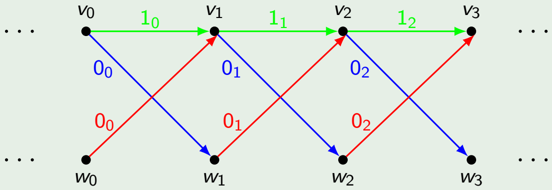
$$\mathcal{L}/G : (E/G)^1 \to \mathcal{A}/G \quad \text{be given by} \quad (\mathcal{L}/G)(Ge) = G\mathcal{L}(e)$$

$$r(Ge) = Gr(e) \qquad s(Ge) = Gs(e)$$

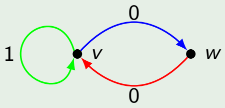
for $Ge\in(E/G)^1$. Then $(E/G,\mathcal{L}/G)$ is a labeled graph over $\mathcal{A}/G$ which we call the quotient labeled graph.

Example

Consider the following labeled graph



where $E^0 = \{\{v_i, w_i\} : i \in \mathbb{Z}\}$, $\mathcal{A} = \{\{0_i, 1_i\} : i \in \mathbb{Z}\}$, $G = \mathbb{Z}$, α is left translation. Then the quotient labeled graph is



Main Theorem (Bates, Pask, W*)

Let $(E, \mathcal{L})$ be a labeled graph and let G be a group acting freely on $(E, \mathcal{L})$ via α. Then (under a technical hypothesis) G acts on $C^*(E, \mathcal{L})$ via $\widetilde{\alpha}$ and

$$C^*(E, \mathcal{L}) \times_{\widetilde{\alpha}, r} G \cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)),$$

i.e. $C^*(E, \mathcal{L}) \times_{\widetilde{\alpha}, r} G$ is strongly Morita equivalent to $C^*(E/G, \mathcal{L}/G)$.

Steps

To prove

$$C^*(E,\mathcal{L}) \times_{\widetilde{\alpha},r} G \cong C^*(E/G,\mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G))$$

we need two theorems.

- The Gross-Tucker theorem for labeled graphs: a free action on a labeled graph is naturally equivariantly isomorphic to a skew product action obtained from the quotient labeled graph.
- A generalization of a theorem of Kaliszewski, Quigg, and Raeburn: the C^*-algebra of a skew product labeled graph is naturally isomorphic to a co-crossed product of a coaction of the group on the C^*-algebra of the labeled graph.

While we are interested in skew products of quotient graphs, for now we will define the skew product of a general labeled graph.

Definition

Let $(E, \mathcal{L})$ be a labeled graph and let $c, d : E^1 \to G$ be functions. The skew product labeled graph $(E \times_c G, \mathcal{L}_d)$ over alphabet $\mathcal{A} \times G$ is the skew product graph $E \times_c G$ defined by:

$$(E \times_c G)^0 := E^0 \times G \qquad (E \times_c G)^1 := E^1 \times G$$

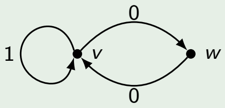
$$r(e, g) := (r(e), gc(e)) \qquad s(e, g) := (s(e), g)$$

together with the labeling $\mathcal{L}_d : E^1 \times G \to \mathcal{A} \times G$ given by

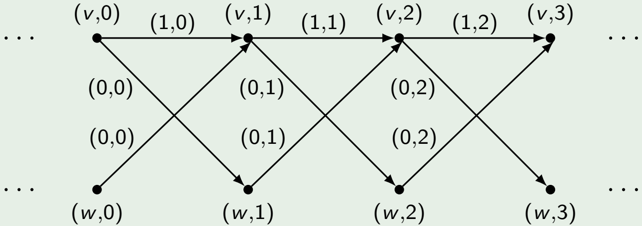
$$\mathcal{L}_d(e, g) := (\mathcal{L}(e), gd(e)).$$

Example

Consider the labeled graph $(E, \mathcal{L})$



Let $G = \mathbb{Z}$ and $c, d : E^1 \to \mathbb{Z}$ be $c(\varepsilon) = 1$ and $d(\varepsilon) = 0$, for $\varepsilon \in E^1$. Then the skew-product labeled graph $(E \times_c \mathbb{Z}, \mathcal{L}_d)$ is:



Example

For $(x, h) \in (E \times_c G)^i$, $(a, h) \in \mathcal{A} \times G$, $g \in G$, and $i = 0, 1$ let

$$\tau^i_g(x, h) = (x, gh) \qquad \text{and} \qquad \tau^{\mathcal{A}}_g(a, h) = (a, gh).$$

The map $\tau = (\tau^0, \tau^1, \tau^{\mathcal{A}}) : G \to \text{Aut}(E \times_c G, \mathcal{L}_d)$ defined by $g \to \tau_g$ is called the (left) labeled graph translation map.

The left labeled graph translation action is a free action since G acts freely on itself by left translation.

Gross-Tucker Theorem for Labeled Graphs

Theorem (Bates, Pask, W*)

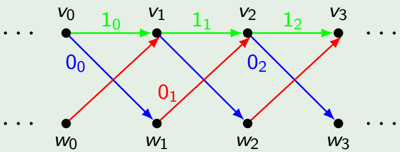
Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action. There are functions $c,d:(E/G)^1 \to G$ such that $((E,\mathcal{L}),G,\alpha)$ is equivariantly isomorphic to $((E/G \times_c G,(\mathcal{L}/G)_d),G,\tau)$, i.e. the isomorphism intertwines the actions.

So by this theorem we have

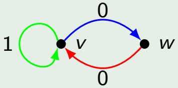
$$C^*(E,\mathcal{L}) \times_{\widetilde{\alpha},r} G \cong C^*(E/G \times_c G,(\mathcal{L}/G)_d) \times_{\widetilde{\tau},r} G.$$

Example

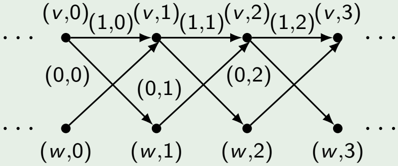
$(E, \mathcal{L}) :=$



$(E/\mathbb{Z}, \mathcal{L}/\mathbb{Z}) :=$



$(E/\mathbb{Z} \times_c \mathbb{Z}, \mathcal{L}_d) :=$

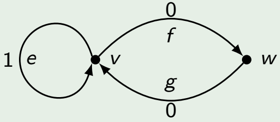


Definition

Let $(E, \mathcal{L})$ be a labeled graph and G be a group. A function $c : E^1 \to G$ is label consistent if whenever $e, f \in E^1$ satisfy $\mathcal{L}(e) = \mathcal{L}(f)$, then $c(e) = c(f)$.

Example

Consider the labeled graph $(E, \mathcal{L})$



Let $G = \mathbb{Z}$ and $c, d : E^1 \to \mathbb{Z}$ be $c(\varepsilon) = 1$ and $d(\varepsilon) = 0$, $\forall \varepsilon \in E^1$.

If $c : E^1 \to G$ is label consistent function, it induces a well-defined function $C : \mathcal{A} \to G$ such that $C(a) = c(e)$ where $\mathcal{L}(e) = a$.

Lemma

Let $(E, \mathcal{L})$ be a labeled graph, G be a discrete group, and $c : E^1 \to G$ be a label consistent function. Then there is a coaction $\delta : C^*(E, \mathcal{L}) \to C^*(E, \mathcal{L}) \otimes C^*(G)$ such that

$$\delta(s_a) = s_a \otimes u_{C(a)} \qquad \textit{and} \qquad \delta(p_{r(\beta)}) = p_{r(\beta)} \otimes u_{1_G}$$

where $\{s_a, p_{r(\beta)}\}$ is a Cuntz-Krieger $(E, \mathcal{L})$-family and $\{u_g : g \in G\}$ are the generators of $C^*(G)$.

Remark

Recall $((E \times_c G, \mathcal{L}_d), G, \tau)$, *the left labeled graph translation action, which is a free action. This induces an action* $\widetilde{\tau} : G \to \text{Aut}\, C^*(E \times_c G, \mathcal{L}_d)$ *such that*

$$\widetilde{\tau}_h(s_{a,g}) = s_{a,hg} \qquad \text{and} \qquad \widetilde{\tau}_h(p_{r(\beta),g}) = p_{r(\beta),hg}.$$

Remark

The C^**-algebra* $C^*(E, \mathcal{L}) \times_\delta G$ *carries an action* $\widehat{\delta}$ *of* G *defined via the formula* $\widehat{\delta}_h(b_g, x) = (b_g, xh^{-1})$.

Theorem (Bates, Pask, W*)

Let $(E, \mathcal{L})$ be a labeled graph, let G be a discrete group, let $c, d : E^1 \to G$ be label consistent functions, and let δ be the coaction. Then

$$C^*(E \times_c G, \mathcal{L}_d) \cong C^*(E, \mathcal{L}) \times_\delta G$$

equivariantly for the action $\widetilde{\tau}$ and the dual action $\widehat{\delta}$.

Remark

$C^*(E \times_c G) \cong C^*(E) \times_\delta G$ was proven by Kaliszewski, Quigg, and Raeburn, so our result that $C^*(E \times_c G, \mathcal{L}_d) \cong C^*(E, \mathcal{L}) \times_\delta G$ is a labeled graph version of their result.

Since the isomorphism is equivariant for $\widetilde{\tau}$ and $\widehat{\delta}$ it follows that

$$C^*(E \times_c G, \mathcal{L}_d) \times_{\widetilde{\tau},r} G \cong C^*(E, \mathcal{L}) \times_\delta G \times_{\widehat{\delta},r} G.$$

By Katayama's duality theorem we know that

$$C^*(E, \mathcal{L}) \times_\delta G \times_{\widehat{\delta},r} G \cong C^*(E, \mathcal{L}) \otimes \mathcal{K}(\ell^2(G)).$$

Therefore we have

Theorem

Let $(E, \mathcal{L})$ be a labeled graph, let G be a discrete group, let $c, d : E^1 \to G$ be label consistent functions, and let τ be the left labeled translation map. Then

$$C^*(E \times_c G, \mathcal{L}_d) \times_{\widetilde{\tau},r} G \cong C^*(E, \mathcal{L}) \otimes \mathcal{K}(\ell^2(G)).$$

If the labeled graph is a quotient labeled graph $(E/G, \mathcal{L}/G)$ then

$$C^*(E/G \times_c G, (\mathcal{L}/G)_d) \times_{\widetilde{\tau},r} G \cong C^*(E/G, \mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)).$$

The Gross-Tucker theorem for labeled graphs proves

$$((E,\mathcal{L}),G,\alpha)\cong((E/G\times_c G,(\mathcal{L}/G)_d),G,\tau)$$

which gives us $C^*(E,\mathcal{L})\times_{\widetilde{\alpha},r} G\cong C^*(E/G\times_c G,(\mathcal{L}/G)_d)\times_{\widetilde{\tau},r} G$. We have also shown that

$$C^*(E/G\times_c G,(\mathcal{L}/G)_d)\times_{\widetilde{\tau},r} G\cong C^*(E/G,\mathcal{L}/G)\otimes\mathcal{K}(\ell^2(G)),$$

therefore we have

$$C^*(E,\mathcal{L})\times_{\widetilde{\alpha},r} G\cong C^*(E/G,\mathcal{L}/G)\otimes\mathcal{K}(\ell^2(G)).$$

Remark

Notice that to define the coaction δ we relied on the assumption that c is label consistent. We need d label consistent to identify $\mathcal{L}_d^*(E\times_c G)$ with $\mathcal{L}^*(E)\times G$. Since the Gross-Tucker theorem only gives the existence of functions c and d, we must determine conditions under which c and d are label consistent.

Definition

Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action. A fundamental domain for $((E,\mathcal{L}),G,\alpha)$ is a set $T \subseteq E^0$ such that

1. *for every $v \in E^0$ there exists $g \in G$ and a unique $w \in T$ such that $v = \alpha_g^0 w$,*
2. *if $r(e), r(f) \in T$, and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$,*
3. *if $s(e), s(f) \in T$, and $G\mathcal{L}(e) = G\mathcal{L}(f)$, then $\mathcal{L}(e) = \mathcal{L}(f)$*

for every $e, f \in E^1$.

Proposition

Let $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action. Then by Gross-Tucker for labeled graphs there exists functions $c, d : (E/G)^1 \to G$ such that $((E,\mathcal{L}),G,\alpha) \cong ((E/G \times_c G, (\mathcal{L}/G)_d), G, \tau)$. If $((E,\mathcal{L}),G,\alpha)$ admits a fundamental domain, then c and d are label consistent.

Theorem (Bates, Pask, W*)

Let $(E,\mathcal{L})$ satisfy the nondegenerate hypothesis and $((E,\mathcal{L}),G,\alpha)$ be a free labeled graph action which admits a fundamental domain. Then

$$C^*(E,\mathcal{L}) \times_{\widetilde{\alpha},r} G \cong C^*(E/G,\mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)).$$

Proof.

There are label consistent functions $c,d : (E/G)^1 \to G$ such that

$$((E,\mathcal{L}),G,\alpha) \cong ((E/G \times_c G,(\mathcal{L}/G)_d),G,\tau),$$

so we have $C^*(E,\mathcal{L}) \times_{\widetilde{\alpha},r} G \cong C^*(E/G \times_c G,(\mathcal{L}/G)_d) \times_{\widetilde{\tau},r} G$. We have shown that

$$C^*(E/G \times_c G,(\mathcal{L}/G)_d) \times_{\widetilde{\tau},r} G \cong C^*(E/G,\mathcal{L}/G) \otimes \mathcal{K}(\ell^2(G)).$$

□