Commutative algebras of Toeplitz operators in action

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Commutative algebras of Toeplitz operators on the unit disk.

Fine structure of the algebra of Toeplitz operators with $PC$-symbols.

From the unit disk to the unit ball.
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- Commutative algebras of Toeplitz operators on the unit disk.
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The Toeplitz operator was originally defined in terms of the so-called Toeplitz matrix

\[ A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots \\ a_1 & a_0 & a_{-1} & \ldots \\ a_2 & a_1 & a_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \]

where \( a_n \in \mathbb{C}, \ n \in \mathbb{Z}. \)

**Theorem (O. Toeplitz, 1911)**

Matrix \( A \) defines a bounded operator on \( l_2 = l_2(\mathbb{Z}_+) \), where \( \mathbb{Z}_+ = \{0\} \cup \mathbb{N} \), if and only if the numbers \( \{a_n\} \) are the Fourier coefficients of a function \( a \in L_\infty(S^1) \), where \( S^1 \) is the unit circle.
The (discrete) Fourier transform $\mathcal{F}$ is a unitary operator which maps $L_2(S^1)$ onto $l_2(\mathbb{Z})$ and the Hardy space $H^2_+(S^1)$ onto $l_2(\mathbb{Z}_+)$. Then for the operator $A$, defined by the matrix $A$ we have

$$\mathcal{F}^{-1} A \mathcal{F} = T_a : H^2_+(S^1) \longrightarrow H^2_+(S^1).$$

The operator $T_a$ acts on the Hardy space $H^2_+(S^1)$ by the rule

$$T_a : f(t) \in H^2_+(S^1) \longmapsto (P_+af)(t) \in H^2_+(S^1),$$

where $P_+ : L_2(S^1) \longrightarrow H^2_+(S^1)$ is the Szegö orthogonal projection, and the Fourier coefficients of the function $a$ are given by the sequence $\{a_n\}$.
Let $H$ be a Hilbert space, $H_0$ be its subspace. Let $P_0 : H \hookrightarrow H_0$ be the orthogonal projection, and let $A$ be a bounded linear operator on $H$.

The Toeplitz operator with symbol $A$ 

$$T_A : x \in H_0 \hookrightarrow P_0(Ax) \in H_0$$

is the compression of $A$ (in our case of a multiplication operator) onto the subspace $H_0$, representing thus an important model case in operator theory.
Consider now $L_2(\mathbb{D})$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. The Bergman space $A^2(\mathbb{D})$ is the subspace of $L_2(\mathbb{D})$ consisting of functions analytic in $\mathbb{D}$. The Bergman orthogonal projection $B_\mathbb{D}$ of $L_2(\mathbb{D})$ onto $A^2(\mathbb{D})$ has the form

$$(B_\mathbb{D}\varphi)(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{\varphi(\zeta) \, d\mu(\zeta)}{(1 - z\zeta)^2},$$

The Toeplitz operator $T_a$ with symbol $a = a(z)$ acts as follows

$$T_a : \varphi(z) \in A^2(\mathbb{D}) \mapsto (B_\mathbb{D} a\varphi)(z) \in A^2(\mathbb{D}).$$
Consider the unit disk $\mathbb{D}$ endowed with the hyperbolic metric

$$g = ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$ 

A geodesic in $\mathbb{D}$ is (a part of) an Euclidean circle or a straight line orthogonal to the boundary $S^1 = \partial \mathbb{D}$.

Each pair of geodesics, say $L_1$ and $L_2$, lie in a geometrically defined object, one-parameter family $\mathcal{P}$ of geodesics, which is called the pencil determined by $L_1$ and $L_2$.

Each pencil has an associated family $\mathcal{C}$ of lines, called cycles, the orthogonal trajectories to geodesics forming the pencil.
There are three types of pencils of hyperbolic geodesics:

- parabolic,
- elliptic,
- hyperbolic.
Each Möbius transformation $g \in \text{Möb}(\mathbb{D})$ is a movement of the hyperbolic plane, determines a certain pencil of geodesics $\mathcal{P}$, and its action is as follows:

Each geodesic $L$ from the pencil $\mathcal{P}$, determined by $g$, moves along the cycles in $\mathcal{C}$ to the geodesic $g(L) \in \mathcal{P}$, while each cycle in $\mathcal{C}$ is invariant under the action of $g$.
Theorem

Given a pencil $\mathcal{P}$ of geodesics, consider the set of symbols which are constant on corresponding cycles. The $C^*$-algebra generated by Toeplitz operators with such symbols is commutative.

That is, each pencil of geodesics generates a commutative $C^*$-algebra of Toeplitz operators.
Model cases

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Consider the upper half-plane $\Pi$, the space $L_2(\Pi)$, and its Bergman subspace $A^2(\Pi)$. We construct the operator

$$R : L_2(\Pi) \longrightarrow L_2(\mathbb{R}),$$

whose restriction onto the Bergman space

$$R|_{A^2(\Pi)} : A^2(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism.

The adjoint operator

$$R^* : L_2(\mathbb{R}) \longrightarrow A^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto $A^2(\Pi)$.

Moreover we have

$$R R^* = I : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}),$$

$$R^* R = B_\Pi : L_2(\Pi) \longrightarrow A^2(\Pi).$$
Theorem

Let \( a = a(\theta) \in L_\infty(\Pi) \) be a homogeneous of order zero function, (a functions depending only on the polar angle \( \theta \)).

Then the Toeplitz operator \( T_a \) acting on \( \mathcal{A}^2(\Pi) \) is unitary equivalent to the multiplication operator \( \gamma_a I = R T_a R^* \), acting on \( L_2(\mathbb{R}) \).

The function \( \gamma_a(\lambda) \) is given by

\[
\gamma_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_{0}^{\pi} a(\theta) e^{-2\lambda\theta} \, d\theta, \quad \lambda \in \mathbb{R}.
\]
We consider the pair \((\mathbb{D}, \omega)\), where \(\mathbb{D}\) is the unit disk and
\[
\omega = \frac{1}{\pi} \frac{dx \wedge dy}{(1 - (x^2 + y^2))^2} = \frac{1}{2\pi i} \frac{d\bar{z} \wedge dz}{(1 - |z|^2)^2}.
\]

Poisson brackets:
\[
\{a, b\} = \pi(1 - (x^2 + y^2))^2 \left(\frac{\partial a}{\partial y} \frac{\partial b}{\partial x} - \frac{\partial a}{\partial x} \frac{\partial b}{\partial y}\right)
= 2\pi i(1 - z\bar{z})^2 \left(\frac{\partial a}{\partial z} \frac{\partial b}{\partial \bar{z}} - \frac{\partial a}{\partial \bar{z}} \frac{\partial b}{\partial z}\right).
\]

Laplace-Beltrami operator:
\[
\Delta = \pi(1 - (x^2 + y^2))^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)
= 4\pi(1 - z\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}}.
\]
Introduce weighted Bergman spaces $A^2_{\hbar}(\mathbb{D})$ with the scalar product

$$(\varphi, \psi) = \left(\frac{1}{\hbar} - 1\right) \int_{\mathbb{D}} \varphi(z) \overline{\psi(z)} (1 - zz)^{\frac{1}{\hbar}} \omega(z).$$

The weighted Bergman projection has the form

$$(B_{\mathbb{D}, \hbar} \varphi)(z) = \left(\frac{1}{\hbar} - 1\right) \int_{\mathbb{D}} \varphi(\zeta) \left(\frac{1 - \zeta \overline{\zeta}}{1 - z \zeta}\right)^{\frac{1}{\hbar}} \omega(\zeta).$$

Let $E = (0, \frac{1}{2\pi})$, for each $\hbar = \frac{h}{2\pi} \in E$, and consequently $h \in (0, 1)$, introduce the Hilbert space $H_{\hbar}$ as the weighted Bergman space $A^2_{\hbar}(\mathbb{D})$. 

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For each function \( a = a(z) \in C^{\infty}(\mathbb{D}) \) consider the family of Toeplitz operators \( T_{a}^{(h)} \) with (anti-Wick) symbol \( a \) acting on \( A_{h}^{2}(\mathbb{D}) \), for \( h \in (0, 1) \), and denote by \( \mathcal{I}_{h} \) the \(*\)-algebra generated by Toeplitz operators \( T_{a}^{(h)} \) with symbols \( a \in C^{\infty}(\mathbb{D}) \).

The Wick symbols of the Toeplitz operator \( T_{a}^{(h)} \) has the form

\[
\tilde{a}_{h}(z, \bar{z}) = \left( \frac{1}{h} - 1 \right) \int_{\mathbb{D}} a(\zeta) \left( \frac{(1 - |z|^{2})(1 - |\zeta|^{2})}{(1 - z\bar{\zeta})(1 - \zeta\bar{z})} \right)^{\frac{1}{h}} \omega(\zeta).
\]
For each $h \in (0, 1)$ define the function algebra

$$\tilde{\mathbb{A}}_h = \{\tilde{a}_h(z, \bar{z}) : a \in C^\infty(\mathbb{D})\}$$

with point wise linear operations, and with the multiplication law defined by the product of Toeplitz operators:

$$\tilde{a}_h \star \tilde{b}_h = \left(\frac{1}{h} - 1\right) \int_{\mathbb{D}} \tilde{a}_h(z, \bar{\zeta}) \tilde{b}_h(\zeta, \bar{z}) \left(\frac{(1 - |z|^2)(1 - |\zeta|^2)}{(1 - z\bar{\zeta})(1 - \zeta\bar{z})}\right)^{\frac{1}{h}} \omega.$$
The correspondence principle is given by

\[ \tilde{a}_h(z, \bar{z}) = a(z, \bar{z}) + O(\hbar), \]
\[ (\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) = i\hbar \{a, b\} + O(\hbar^2). \]
Three term asymptotic expansion

\[(\tilde{a}_h \star \tilde{b}_h - \tilde{b}_h \star \tilde{a}_h)(z, \bar{z}) =
\]
\[i\hbar \{a, b\} + \]
\[i\frac{\hbar^2}{4} (\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\} + 8\pi\{a, b\}) + \]
\[i\frac{\hbar^3}{24} \left[ \{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2\{a, b\} + \right. \]
\[\left. \Delta\{a, \Delta b\} + \Delta\{\Delta a, b\} + 28\pi (\Delta\{a, b\} + \{a, \Delta b\} + \{\Delta a, b\}) + 96\pi^2\{a, b\} \right] + \]
\[o(\hbar^3)\]
**Corollary**

Let $A(\mathbb{D})$ be a subspace of $C^\infty(\mathbb{D})$ such that for each $h \in (0, 1)$ the Toeplitz operator algebra $T_h(A(\mathbb{D}))$ is commutative. Then for all $a, b \in A(\mathbb{D})$ we have

\[
\{a, b\} = 0,
\{a, \Delta b\} + \{\Delta a, b\} = 0,
\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0.
\]
Let $\mathcal{A}(\mathbb{D})$ be a linear space of smooth functions which generates the commutative $C^*$-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators for each $h \in (0, 1)$.
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**First term:** $\{a, b\} = 0$:

**Lemma**

*All functions in $\mathcal{A}(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.*

**Second term:** $\{\Delta a, b\} + \{a, \Delta^2 b\} = 0$:

**Theorem**

The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common gradient lines are geodesics in the hyperbolic geometry of the unit disk $\mathbb{D}$.

**Third term:** $\{\Delta a, \Delta b\} + \{a, \Delta^2 b\} + \{\Delta^2 a, b\} = 0$:

**Theorem**

The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common level lines are cycles.
Let $\mathcal{A}(\mathbb{D})$ be a linear space of smooth functions which generates the commutative $C^*$-algebra $T_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators for each $h \in (0, 1)$.

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**Theorem**

*The space $\mathcal{A}(\mathbb{D})$ consists of functions whose common level lines are cycles.*
Theorem

Let $\mathcal{A}(\mathbb{D})$ be a space of smooth functions. Then the following two statements are equivalent:

1. there is a pencil $\mathcal{P}$ of geodesics in $\mathbb{D}$ such that all functions in $\mathcal{A}(\mathbb{D})$ are constant on the cycles of $\mathcal{P}$;

2. the $C^*$-algebra generated by Toeplitz operators with $\mathcal{A}(\mathbb{D})$-symbols is commutative on each weighted Bergman space $\mathcal{A}^2_h(\mathbb{D})$, $h \in (0, 1)$.
Commutative algebras of Toeplitz operators on the unit disk.

Fine structure of the algebra of Toeplitz operators with $PC$-symbols.

From the unit disk to the unit ball.
Let $\mathcal{T}(C(\overline{D}))$ be the $C^*$-algebra generated by $T_a$, with $a \in C(\overline{D})$.

**Theorem**

The algebra $\mathcal{T} = \mathcal{T}(C(\overline{D}))$ is irreducible and contains the whole ideal $\mathcal{K}$ of compact on $A^2(\mathbb{D})$ operators. Each operator $T \in \mathcal{T}(C(\overline{D}))$ is of the form

$$T = T_a + K,$$

where $a \in C(\overline{D})$, $K \in \mathcal{K}$.

**The homomorphism**

$$\text{sym} : \mathcal{T} \longrightarrow \text{Sym} \ \mathcal{T} = \mathcal{T} / \mathcal{K} \cong C(\partial \mathbb{D})$$

is generated by

$$\text{sym} : T_a \longmapsto a|_{\partial \mathbb{D}}.$$
Fix a finite number of distinct points $T = \{t_1, \ldots, t_m\}$ on $\gamma = \partial \mathbb{D}$. Let $\ell_k, k = 1, \ldots, m$, be the part of the radius of $\mathbb{D}$ starting at $t_k$. Let $\mathcal{L} = \bigcup_{k=1}^{m} \ell_k$. 

Denote by $\text{PC}(\mathbb{D}, T)$ the set (algebra) of all piece-wise continuous functions on $\mathbb{D}$ which are continuous in $\mathbb{D} \setminus \mathcal{L}$, have one-sided limit values at each point of $\mathcal{L}$. 

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Fix a finite number of distinct points \( T = \{t_1, \ldots, t_m\} \) on \( \gamma = \partial \mathbb{D} \). Let \( \ell_k, \ k = 1, \ldots, m \), be the part of the radius of \( \mathbb{D} \) starting at \( t_k \). Let \( \mathcal{L} = \bigcup_{k=1}^{m} \ell_k \).

Denote by \( PC(\mathbb{D}, T) \) the set (algebra) of all piece-wise continuous functions on \( \mathbb{D} \) which are

- continuous in \( \overline{\mathbb{D}} \setminus \mathcal{L} \),
- have one-sided limit values at each point of \( \mathcal{L} \).

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We consider the $C^*$-algebra $\mathcal{T}_{PC} = \mathcal{T}(PC(\overline{D}, \ell))$ generated by all Toeplitz operators $T_a$ with symbols $a(z) \in PC(\overline{D}, \ell)$.\[\text{Bad news:}\] Let $a(z), b(z) \in PC(\overline{D}, \ell)$, then $[T_a, T_b] = T_a T_b - T_{ab}$ is not compact in general. That is $T_a T_b \neq T_{ab} + K$. The algebra $\mathcal{T}_{PC}$ has a more complicated structure.
We consider the $C^*$-algebra $\mathcal{T}_{PC} = \mathcal{T}(PC(\overline{D}, \ell))$ generated by all Toeplitz operators $T_a$ with symbols $a(z) \in PC(\overline{D}, \ell)$.

Bad news: Let $a(z), b(z) \in PC(\overline{D}, \ell)$, then

$$[T_a, T_b] = T_a T_b - T_{ab}$$

is not compact in general.

That is

$$T_a T_b \neq T_{ab} + K.$$ 

The algebra $\mathcal{T}_{PC}$ has a more complicated structure.
For piece-wise continuous symbols the $C^*$-algebra $\mathcal{T}_{PC}$ contains:

- initial generators $T_a$, where $a \in PC$,

$$\sum_{k=1}^{p} \prod_{j=1}^{q_k} T_{a_{j,k}}, \quad a_{j,k} \in PC,$$

- uniform limits of sequences of such elements.
Compact set $\Gamma$

For each $a_1, a_2 \in PC(\overline{D}, \ell)$ the commutator $[T_{a_1}, T_{a_2}]$ is compact, thus the algebra $\text{Sym} \mathcal{T}_{PC}$ is commutative. And thus

$$\text{Sym} \mathcal{T}_{PC} \cong C(\text{over certain compact set } \Gamma).$$
Compact set $\Gamma$

For each $a_1, a_2 \in PC(\mathbb{D}, \ell)$ the commutator $[T_{a_1}, T_{a_2}]$ is compact, thus the algebra $\text{Sym} T_{PC}$ is commutative. And thus

$$\text{Sym} T_{PC} \cong C(\text{over certain compact set } \Gamma).$$

The set $\Gamma$ is the union $\widehat{\gamma} \cup (\bigcup_{k=1}^{m} [0, 1)_k)$, where $\widehat{\gamma}$ be the boundary $\gamma$, cut by points $t_k \in T$, with the following point identification

$$t_k - 0 \equiv 0_k, \quad t_k + 0 \equiv 1_k.$$
Theorem
The symbol algebra $\text{Sym} \, \mathcal{T}(PC(\overline{D}, \ell)) = \mathcal{T}(PC(\overline{D}, \ell))/\mathcal{K}$ is isomorphic and isometric to $C(\Gamma)$.

The homomorphism

$$\text{sym} : \mathcal{T}(PC(\overline{D}, \ell)) \to \text{Sym} \, \mathcal{T}(PC(\overline{D}, \ell)) = C(\Gamma)$$

is generated by

$$\text{sym} : \mathcal{T}_a \mapsto \begin{cases} a(t), & t \in \hat{\gamma} \\ a(t_k - 0)(1 - x) + a(t_k + 0)x, & x \in [0, 1] \end{cases},$$

where $t_k \in T$, $k = 1, 2, \ldots, m$. 

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For each $k = 1, \ldots, m$, let

$$\chi_k = \chi_k(z)$$

be the characteristic function of the half-disk obtained by cutting $\mathbb{D}$ by the diameter passing through $t_k \in T$, and such that $\chi_k^+(t_k) = 1$, and thus $\chi_k^-(t_k) = 0$. 

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Auxiliary functions: $v_k$

For two small neighborhoods $V'_k \subset V''_k$ of the point $t_k \in T$, let

$$v_k = v_k(z) : \overline{D} \rightarrow [0, 1]$$

be a continuous function such that

$$v_k|_{\overline{V'_k}} \equiv 1, \quad v_k|_{\overline{D \setminus V''_k}} \equiv 0.$$
Let $a \in PC(\mathbb{D}, T)$. Then

$$T_a = T_{s_a} + \sum_{k=1}^{m} T_{v_k} p_{a,k}(T_{\chi_k}) T_{v_k} + K,$$

where $K$ is compact, $s_a \in C(\mathbb{D})$,

$$s_a(z)|_{\gamma} \equiv \left[ a(z) - \sum_{k=1}^{m} [a^{-}(t_k) + (a^{+}(t_k) - a^{-}(t_k))\chi_k(z)]v_k^2(z) \right]_{\gamma},$$

$$p_{a,k}(x) = a^{-}(t_k)(1-x) + a^{+}(t_k)x.$$
Let

\[ A = \sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}}, \]

then

\[ A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} p_{A,k}(T_{\chi_k}) T_{v_k} + K_A, \]

where \( s_A \in C(\overline{D}) \), \( p_{A,k} = p_{A,k}(x), \) \( k = 1, \ldots, m, \) are polynomials, and \( K_A \) is compact.
Theorem

Every operator $A \in \mathcal{T}(PC(\overline{D}, T))$ admits the canonical representations

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T\chi_k) T_{v_k} + K,$$

where $s_A(z) \in C(\overline{D})$, $f_{A,k}(x) \in C[0,1]$, $k = 1, \ldots, m$, $K$ is compact.
Theorem

An operator

\[ A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K \]

is a compact perturbation of a Toeplitz operator if and only if every operator \( f_{A,k}(T_{\chi_k}) \) is a Toeplitz operator, where \( k = 1, \ldots, m \).
Theorem

An operator

\[
A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K
\]

is a compact perturbation of a Toeplitz operator if and only if every operator \( f_{A,k}(T_{\chi_k}) \) is a Toeplitz operator, where \( k = 1, \ldots, m \).

Let \( f_{A,k}(T_{\chi_k}) = T_{a_k} \) for some \( a_k \in L_\infty(\mathbb{D}) \). Then \( A = T_a + K_A \), where

\[
a(z) = s_A(z) + \sum_{k=1}^{m} a_k(z)v_k^2(z).
\]
Example

The Toeplitz operator $T_{\chi^+}$ is self-adjoint and $\text{sp } T_{\chi^+} = [0, 1]$. By functional calculus, for each $f \in C([0, 1])$, the operator $f(T_{\chi^+})$ is well defined and belongs to the $C^*$-algebra generated by $T_{\chi^+}$.
The Toeplitz operator $T_{\chi^+}$ is self-adjoint and $\text{sp } T_{\chi^+} = [0, 1]$. By functional calculus, for each $f \in C([0, 1])$, the operator $f(T_{\chi^+})$ is well defined and belongs to the $C^*$-algebra generated by $T_{\chi^+}$.

For any $\alpha \in (0, 1)$, introduce

$$f_\alpha(x) = x^{2(1-\alpha)} \frac{(1-x)^{2\alpha} - x^{2\alpha}}{(1-x) - x}, \quad x \in [0, 1].$$

Then

$$f_\alpha(T_{\chi^+}) = T_{\chi[0,\alpha\pi]}.$$
Example
Let \( p(x) = \sum_{k=1}^{n} a_k x^k \) be a polynomial of degree \( n \geq 2 \). Then the bounded operator \( p(T_{\chi_+}) \) is not a Toeplitz operator.
Example

Let \( p(x) = \sum_{k=1}^{n} a_k x^k \) be a polynomial of degree \( n \geq 2 \). Then the bounded operator \( p(T_{\chi^+}) \) is not a Toeplitz operator.

Corollary

Let

\[ A = \sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}} \in \mathcal{T}(PC(\mathbb{D}, T)). \]

Then \( A \) is a compact perturbation of a Toeplitz operator if and only if \( A \) is a compact perturbation an initial generator \( T_a \), for some \( a \in PC(\mathbb{D}, T) \).
Each operator $A \in \mathcal{T}(PC(\mathbb{D}, T))$ admits a transparent canonical representation

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K.$$
Each operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ admits a transparent canonical representation

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K.$$ 

All initial generators $T_a, a \in PC(\overline{\mathbb{D}}, T)$ are Toeplitz operators.
Each operator $A \in T(PC(\mathbb{D}, T))$ admits a transparent canonical representation

$$A = T_{sA} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K.$$

All initial generators $T_a$, $a \in PC(\mathbb{D}, T)$ are Toeplitz operators.

None of the (non trivial) elements

$$\sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}},$$

is a compact perturbation of a Toeplitz operator.
The uniform closure contains a huge amount of Toeplitz operators, with bounded and even unbounded symbols, which are drastically different from the initial generators.
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All these Toeplitz operators are uniform limits of sequences of non-Toeplitz operators.
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The uniform closure contains as well many non-Toeplitz operators.
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Model cases

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Model Maximal Commutative Subgroups

- **Elliptic**: $\mathbb{T}$, with $z \in \mathcal{D} \mapsto tz \in \mathcal{D}$, $t \in \mathbb{T}$.

- **Hyperbolic**: $\mathbb{R}^+$, with $z \in \Pi \mapsto rz \in \Pi$, $r \in \mathbb{R}^+$.

- **Parabolic**: $\mathbb{R}$, with $z \in \Pi \mapsto z + h \in \Pi$, $h \in \mathbb{R}$.

We consider the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$,

$$\mathbb{B}^n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z|^2 = |z_1|^2 + \ldots + |z_n|^2 < 1 \}.$$
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For each $\lambda \in (-1, \infty)$, introduce the measure

$$d\mu_\lambda(z) = c_\lambda (1 - |z|^2)^\lambda \, dv(z),$$

where $dv(z) = dx_1 dy_1 \ldots dx_n dy_n$ and

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}.$$

The (weighted) Bergman space $A^2_\lambda(\mathbb{B}^n)$ is the subspace of $L^2(\mathbb{B}^n, d\mu_\lambda)$ consisting of functions analytic in $\mathbb{B}^n$.

The orthogonal Bergman projection has the form

$$(B_{\mathbb{B}^n} \varphi)(z) = \int_{\mathbb{B}^n} \varphi(\zeta) \frac{(1 - |\zeta|^2)^\lambda}{(1 - z \cdot \zeta)^{n+\lambda+1}} \, c_\lambda \, dv(\zeta).$$
The standard unbounded realization of the unit disk $\mathbb{D}$ is the upper half-plane

$$\Pi = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \}.$$ 

The standard unbounded realization of the unit ball $\mathbb{B}^n$ is the Siegel domain in $\mathbb{C}^n$

$$D_n = \{ z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im} \, z_n - |z'|^2 > 0 \},$$

where we use the following notation for the points of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$:

$$z = (z', z_n), \quad \text{where} \quad z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}, \quad z_n \in \mathbb{C}.$$
Model Maximal Commutative Subgroups

- **Quasi-elliptic:** $\mathbb{T}^n$, for each $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$:
  $$z = (z_1, \ldots, z_n) \in \mathbb{B}^n \mapsto tz = (t_1z_1, \ldots, t_nz_n) \in \mathbb{B}^n;$$

- **Quasi-hyperbolic:** $\mathbb{T}^{n-1} \times \mathbb{R}_+$, for each $(t, r) \in \mathbb{T}^{n-1} \times \mathbb{R}_+$:
  $$(z', z_n) \in D_n \mapsto (r^{1/2}tz', rz_n) \in D_n;$$

- **Quasi-parabolic:** $\mathbb{T}^{n-1} \times \mathbb{R}$, for each $(t, h) \in \mathbb{T}^{n-1} \times \mathbb{R}$:
  $$(z', z_n) \in D_n \mapsto (tz', z_n + h) \in D_n;$$

- **Nilpotent:** $\mathbb{R}^{n-1} \times \mathbb{R}$, for each $(b, h) \in \mathbb{R}^{n-1} \times \mathbb{R}$:
  $$(z', z_n) \in D_n \mapsto (z' + b, z_n + h + 2iz' \cdot b + i|b|^2) \in D_n;$$

- **Quasi-nilpotent:** $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, $0 < k < n - 1$, for each $(t, b, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$:
  $$(z', z'', z_n) \in D_n \mapsto (tz', z'' + b, z_n + h + 2iz'' \cdot b + i|b|^2) \in D_n.$$
Theorem

Given any maximal commutative subgroup $G$ of biholomorphisms of the unit ball $\mathbb{B}^n$, denote by $\mathcal{A}_G$ the set of all $L_\infty(\mathbb{B}^n)$-functions which are invariant under the action of $G$. Then the $C^*$-algebra generated by Toeplitz operators with symbols from $\mathcal{A}_G$ is commutative on each weighted Bergman space $A^2_\lambda(\mathbb{B}^n)$, $\lambda \in (-1, \infty)$. 
Theorem

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Then the $C^*$-algebra generated by Toeplitz operators with symbols from $\mathcal{A}_G$ is commutative on each weighted Bergman space $A^2_\lambda(\mathbb{B}^n)$, $\lambda \in (-1, \infty)$.

The result can be alternatively formulated in terms of the so-called Lagrangian frames, the multidimensional analog of pencils of geodesics and cycles of the unit disk.
It was firmly expected that the situation for the unit ball is pretty much the same as in the case of the unit disk, that is:

The above algebras exhaust all possible algebras of Toeplitz operators on the unit ball which are commutative on each weighted Bergman space.
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But:

It turns out that there exist many other Banach algebras generated by Toeplitz operators which are commutative on each weighted Bergman space, non of them is a $C^*$-algebra, and for $n = 1$ all of them collapse to known commutative $C^*$-algebras of the unit disk.